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Strong convergence theorem of two-step iterative algorithm for split feasibility problems

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Dedicated to Professor Shih-sen Chang on the occasion of his 80th birthday

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Abstract

The main purpose of this paper is to introduce a two-step iterative algorithm for split feasibility problems such that the strong convergence is guaranteed. Our result extends and improves the corresponding results of He *et al.* and some others.

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1 Introduction

The split feasibility problem (SFP) was proposed by Censor and Elfving in [1]. It can be formulated as the problem of finding a point x satisfying the property

$$x \in C, \qquad Ax \in Q, \tag{1.1}$$

where A is a given $M \times N$ real matrix, C and Q are nonempty, closed and convex subsets in $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively.

Due to its extraordinary utility and broad applicability in many areas of applied mathematics (most notably, fully-discretized models of problems in image reconstruction from projections, in image processing, and in intensity-modulated radiation therapy), algorithms for solving convex feasibility problems continue to receive great attention (see, for instance, [2–5] and also [6–10]).

We assume that SFP (1.1) is consistent, and let Γ be the solution set, *i.e.*,

$$\Gamma = \{x \in C : Ax \in Q\}.$$

It is not hard to see that Γ is closed convex and $x \in \Gamma$ if and only if it solves the fixed-point equation

$$x = P_C\big(I - \gamma A^*(I - P_Q)A\big)x, \tag{1.2}$$

where P_C and P_Q are the orthogonal projections onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A.

To solve (1.2), Byrne [11] proposed his CQ algorithm which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C\big(x_n - \tau_n A^*(I - P_Q)Ax_n\big), \tag{1.3}$$

where $\tau_n \in (0, \frac{2}{\|A\|^2})$.

The CQ algorithm (1.3) can be obtained from optimization. In fact, if we introduce the convex objective function

$$f(x) = \frac{1}{2}\big\|(I - P_Q)Ax\big\|^2, \tag{1.4}$$

and analyze the minimization problem

$$\min_{x\in C} f(x), \tag{1.5}$$

then the CQ algorithm (1.3) comes immediately as a special case of the gradient projection algorithm (GPA). Since the convex objective function $f(x)$ is differentiable and has a Lipschitz gradient, which is given by

$$\nabla f(x) = A^*(I - P_Q)Ax, \tag{1.6}$$

the GPA for solving the minimization problem (1.4) generates a sequence $\{x_n\}$ recursively as

$$x_{n+1} = P_C\big(x_n - \tau_n \nabla f(x_n)\big), \tag{1.7}$$

where τ_n is chosen in the interval $(0, \frac{2}{L})$, and L is the Lipschitz constant of ∇f.

Observe that in algorithms (1.3) and (1.7) mentioned above, in order to implement the CQ algorithm, one has to compute the operator norm $\|A\|$, which is in general not an easy work in practice. To overcome this difficulty, some authors proposed different adaptive choices of selecting the τ_n (see [11–13]). For instance, López *et al.* introduced a new way of selecting the stepsize [12] as follows:

$$\tau_n := \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \quad 0 < \rho_n < 4. \tag{1.8}$$

The computation of a projection onto a general closed convex subset is generally difficult. To overcome this difficulty, Fukushima [14] suggested the so-called relaxed projection method to calculate the projection onto a level set of a convex function by computing a sequence of projections onto half-spaces containing the original level set. In the setting of finite-dimensional Hilbert spaces, this idea was followed by Yang [9], who introduced the relaxed CQ algorithms for solving SFP (1.1) where the closed convex subsets C and Q are level sets of convex functions.

Recently, for the purpose of generality, SFP (1.1) has been studied in a more general setting. For instance, see [8, 12]. However, their algorithm has only weak convergence in the setting of infinite-dimensional Hilbert spaces. Very recently, He and Zhao [15] introduced a new relaxed CQ algorithm (1.9) such that the strong convergence is guaranteed in infinite-dimensional Hilbert spaces:

$$x_{n+1} = P_{C_n}\big(\alpha_n u + (1-\alpha_n)\big(x_n - \tau_n \nabla f_n(x_n)\big)\big). \tag{1.9}$$

Motivated and inspired by the research going on in this section, the purpose of this article is to study a two-step iterative algorithm for split feasibility problems such that the strong convergence is guaranteed in infinite-dimensional Hilbert spaces. Our result extends and improves the corresponding results of He and Zhao [15] and some others.

