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Research Article

Calibration of the Volatility in Option Pricing Using the Total Variation Regularization

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In market transactions, volatility, which is a very important risk measurement in financial economics, has significantly intimate connection with the future risk of the underlying assets. Identifying the implied volatility is a typical PDE inverse problem. In this paper, based on the total variation regularization strategy, a bivariate total variation regularization model is proposed to estimate the implied volatility. We not only prove the existence of the solution, but also provide the necessary condition of the optimal control problem—Euler-Lagrange equation. The stability and convergence analyses for the proposed approach are also given. Finally, numerical experiments have been carried out to show the effectiveness of the method.

1. Introduction

Volatility is a very important risk measurement in financial economics. The estimation of it is critical for option pricing and management of the derivative positions. In order to estimate the volatility effectively, two main classes of parametric approaches have been developed: discrete-time models and continuous-time models.

There are numerous literatures on the discrete-time models and here we provide only a partial overview related to our studies. The ARCH model developed by Engle [1] is the first model that provided a systematic framework for volatility modeling. Based on the ARCH model, Bollerslev [2] proposed the GARCH model and Nelson [3] and Glosten et al. [4] argued that the GARCH model provides more flexibility. There are many popular extensions including EGARCH [3], GJR-GARCH [4], QGARCH [5], TGARCH [6], and GARCH-M [7]. Moreover, a multifactors volatility structure has been studied in Engel and Lee [8], Christoffersen et al. [9], Li and Zhang [10], and Adrian and Rosenbery [11]. On the other hand, models for asset pricing under risk-neutral measure have been dominated traditionally by continuous-time processes. Heston [12] proposed an option pricing model with stochastic volatility. Duan [13] and Heston and Nandi [14] developed an option pricing model based on the GARCH process. However, those models fail to address the smile and the smirk quantitatively. Existing literatures have attempted to cope with this by combining stochastic volatility specifications with jump process or by using nonnormal innovations in GARCH models; see, for example, Bates [15, 16], Pan [17], Duan et al. [18, 19], Eraker [20], Broadie et al. [21], Christoffersen et al. [22–24], and so forth.

However, those models generally suffer from a curse of dimension that severely constrains their practice and the coming of high frequency financial data makes it worse. Nowadays, the availability of intraday data has facilitated the use of the so-called Realized Volatility (RV) which was introduced in the literature by Taylor and Xu [25] and Anderson and Bollerslev [26] and which is grounded in the framework of continuous time finance with the notion of quadratic variation of a martingale. The literature on RV models has grown remarkably over the last decade; see, for example, Andersen et al. [27], Andersen et al. [28, 29], Barndorff-Nielsen and Shephard [30, 31], Bandi et al. [32], and references therein. The RV model has the clear advantage of providing a precise nonparametric measure of daily volatility which leads to simplicity in model estimation and superior forecasting performance. Corsi et al. [33] followed a similar approach by jointly modeling returns and the two-scale realized volatility [34]. Christoffersen et al. [35] developed a new class of affine discrete-time models that allow for closed-form option valuation formulas using the conditional moment generation function and modeled daily returns as well as realized volatility.

There is also a common practice to infer the volatility from quoted option prices based on the Black-Scholes theoretical framework [36], called implied volatility; see, for example, Dupier [37], Lagnado and Osher [38], Chiarella et al. [39], Jiang and Tao [40], Crépey [41], Isakov [42], Egger and Engl [43], Ngnepieba [44], Deng et al. [45], and so forth. The volatility value implied by an observed market option price (implied volatility) indicates a consensual view about the volatility level determined by the market. This paper is devoted to studying the regularization method of identifying the implied volatility.

The stochastic process of the asset price S_t is modeled to satisfy the Geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t d\omega(t), \tag{1}$$

where μ is the expected rate of return, σ is the volatility, and $\omega(t)$ is the standard Brownian process; here $E[\omega(t)^2] = t$.

An option is classified either as a call option or a put option. A call (put) option is a contract which gives the buyer (the owner) the right, but not the obligation, to buy (or sell) an underlying asset or instrument at a specified strike price on or before a specified date.