2 Preliminaries and lemmas

Throughout the rest of this paper, we assume that H, H_1 and H_2 all are Hilbert spaces, A is a bounded linear operator from H_1 to H_2, and I is the identity operator on H, H_1 or H_2. If $f : H \to \mathbb{R}$ is a differentiable function, then we denote by ∇f the gradient of the function f. We will also use the following notations: $\to$ to denote the strong convergence, $\rightharpoonup$ to denote the weak convergence and

$$w_\omega(x_n) = \big\{x \mid \exists \{x_{n_k}\} \subset \{x_n\} \text{ such that } x_{n_k} \rightharpoonup x\big\}$$

to denote the weak ω-limit set of $\{x_n\}$.

Recall that a mapping $T : H \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad x, y \in H.$$

$T : H \to H$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \big\|(I-T)x - (I-T)y\big\|^2, \quad x, y \in H.$$

A mapping $T : H \to H$ is said to be demiclosed at origin if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$ and $\limsup_{n\to\infty} \|(I-T)x_n\| = 0$, then $x^* = Tx^*$.

It is easy to prove that if $T : H \to H$ is a firmly nonexpansive mapping, then T is demiclosed at origin.

A function $f : H \to \mathbb{R}$ is called convex if

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y), \quad \forall \lambda \in (0,1), \forall x, y \in H.$$

A function $f : H \to \mathbb{R}$ is said to be weakly lower semi-continuous (w-lsc) at x if $x_n \rightharpoonup x$ implies

$$f(x) \le \liminf_{n\to\infty} f(x_n).$$

Lemma 2.1 *Let $T : H_2 \to H_2$ be a firmly nonexpansive mapping such that $\|(I-T)x\|$ is a convex function from H_2 to $\mathbb{R}$, let $A : H_1 \to H_2$ be a bounded linear operator and*

$$f(x) = \frac{1}{2}\big\|(I-T)Ax\big\|^2, \quad \forall x \in H_1.$$

Then

(i) $\nabla f(x) = A^*(I-T)Ax$, $x \in H_1$.
(ii) ∇f *is* $\|A\|^2$*-Lipschitz*: $\|\nabla f(x) - \nabla f(y)\| \leq \|A\|^2\|x-y\|$, $x, y \in H_1$.

Proof (i) From the definition of f, we know that f is convex. For any given $x \in H_1$ and for any $v \in H_1$, first we prove that the limit

$$\langle \nabla f(x), v \rangle = \lim_{h\to 0+} \frac{f(x+hv) - f(x)}{h}$$

exists in $\bar{\mathscr{R}} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ and satisfies

$$\langle \nabla f(x), v \rangle \leq f(x+v) - f(x), \quad \forall v \in H_1.$$

If fact, if $0 < h_1 \leq h_2$, then

$$f(x+h_1 v) - f(x) = f\left(\frac{h_1}{h_2}(x+h_2 v) + \left(1 - \frac{h_1}{h_2}\right)x\right) - f(x).$$

Since f is convex and $\frac{h_1}{h_2} \leq 1$, it follows that

$$f(x+h_1 v) - f(x) \leq \frac{h_1}{h_2} f(x+h_2 v) + \left(1 - \frac{h_1}{h_2}\right) f(x) - f(x),$$

and

$$\frac{f(x+h_1 v) - f(x)}{h_1} \leq \frac{f(x+h_2 v) - f(x)}{h_2}.$$

This shows that this difference quotient is increasing, therefore it has a limit in $\bar{\mathscr{R}}$ as $h \to 0+$:

$$\langle \nabla f(x), v \rangle = \inf_{h>0} \frac{f(x+hv) - f(x)}{h} = \lim_{h\to 0+} \frac{f(x+hv) - f(x)}{h}. \tag{2.1}$$

This implies that f is differential. Taking $h = 1$ in (2.1), we have

$$\langle \nabla f(x), v \rangle \leq f(x+v) - f(x).$$

Next we prove that

$$\nabla f(x) = A^*(I-T)Ax, \quad x \in H_1.$$

In fact, since

$$\lim_{h\to 0+} \frac{f(x+hv) - f(x)}{h} = \lim_{h\to 0+} \frac{\|Ax + hAv - TA(x+hv)\|^2 - \|(I-T)Ax\|^2}{2h} \tag{2.2}$$

and

$$\begin{aligned}&\big\|Ax + hAv - TA(x + hv)\big\|^2 - \big\|(I - T)Ax\big\|^2\\&\quad= \|Ax\|^2 + h^2\|Av\|^2 + 2h\langle A^*Ax, v\rangle\\&\qquad+ \big\|TA(x + hv)\big\|^2 - \|Ax\|^2 - \|TAx\|^2 - 2\langle Ax, TA(x + hv) - TAx\rangle\\&\qquad- 2h\langle A^*TA(x + hv), v\rangle.\end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), simplifying it and then letting $h \to 0+$, we have

$$\begin{aligned}\lim_{h\to 0+}\frac{f(x + hv) - f(x)}{h} &= \lim_{h\to 0+}\frac{2h\{\langle A^*Ax, v\rangle - \langle A^*TA(x + hv), v\rangle\}}{2h}\\&= \langle A^*(I - T)Ax, v\rangle, \quad \forall v \in H_1.\end{aligned}$$

It follows from (2.1) that

$$\nabla f(x) = A^*(I - T)Ax, \quad x \in H_1.$$

Now we prove conclusion (ii). Indeed, it follows from (i) that

$$\begin{aligned}\big\|\nabla f(x) - \nabla f(y)\big\| &= \big\|A^*(I - T)Ax - A^*(I - T)Ay\big\| = \big\|A^*\big[(I - T)Ax - (I - T)Ay\big]\big\|\\&\le \|A\|\|Ax - Ay\| \le \|A\|^2\|x - y\|, \quad x, y \in H_1.\end{aligned}$$

Lemma 2.1 is proved. □

Lemma 2.2 *Let $T : H \to H$ be an operator. The following statements are equivalent.*
(i) *T is firmly nonexpansive.*
(ii) *$\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle$, $\forall x, y \in H$.*
(iii) *$I - T$ is firmly nonexpansive.*

Proof (i) ⇒ (ii): Since T is firmly nonexpansive, we have, for all $x, y \in H$,

$$\begin{aligned}\|Tx - Ty\|^2 &\le \|x - y\|^2 - \big\|(I - T)x - (I - T)y\big\|^2 = \|x - y\|^2 - \big\|(x - y) - (Tx - Ty)\big\|^2\\&= \|x - y\|^2 - \|x - y\|^2 - \|Tx - Ty\|^2 + 2\langle x - y, Tx - Ty\rangle\\&= 2\langle x - y, Tx - Ty\rangle - \|Tx - Ty\|^2,\end{aligned}$$

hence

$$\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle, \quad \forall x, y \in H.$$

(ii) ⇒ (iii): From (ii) we know that for all $x, y \in H$,

$$\begin{aligned}\big\|(I - T)x - (I - T)y\big\|^2 &= \big\|(x - y) + (Tx - Ty)\big\|^2\\&= \|x - y\|^2 - 2\langle x - y, Tx - Ty\rangle + \|Tx - Ty\|^2\end{aligned}$$

$$\begin{aligned} &\le \|x-y\|^2 - \langle x-y, Tx-Ty\rangle \\ &= \bigl\langle x-y, (I-T)x-(I-T)y\bigr\rangle. \end{aligned}$$

This implies that $I-T$ is firmly nonexpansive.

(iii) $\Rightarrow$ (i): From (iii) we immediately know that T is firmly nonexpansive. □

Lemma 2.3 [16] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\sigma_n, \quad n = 0,1,2,\ldots,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n \le 0$, *or* $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 [17] *Let X be a real Banach space and $J : X \to 2^{X^*}$ be the normalized duality mapping, then, for any $x, y \in X$, the following inequality holds*:

$$\|x+y\|^2 \le \|x\|^2 + 2\bigl\langle y, j(x+y)\bigr\rangle, \quad \forall j(x+y) \in J(x,y).$$

Especially, if X is a real Hilbert space, then we have

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,y \in X.$$

3 Main results

In this section, we shall prove our main theorem.