Suppose $V(S_t, t)$ is the price of a European option, the differential of which is given by

$$dV = \left(\frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial S}\sigma S dw(t). \tag{2}$$

Consider a portfolio that involves short selling of one unit of a European call option and long holding of Δ_t units of the underlying asset. The portfolio value $\Pi(S_t, t)$ at time t is given by

$$\Pi = -V + \Delta_t S. \tag{3}$$

By virtue of the no-arbitrage principle, we have

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV, \tag{4}$$

$$V_T = (S_T - K)^+ = \max(S_T - K, 0), \text{ call option} \tag{5}$$

$$V_T = (K - S_T)^+ = \max(K - S_T, 0), \text{ put option,} \tag{6}$$

where r is the riskless interest rate, T is the maturity, and K is the strike price. The above parabolic partial differential equation is the famous Black-Scholes equation. With the boundary condition $V(0, t) = 0$, that is, the option is worthless if the stock is valued at nothing, the analytical solution of the European call option is given by

$$V(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{7}$$

where

$$\begin{aligned} N(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\omega^2/2} d\omega, \\ d_1 &= \frac{\ln(S/K) + \left(r + \left(\sigma^2/2\right)\right)(T-t)}{\sigma\sqrt{(T-t)}}, \\ d_2 &= d_1 - \sigma\sqrt{(T-t)}. \end{aligned} \tag{8}$$

The option prices obtained from the Black-Scholes pricing model are functions of five parameters: S, K, r, T, and σ. Except for the volatility parameter, the other four parameters T, K, q, and r are observable quantities. There is evidence that the volatility is time varying [46, 47] in actual markets. For any fixed maturity, implied volatility varies with the strike price in a parabolic shape that is often called the volatility smile. The pattern of implied volatilities across maturities is known as the volatility term structure. One possibility to explain the volatility smiles in the Black-Scholes model is to use a deterministic function of underlying asset price S_t and time t; that is, $\sigma = \sigma(S, t)$.

A natural question then arises: how can we get the implied volatility of the future underlying asset by option quotes? This is the typical IPOP (inverse problem of option pricing).

The PDE inverse problem of option pricing was first considered by Dupire in [37] where he obtained a formula of the local volatility with all possible strike prices and maturities. However, the formula was instable and could not be used in practice. The inverse problem which consists in using the results of actual measurements to infer the values of the parameters is usually ill-posed. The fact that the solution fails to depend continuously upon the given data is the source of many difficulties inherently in solving the inverse problem. Ill-posed problems require the use of regularization techniques for any practical application. The most widely known and applicable regularization methods is Tikhonov regularization [48], where regular items play a critical role of stability. Over the past decades, the inverse problem of determining the implied volatility has already obtained widespread development; see, for example, [38–45, 49] and references therein. However, the traditional Tikhonov regularization strategy may oversmooth the solution, so that the regularized solution cannot effectively approximate the exact solution of the original problem, when the exact solution is nonsmooth or even has some singularities. This shortcoming will blur the edge of the restored image in image processing. To overcome the shortcoming, Rudin et al. [50] proposed the total variation regularization strategy (TV-L^2 model):

$$\min_{u\in\Omega} \frac{\lambda}{2}\|u - f\|^2_{L^2(\Omega)} + |\nabla u|_{L^1(\Omega)}. \tag{9}$$

The total variation regularization might be able to characterize the properties (the jump, overnight, weekend effect, etc.) of the volatility better. So whether the total variation regularization strategy could be applied to identify the implied volatility is a question worth pondering.

This paper is organized as follows. Section 2 introduces the total variation regularization item in the inverse problem of option pricing and puts forward a new model with terminal observations. In Section 3, we give mathematical analysis of the existence of the solution and the necessary condition of the optimal control problem. The stability and convergence of the proposed regularized approach are analyzed in Section 4. Section 5 presents a selection of numerical experiments. Section 6 concludes the paper.

2. Total Variation Regularization Model

In [38] Lagnado and Osher determined this inverse problem by using Tikhonov regularization strategy that is attempting to minimize

$$\widetilde{G}(\sigma) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{M_i}\left(V\left(S_0, 0, K_{ij}, T_i, \sigma\right) - V_{ij}\right)^2 + \|\nabla\sigma\|_2^2, \quad (10)$$

where ∇ denotes the gradient operator. This regularization strategy is just for one fixed value of underlying asset S_0, at one fixed point at time $t = 0$. There is no guarantee that the value of σ calculated by this approach will be correct either for other underlying assets or at future times, and the estimated volatility may be negative in some cases.

Based on their work, Chiarella et al. [39] modified the objective functional as follow:

$$\widehat{G}(\sigma) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{M_i}\int_0^{\infty}\int_0^{T_{\text{cur}}}\left[V\left(S, t, K_{ij}, T_i, \sigma\right) - V_{ij}\right]^2 dS\,dt + \|\nabla\sigma\|_2^2, \quad (11)$$

where T_{cur} is the current time.

Tikhonov regularization strategy may oversmooth the solution, so it may not preserve the singularities of the solution well. We adopt total variation regularization strategy proposed by Rudin et al. [50] to maintain the singularity (the jump, overnight, weekend effect, etc.) of volatility. In fact, total variation regularization strategy can preserve the edge of the restored image and has become a standard approach for the computation of discontinuous solutions of inverse problems.

Set $V(S, t, \sigma(S, t))$ (hereafter denote $V(S, t, \sigma(S, t))$ by $V(\sigma)$ for convenience sake) to be the solution of the Black-Scholes equations (4) and (5); then we regard $V(\sigma)$ as a nonlinear operator with respect to σ:

$$L^2(\Omega) \supseteq \mathscr{D} \ni \sigma \xrightarrow{V} V(\sigma) \in L^2(\Omega). \quad (12)$$

Consider the following bivariate total variation regularization problem:

$$\min_{\sigma\in\mathscr{D}} \widehat{J}(\sigma) = \frac{1}{2}\|V(\sigma) - \nu\|^2 + \alpha J(\sigma), \quad (13)$$

where $J(\sigma)$ is the seminorm

$$J(\sigma) = \int_{\Omega} |\nabla\sigma|\, dS\, dt, \quad (14)$$

α denotes the regularization parameter, and ν is the vector of market observed prices at the calibration time. $\mathscr{D} : \mathscr{D}(J) \neq \emptyset$ for "operator" $V(\sigma)$, $\Omega : (0, S_{\max}) \times (0, T_{\text{cur}}]$ and

$$\mathscr{D}(J) := \{\sigma \in \Lambda : J(\sigma) \neq \infty\},$$
$$\Lambda = \left\{\sigma \equiv \sigma(S, t) \mid 0 \le \sigma_{\min} \le \sigma \le \sigma_{\max}, \sigma \in L^2(\Omega)\right\}, \quad (15)$$

where $\sigma_{\min}$, $\sigma_{\max}$ are given constants.

The term $|\nabla\sigma|^{-1}$ will appear in later necessary optimality condition. To avoid $|\nabla\sigma| \approx 0$ in the flat area, as is done in the image processing, the problem (13) is usually approximated by

$$\min_{\sigma\in D} \overline{J}(\sigma) = \frac{1}{2}\|V(\sigma) - \nu\|^2 + \alpha J_\beta(\sigma), \quad (16)$$

where

$$J_\beta(\sigma) = \int_{\Omega} \sqrt{|\nabla\sigma|^2 + \beta^2}\, dS\, dt. \quad (17)$$

β is a (typically small) positive parameter which usually can be taken as a constant, for example, $\beta = 10^{-6}$.

Our total variation regularization strategy has two advantages compared with Tikhonov regularization strategy proposed by Lagnado and Osher: one is that it contains no terms involving the Dirac delta function [51]; the other is that the total variation regularization strategy may maintain the singularities of the solution better. Next we will investigate mathematical properties of the solution such as the existence, necessary condition, stability, and convergence.

3. Existence and Necessary Optimality Condition

The minimization problem (16) is quite different from the standard Tikhonov regularization strategy since the regularization item involves $J_\beta(\sigma)$.