Theorem 3.1 *Let H_1, H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_1 \to H_1$ be a firmly nonexpansive mapping, $T_1, T_2 : H_2 \to H_2$ be two firmly nonexpansive mappings such that $\|(I-T_i)x\|$ $(i = 1,2)$ is a convex function from H_2 to $\mathbb{R}$ with $C := F(S) \ne \emptyset$ and $Q := F(T_1) \cap F(T_2) \ne \emptyset$. Let $u \in H_1$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in H_1 \text{ chosen arbitrarily}, \\ x_{n+1} = S(\alpha_n u + (1-\alpha_n)(y_n - \xi_n \nabla g(y_n))), \\ y_n = \beta_n u + (1-\beta_n)(x_n - \tau_n \nabla f(x_n)), \end{cases} \tag{3.1}$$

where

$$f(x_n) = \frac{1}{2}\bigl\|(I-T_1)Ax_n\bigr\|^2, \qquad \nabla f(x_n) = A^*(I-T_1)Ax_n, \qquad \tau_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}$$

and

$$g(y_n) = \frac{1}{2}\bigl\|(I-T_2)Ay_n\bigr\|^2, \qquad \nabla g(y_n) = A^*(I-T_2)Ay_n, \qquad \xi_n = \frac{\rho_n g(y_n)}{\|\nabla g(y_n)\|^2}.$$

If the solution set Γ of SPF (1.1) *is not empty, and the sequences $\{\rho_n\} \subset (0,4)$ and $\{\alpha_n\}, \{\beta_n\} \subset$* $(0,1)$ *satisfy the following conditions*:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$,

(ii) *the sequence* $\{\frac{\alpha_n}{\beta_n}\}$ *is bounded and* $\Sigma_{n=1}^{\infty}\beta_n = \infty$,

then the sequence $\{x_n\}$ *converges strongly to* $P_\Gamma u$.

Proof Since Γ ($\neq\emptyset$) is the solution set of SPF (1.1), Γ is closed and convex. Thus, the metric projection P_Γ is well defined. Letting $p = P_\Gamma u$, it follows from Lemma 2.4 that

$$\begin{aligned}\|y_n - p\|^2 &= \big\|\beta_n u + (1-\beta_n)\big(x_n - \tau_n\nabla f(x_n)\big) - p\big\|^2 \\ &\le (1-\beta_n)\big\|x_n - \tau_n\nabla f(x_n) - p\big\|^2 + 2\beta_n\langle u - p, y_n - p\rangle. \end{aligned} \tag{3.2}$$

Since $p \in \Gamma \subset C$, $\nabla f(p) = 0$. Observe that $I - T_1$ is firmly nonexpansive, from Lemma 2.2(ii) we have that

$$\begin{aligned}\big\langle\nabla f(x_n), x_n - p\big\rangle &= \big\langle (I - T_1)Ax_n, Ax_n - Ap\big\rangle \\ &\ge \big\|(I - T_1)Ax_n\big\|^2 = 2f(x_n),\end{aligned}$$

which implies that

$$\begin{aligned}\big\|x_n - \tau_n\nabla f(x_n) - p\big\|^2 &= \|x_n - p\|^2 + \big\|\tau_n\nabla f(x_n)\big\|^2 - 2\tau_n\big\langle\nabla f(x_n), x_n - p\big\rangle \\ &\le \|x_n - p\|^2 + \tau_n^2\big\|\nabla f(x_n)\big\|^2 - 4\tau_n f(x_n) \\ &= \|x_n - p\|^2 - \rho_n(4-\rho_n)\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|y_n - p\|^2 &\le (1-\beta_n)\big\|x_n - \tau_n\nabla f(x_n) - p\big\|^2 + 2\beta_n\langle u - p, y_n - p\rangle \\ &\le (1-\beta_n)\|x_n - p\|^2 + 2\beta_n\langle u - p, y_n - p\rangle \\ &\quad - (1-\beta_n)\rho_n(4-\rho_n)\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2},\end{aligned} \tag{3.3}$$

and so we have

$$\begin{aligned}\|y_n - p\|^2 &\le (1-\beta_n)\|x_n - p\|^2 + 2\beta_n\left\langle 2(u-p), \frac{(y_n - p)}{2}\right\rangle \\ &\le (1-\beta_n)\|x_n - p\|^2 + \frac{1}{4}\beta_n\|y_n - p\|^2 + 4\beta_n\|u - p\|^2.\end{aligned} \tag{3.4}$$

Consequently, we have

$$\|y_n - p\|^2 \le \frac{1-\beta_n}{1-\frac{1}{4}\beta_n}\|x_n - p\|^2 + \frac{\frac{3}{4}\beta_n}{1-\frac{1}{4}\beta_n}\frac{16}{3}\|u - p\|^2.$$