Lemma 1. *Under the constraints of the total variation regularization problem* (16), *if* $\{\sigma_n\} \rightharpoonup \sigma^*$, *then* $\{V(\sigma_n)\} \rightharpoonup V(\sigma^*)$, *where* $V(\sigma_n)$ *is the solution to* (4) *when* $\sigma = \sigma_n$.

This lemma can easily be similarly proved like proposition A.3 in [43].

Theorem 2. *The total variation minimization problem* (16) *at least attains a minimizer* $\widetilde{\sigma} \in \mathscr{D}$.

Proof. The weak lower semicontinuity of the norm and weakly continuity of the operator $V(\sigma)$ imply the lower semicontinuity of the functionals $\|V(\sigma) - \nu\|^2$ and $J_\beta(\sigma)$. Moreover, the level sets of the functional $J_\beta(\sigma)$ are compact in $L^2(\Omega)$. So the total variation minimization problem (16) has a compact set of minimizers by Theorem 2 in [48]. □

We can calculate approximate solutions by solving the Euler-Lagrange equation. Generally speaking, the total variational regularization problem (16) is not strictly convex or even nonconvex. Next we deduce the necessary condition Euler-Lagrange equation which has to be satisfied by each optimal control minimum.

Set

$$F = \frac{1}{2}[V(\sigma) - v]^2 + \alpha |\nabla\sigma(S,t)|, \quad (18)$$

and further assume that F is the third-order differentiable function and $\sigma = \sigma(S,t)$ is the second-order differentiable function.

Theorem 3. *Necessary optimality condition: let σ be a solution of the total variation regularization problem* (16)*; then σ satisfies*

$$\frac{\partial V}{\partial\sigma}(S,t,\sigma)[V(\sigma) - v] - \alpha\nabla\cdot\left(\frac{\nabla\sigma}{\sqrt{|\nabla\sigma|^2+\beta^2}}\right) = 0. \quad (19)$$

Proof. By using the variational method, the corresponding Euler-Lagrange partial differential equation is

$$F_\sigma - \frac{\partial}{\partial S}\{F_p\} - \frac{\partial}{\partial t}\{F_q\} = 0, \quad (20)$$

where

$$p = \frac{\partial\sigma(S,t)}{\partial S}, \qquad q = \frac{\partial\sigma(S,t)}{\partial t}. \quad (21)$$

Combining (18) and (20), we have

$$\begin{aligned} F_\sigma &= \frac{\partial V}{\partial\sigma}(S,t,\sigma)[V(\sigma)-v], \\ F_p &= \frac{\partial\sigma/\partial S}{|\nabla\sigma|}, \qquad F_q = \frac{\partial\sigma/\partial t}{|\nabla\sigma|}. \end{aligned} \quad (22)$$

Therefore

$$\begin{aligned} &\frac{\partial V}{\partial\sigma}(S,t,\sigma)[V(\sigma)-v] \\ &\quad - \alpha\left\{\frac{\partial}{\partial S}\left\{\left(\frac{\partial\sigma/\partial S}{|\nabla\sigma|}\right)\right\} + \frac{\partial}{\partial t}\left\{\left(\frac{\partial\sigma/\partial t}{|\nabla\sigma|}\right)\right\}\right\} = 0, \\ &\Longrightarrow \frac{\partial V}{\partial\sigma}(S,t,\sigma)[V(\sigma)-v] \\ &\quad - \alpha\left(\frac{\partial}{\partial S},\frac{\partial}{\partial t}\right)\cdot\left\{\frac{\partial\sigma/\partial S}{|\nabla\sigma|},\frac{\partial\sigma/\partial t}{|\nabla\sigma|}\right\} = 0, \\ &\Longrightarrow \frac{\partial V}{\partial\sigma}(S,t,\sigma)[V(\sigma)-v] \\ &\quad - \alpha\left(\frac{\partial}{\partial S},\frac{\partial}{\partial t}\right)\cdot\left\{\frac{1}{|\nabla\sigma|}\left(\frac{\partial\sigma}{\partial S},\frac{\partial\sigma}{\partial t}\right)\right\} = 0, \\ &\Longrightarrow \frac{\partial V}{\partial\sigma}(S,t,\sigma)[V(\sigma)-v] - \alpha\nabla\cdot\left(\frac{\nabla\sigma}{|\nabla\sigma|}\right) = 0. \end{aligned} \quad (23)$$

The corresponding Euler-Lagrange equation related to the total variation model with $|\nabla\sigma|$ replaced by $|\nabla\sigma|_\beta$ is given by

$$\frac{\partial V}{\partial\sigma}(S,t,\sigma)[V(\sigma) - v] - \alpha\nabla\cdot\left(\frac{\nabla\sigma}{\sqrt{|\nabla\sigma|^2+\beta^2}}\right) = 0. \quad (24)$$

This completes the proof. □

The next theorem states well posedness of the regularized problem.

Theorem 4. *Under the constraints of the total variation regularization problem* (16)*, the minimization of*

$$\overline{J}_\beta^\delta(\sigma) = \frac{1}{2}\left\|V(\sigma) - v^\delta\right\|^2 + \alpha J_\beta(\sigma) \quad (25)$$

is stable with respect to perturbations in the data; that is, $\alpha > 0$, if $\{v^k\} \to v^\delta$ in $L^2(\Omega)$ and σ_k denotes the solution to the problem (25) *with v^δ replaced by v^k, then*

$$\{\sigma_k\} \longrightarrow \sigma, \qquad \{J_\beta(\sigma_k)\} \longrightarrow J_\beta(\sigma). \quad (26)$$

Proof. $\{v^k\} \to v^\delta$ in $L^2(\Omega)$ implies that $\{\sigma_k, V_k(\sigma_k)\}$ satisfies

$$\begin{aligned} &\frac{1}{2}\left\|V_k(\sigma_k) - v^k\right\|^2 + \alpha J_\beta(\sigma_k) \\ &\quad \le \frac{1}{2}\left\|V(\sigma) - v^k\right\|^2 + \alpha J_\beta(\sigma) \end{aligned} \quad (27)$$

for every $\sigma \in \mathcal{D}$. Thus $\{\sigma_k\}$ is bounded in $\mathcal{D}$ and therefore has a weakly convergent subsequence $\{\sigma_m\} \rightharpoonup \widehat{\sigma}$. Similarly, there exists a subsequence $\{V_m\}$ corresponding to $\{\sigma_m\}$ such that $\{V_m\} \rightharpoonup \widehat{V}$, where $\widehat{V}$ is the solution to (4) when $\sigma = \widehat{\sigma}$. By the weak lower semicontinuity of $J_\beta(\sigma_m)$ and $\|\cdot\|$, we have

$$\begin{aligned} J_\beta(\widehat{\sigma}) &\le \limsup J_\beta(\sigma_m), \\ \frac{1}{2}\left\|\widehat{V} - v^\delta\right\|^2 &\le \limsup \frac{1}{2}\|V_m - v^m\|^2, \end{aligned} \quad (28)$$

and therefore by (27),

$$\begin{aligned} &\frac{1}{2}\left\|\widehat{V} - v^\delta\right\| + \alpha J_\beta(\widehat{\sigma}) \\ &\quad \le \liminf\left\{\frac{1}{2}\|V_m - v^m\|^2 + \alpha J_\beta(\sigma_m)\right\} \\ &\quad \le \limsup\left\{\frac{1}{2}\|V_m - v^m\|^2 + \alpha J_\beta(\sigma_m)\right\} \\ &\quad \le \lim_{m\to\infty}\left\{\frac{1}{2}\|V - v^m\|^2 + \alpha J_\beta(\sigma)\right\} \\ &\quad = \frac{1}{2}\left\|V - v^\delta\right\|^2 + \alpha J_\beta(\sigma) \end{aligned} \quad (29)$$

for all $\sigma \in \mathcal{D}$. This implies that $\widehat{\sigma}$ is a minimizer of the total variation regularization problem (25) and that

$$\begin{aligned} &\lim_{m\to\infty}\left\{\frac{1}{2}\|V_m - v^m\|^2 + \alpha J_\beta(\sigma_m)\right\} \\ &\quad = \frac{1}{2}\left\|\widehat{V} - v^\delta\right\|^2 + \alpha J_\beta(\widehat{\sigma}). \end{aligned} \quad (30)$$