It turns out that

$$\|y_n - p\| \le \max\left\{\|x_n - p\|, \frac{16}{3}\|u - p\|\right\},$$

and inductively

$$\|y_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{16}{3}\|u - p\|\right\}.$$

This implies that the sequence $\{y_n\}$ is bounded. Since the mapping S is firmly nonexpansive, it follows from Lemma 2.4 that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \left\|S\big(\alpha_n u + (1-\alpha_n)\big(y_n - \xi_n \nabla g(y_n)\big)\big) - p\right\|^2 \\ &\leq \left\|\alpha_n(u-p) + (1-\alpha_n)\big(y_n - \xi_n\nabla g(y_n) - p\big)\right\|^2 \\ &\leq (1-\alpha_n)\left\|y_n - \xi_n \nabla g(y_n) - p\right\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle.\end{aligned}$$

Since $p \in \Gamma \subset C$, $\nabla g(p) = 0$. Observe that $I - T_2$ is firmly nonexpansive, it is deduced from Lemma 2.2(ii) that

$$\big\langle \nabla g(y_n), y_n - p\big\rangle = \big\langle (I - T_2)Ay_n, Ay_n - Ap\big\rangle \geq \left\|(I - T_2)Ay_n\right\|^2 = 2g(y_n),$$

which implies that

$$\begin{aligned}\left\|y_n - \xi_n\nabla g(y_n) - p\right\|^2 &= \|y_n - p\|^2 + \left\|\xi_n \nabla g(y_n)\right\|^2 - 2\xi_n\big\langle \nabla g(y_n), y_n - p\big\rangle \\ &\leq \|y_n - p\|^2 + \xi_n^2\left\|\nabla g(y_n)\right\|^2 - 4\xi_n g(y_n) \\ &= \|y_n - p\|^2 - \rho_n(4 - \rho_n)\frac{g^2(y_n)}{\|\nabla g(y_n)\|^2}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\alpha_n)\left\|y_n - \xi_n\nabla g(y_n) - p\right\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle \\ &\leq (1-\alpha_n)\|y_n - p\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle \\ &\quad - (1-\alpha_n)\rho_n(4 - \rho_n)\frac{g^2(y_n)}{\|\nabla g(y_n)\|^2}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\alpha_n)\|y_n - p\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle \\ &\leq (1-\alpha_n)\|y_n - p\|^2 + \frac{1}{4}\alpha_n\|x_{n+1} - p\|^2 + 4\alpha_n\|u - p\|^2.\end{aligned}$$

Consequently, we have

$$\|x_{n+1} - p\|^2 \leq \frac{1-\alpha_n}{1 - \frac{1}{4}\alpha_n}\|y_n - p\|^2 + \frac{\frac{3}{4}\alpha_n}{1 - \frac{1}{4}\alpha_n}\frac{16}{3}\|u - p\|^2.$$

It turns out that

$$\|x_{n+1} - p\| \leq \max\left\{\|y_n - p\|, \frac{16}{3}\|u - p\|\right\}.$$

Since $\{y_n\}$ is bounded, so is $\{x_n\}$. Since $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\beta_n = 0$, without loss of generality, we may assume that there is $\sigma > 0$ such that

$$\rho_n(4-\rho_n)(1-\alpha_n)(1-\beta_n) > \sigma, \quad \forall n \geq 1.$$

Substituting (3.3) into (3.5), we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq (1-\beta_n)\|x_n-p\|^2 + 2\beta_n\langle u-p, y_n-p\rangle \\ &\quad + 2\alpha_n\langle u-p, x_{n+1}-p\rangle - \frac{\sigma f^2(x_n)}{\|\nabla f(x_n)\|^2} - \frac{\sigma g^2(y_n)}{\|\nabla g(y_n)\|^2}. \end{aligned} \tag{3.6}$$

Setting $s_n = \|x_n - p\|^2$, we get the following inequality:

$$\begin{aligned} &s_{n+1} - s_n + \beta_n s_n + \frac{\sigma f^2(x_n)}{\|\nabla f(x_n)\|^2} + \frac{\sigma g^2(y_n)}{\|\nabla g(y_n)\|^2} \\ &\quad \leq 2\alpha_n\langle u-p, x_{n+1}-p\rangle + 2\beta_n\langle u-p, y_n-p\rangle. \end{aligned} \tag{3.7}$$

Now, we prove $s_n \to 0$. For the purpose, we consider two cases.

Case 1: $\{s_n\}$ is eventually decreasing, *i.e.*, there exists a sufficiently large positive integer $k \geq 1$ such that $s_n > s_{n+1}$ holds for all $n \geq k$. In this case, $\{s_n\}$ must be convergent, and from (3.7) it follows that

$$\frac{\sigma f^2(x_n)}{\|\nabla f(x_n)\|^2} + \frac{\sigma g^2(y_n)}{\|\nabla g(y_n)\|^2} \leq (s_n - s_{n+1}) + (\alpha_n + \beta_n)M, \tag{3.8}$$

where M is a constant such that $M \geq 2(\|x_{n+1}-p\| + \|y_n - p\|)\|u-p\|$ for all $n \in \mathbb{N}$. Using condition (i) and (3.8), we have that

$$\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2} + \frac{g^2(y_n)}{\|\nabla g(y_n)\|^2} \to 0 \quad (n \to \infty).$$

To verify that $f(x_n) \to 0$ and $g(y_n) \to 0$, it suffices to show that $\{\|\nabla f(x_n)\|\}$ and $\{\|\nabla g(y_n)\|\}$ are bounded. In fact, it follows from Lemma 2.1(ii) that for all $n \in \mathbb{N}$,

$$\|\nabla f(x_n)\| = \|\nabla f(x_n) - \nabla f(p)\| \leq \|A\|^2\|x_n - p\|$$

and

$$\|\nabla g(y_n)\| = \|\nabla g(y_n) - \nabla g(p)\| \leq \|A\|^2\|y_n - p\|.$$

These imply that $\{\|\nabla f(x_n)\|\}$ and $\{\|\nabla g(y_n)\|\}$ are bounded. It yields $f(x_n) \to 0$ and $g(y_n) \to 0$, namely

$$\|(I-T_1)Ax_n\| \to 0, \qquad \|(I-T_2)Ay_n\| \to 0. \tag{3.9}$$

From (3.1) we have

$$\begin{aligned}\|x_n - y_n\| &= \big\|\beta_n u + (1-\beta_n)\big(x_n - \tau_n \nabla f(x_n)\big) - x_n\big\| \\ &= \big\|\beta_n\big(u - x_n + \tau_n \nabla f(x_n)\big)\big\| \\ &= \beta_n \big\|u - x_n + \tau_n \nabla f(x_n)\big\|.\end{aligned}$$

Noting that $\{x_n\}$, $\{\|\nabla f(x_n)\|\}$ are bounded and $\beta_n \to 0$, $\tau_n \to 0$, we get

$$\|x_n - y_n\| \to 0 \quad (n \to \infty). \tag{3.10}$$

For any $x^* \in w_\omega(x_n)$, and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in H_1$, then it follows from (3.10) that $y_{n_k} \rightharpoonup x^*$. Thus we have

$$Ax_{n_k} \rightharpoonup Ax^*, \qquad Ay_{n_k} \rightharpoonup Ax^*. \tag{3.11}$$

On the other hand, from (3.9) we have

$$\big\|(I - T_1)Ax_{n_k}\big\| \to 0, \qquad \big\|(I - T_2)Ay_{n_k}\big\| \to 0. \tag{3.12}$$

Since T_1, T_2 are demiclosed at origin, from (3.11) and (3.12) we have that $Ax^* \in F(T_1) \cap F(T_2)$, *i.e.*, $Ax^* \in Q$.

Next, we turn to prove $x^* \in C$. For convenience, we set $v_n := \alpha_n u + (1-\alpha_n)(y_n - \xi_n \nabla g(y_n))$. Since S is firmly nonexpansive, it follows from Lemma 2.4 that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|Sv_n - Sp\|^2 = \|Sv_n - Sy_n + Sy_n - Sp\|^2 \\ &\le \|Sy_n - Sp\|^2 + 2\langle Sv_n - Sy_n, x_{n+1} - p\rangle \\ &\le \|Sy_n - p\|^2 + 2\|Sv_n - Sy_n\|\|x_{n+1} - p\| \\ &\le \|y_n - p\|^2 - \big\|(I - S)y_n\big\|^2 + 2\|v_n - y_n\|\|x_{n+1} - p\|.\end{aligned} \tag{3.13}$$

In view of the definition of v_n, we have

$$\|v_n - y_n\| \le \alpha_n\|y_n - u\| + \xi_n\big\|\nabla g(y_n)\big\| = \alpha_n\|y_n - u\| + \frac{\rho_n g(y_n)}{\|\nabla g(y_n)\|}. \tag{3.14}$$

From (3.4), (3.13) and (3.14), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le (1-\beta_n)\|x_n - p\|^2 + \frac{1}{4}\beta_n\|y_n - p\|^2 + 4\beta_n\|u - p\|^2 \\ &\quad - \big\|(I - S)y_n\big\|^2 + \left(\alpha_n + \frac{g(y_n)}{\|\nabla g(y_n)\|}\right)N,\end{aligned} \tag{3.15}$$

where $N \ge 2\|x_{n+1} - p\|(\|y_n - u\| + 2\rho_n)$ is a suitable constant. Clearly, from (3.15) we have

$$\begin{aligned}\big\|(I - S)y_n\big\|^2 &\le s_n - s_{n+1} + \frac{1}{4}\beta_n\|y_n - p\|^2 + 4\beta_n\|u - p\|^2 \\ &\quad + \left(\alpha_n + \frac{g(y_n)}{\|\nabla g(y_n)\|}\right)N.\end{aligned} \tag{3.16}$$

Thus, we assert that $\|(I-S)y_n\| \to 0$. In view of $y_{n_k} \rightharpoonup x^*$ and S is demiclosed at origin, we get $x^* \in F(S)$, *i.e.*, $x^* \in C$. Consequently, $w_w(x_n) \subset \Gamma$. Furthermore, we have

$$\limsup_{n\to\infty}\langle u-p, x_n-p\rangle = \max_{w\in w_w(x_n)}\langle u-P_\Gamma u, w-P_\Gamma u\rangle \le 0$$

and

$$\limsup_{n\to\infty}\langle u-p, y_n-p\rangle = \max_{w\in w_w(y_n)}\langle u-P_\Gamma u, w-P_\Gamma u\rangle \le 0.$$

From (3.7) we have

$$\begin{aligned} s_{n+1} &\le (1-\beta_n)s_n + 2\alpha_n\langle u-p, x_{n+1}-p\rangle + 2\beta_n\langle u-p, y_n-p\rangle \\ &= (1-\beta_n)s_n + \beta_n\left(\frac{2\alpha_n}{\beta_n}\langle u-p, x_{n+1}-p\rangle + 2\langle u-p, y_n-p\rangle\right). \end{aligned} \tag{3.17}$$

From condition (ii) and Lemma 2.3, we obtain $s_n \to 0$.

Case 2: $\{s_n\}$ is not eventually decreasing, that is, we can find a positive integer n_0 such that $s_{n_0} \le s_{n_0+1}$. Now we define

$$U_n := \{n_0 \le k \le n : s_k \le s_{k+1}\}, \quad n > n_0.$$

It easy to see that U_n is nonempty and satisfies $U_n \subseteq U_{n+1}$. Let

$$\psi(n) := \max U_n, \quad n > n_0.$$

It is clear that $\psi(n) \to \infty$ as $n \to \infty$ (otherwise, $\{s_n\}$ is eventually decreasing). It is also clear that $s_{\psi(n)} \le s_{\psi(n)+1}$ for all $n > n_0$. Moreover, we prove that

$$s_n \le s_{\psi(n)+1}, \quad \forall n > n_0. \tag{3.18}$$

In fact, if $\psi(n) = n$, then inequality (3.18) is trivial; if $\psi(n) < n$, from the definition of $\psi(n)$, there exists some $i \in \mathbb{N}$ such that $\psi(n) + i = n$, we deduce that

$$s_{\psi(n)+1} > s_{\psi(n)+2} > \cdots > s_{\psi(n)+i} = s_n,$$

and inequality (3.18) holds again. Since $s_{\psi(n)} \le s_{\psi(n)+1}$ for all $n > n_0$, it follows from (3.8) that

$$\frac{\sigma f^2(x_{\psi(n)})}{\|\nabla f(x_{\psi(n)})\|^2} + \frac{\sigma g^2(y_{\psi(n)})}{\|\nabla g(y_{\psi(n)})\|^2} \le (\alpha_{\psi(n)} + \beta_{\psi(n)})M \to 0.$$

Noting that $\{\|\nabla f(x_{\psi(n)})\|\}$ and $\{\|\nabla g(y_{\psi(n)})\|\}$ are both bounded, we get

$$f(x_{\psi(n)}) \to 0, \qquad g(y_{\psi(n)}) \to 0.$$

By the same argument to the proof in case 1, we have $w_w(x_{\psi(n)}) \subset \Gamma$ and

$$\|x_{\psi(n)} - y_{\psi(n)}\| \to 0. \tag{3.19}$$

On the other hand, noting $s_{\psi(n)} \leq s_{\psi(n)+1}$ again, we have from (3.14) and (3.16) that

$$\begin{aligned}
&\|x_{\psi(n)+1} - y_{\psi(n)}\| \\
&\quad = \|Sv_{\psi(n)} - Sy_{\psi(n)} + Sy_{\psi(n)} - y_{\psi(n)}\| \\
&\quad \leq \|Sv_{\psi(n)} - Sy_{\psi(n)}\| + \|Sy_{\psi(n)} - y_{\psi(n)}\| \\
&\quad \leq \|v_{\psi(n)} - y_{\psi(n)}\| + \|(I-S)y_{\psi(n)}\| \\
&\quad \leq \alpha_{\psi(n)}\|y_{\psi(n)} - u\| + \frac{\rho_{\psi(n)}g(y_{\psi(n)})}{\|\nabla g(y_{\psi(n)})\|} \\
&\qquad + \sqrt{\frac{1}{4}\beta_{\psi(n)}\|y_{\psi(n)} - p\|^2 + 4\beta_{\psi(n)}\|u-p\|^2 + \left(\alpha_{\psi(n)} + \frac{g(y_{\psi(n)})}{\|\nabla g(y_{\psi(n)})\|}\right)N}. \qquad (3.20)
\end{aligned}$$

Letting $n \to \infty$, we get

$$\|x_{\psi(n)+1} - y_{\psi(n)}\| \to 0. \qquad (3.21)$$

From (3.19) and (3.21), we have

$$\|x_{\psi(n)} - x_{\psi(n)+1}\| \leq \|x_{\psi(n)} - y_{\psi(n)}\| + \|y_{\psi(n)} - x_{\psi(n)+1}\| \to 0 \quad (n \to \infty). \qquad (3.22)$$

Furthermore, we can deduce that

$$\begin{aligned}
\limsup_{n\to\infty}\langle u-p, x_{\psi(n)+1} - p\rangle &= \limsup_{n\to\infty}\langle u-p, x_{\psi(n)} - p\rangle \\
&= \max_{w\in w_w(x_{\psi(n)})}\langle u - P_\Gamma u, w - P_\Gamma u\rangle \leq 0 \qquad (3.23)
\end{aligned}$$

and

$$\limsup_{n\to\infty}\langle u-p, y_{\psi(n)} - p\rangle = \max_{w\in w_w(y_{\psi(n)})}\langle u - P_\Gamma u, w - P_\Gamma u\rangle \leq 0. \qquad (3.24)$$

Since $s_{\psi(n)} \leq s_{\psi(n)+1}$, it follows from (3.7) that

$$s_{\psi(n)} \leq 2\langle u-p, y_{\psi(n)} - p\rangle + \frac{2\alpha_{\psi(n)}}{\beta_{\psi(n)}}\langle u-p, x_{\psi(n)+1} - p\rangle, \quad n > n_0. \qquad (3.25)$$

Combining (3.23), (3.24) and (3.25), we have

$$\limsup_{n\to\infty} s_{\psi(n)} \leq 0, \qquad (3.26)$$

and hence $s_{\psi(n)} \to 0$, which together with (3.22) implies that

$$\begin{aligned}
\sqrt{s_{\psi(n)+1}} &\leq \|(x_{\psi(n)} - p) + (x_{\psi(n)+1} - x_{\psi(n)})\| \\
&\leq \sqrt{s_{\psi(n)}} + \|x_{\psi(n)+1} - x_{\psi(n)}\| \to 0. \qquad (3.27)
\end{aligned}$$

Noting inequality (3.18), this shows that $s_n \to 0$, that is, $x_n \to p$. This completes the proof of Theorem 3.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally to the writing of the present article. They also read and approved final manuscript.

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