If $\{\sigma_m, V_m\}$ does not converge strongly to $\{\widehat{\sigma}, \widehat{V}\}$, then

$$C := \limsup \left\{ \frac{1}{2} \left\| V_m - v^\delta \right\|^2 \right\} > \frac{1}{2} \left\| \widehat{V} - v^\delta \right\|^2, \tag{31}$$

and there exists a subsequence $\{\sigma_n, V_n\}$ of $\{\sigma_m, V_m\}$ satisfying

$$\begin{gathered} \sigma_n \rightharpoonup \widehat{\sigma}, \qquad V_n \rightharpoonup \widehat{V}, \qquad J_\beta(\widehat{\sigma}) \le \lim_{n \to \infty} J_\beta(\sigma_n), \\ \frac{1}{2} \left\| V_n - v^\delta \right\|^2 \longrightarrow C. \end{gathered} \tag{32}$$

This combined with (30) implies

$$\alpha \lim_{n \to \infty} J_\beta(\sigma_n) = \alpha J_\beta(\widehat{\sigma}) + \frac{1}{2} \left\| \widehat{V} - v^\delta \right\|^2 - C < \alpha J_\beta(\widehat{\sigma}), \tag{33}$$

which is a contradiction to (32), so we have

$$\{\sigma_n\} \longrightarrow \widehat{\sigma}, \qquad J_\beta(\sigma_n) \longrightarrow J_\beta(\widehat{\sigma}). \tag{34}$$

This completes the proof. □

In the next theorem, we show that under the same conditions on $\alpha(\delta)$ as in the linear case solutions of (25) converge to a minimum-norm-solution, that is, a least squares solution.

Theorem 5. *Under the constraints of the total variation regularization problem* (16)*, there exists at least one minimizing solution of* (25)*. Assume that the sequence $\{\delta_k\}$ converges monotonically to 0 and $v^k := v^{\delta_k}$ satisfies $\|v - v^k\| \le \delta_k$; here v denotes the solution of the Black-Scholes model with respect to the minimum solution.*

Moreover, assume that $\alpha(\delta)$ satisfies

$$\alpha(\delta) \longrightarrow 0, \qquad \frac{\delta^2}{\alpha(\delta)} \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0, \tag{35}$$

and $\alpha(\cdot)$ is monotonically increasing. Then every sequence $\{\sigma_{\alpha_k}^{\delta_k}\}$, where $\delta_k \to 0, \alpha_k := \alpha(\delta_k)$,

$$\sigma_{\alpha_k}^{\delta_k} \in \operatorname{argmin} \left\{ \left\| V(\sigma) - v^k \right\|^2 + \alpha J(\sigma) : \sigma \in \mathscr{D} \right\}, \tag{36}$$

has a convergent subsequence. The limit of every convergent subsequence is a minimum solution. If, in addition, the minimum solution σ^+ is unique, then

$$\lim_{\delta \to 0} \sigma_{\alpha(\delta)}^\delta = \sigma^+. \tag{37}$$

Proof. Let α_k and $\{\sigma_{\alpha_k}^{\delta_k}\}$ be as above, and let σ^+ be a minimum solution. Then by the definition of $\{\sigma_{\alpha_k}^{\delta_k}\}$, we have

$$\begin{aligned} & \left\| V\left(\sigma_{\alpha_k}^{\delta_k}\right) - v^k \right\|^2 + \alpha_k J\left(\sigma_{\alpha_k}^{\delta_k}\right) \\ & \quad \le \left\| V(\sigma^+) - v^k \right\|^2 + \alpha_k J(\sigma^+) \\ & \quad = \delta_k^2 + \alpha_k J(\sigma^+), \end{aligned} \tag{38}$$

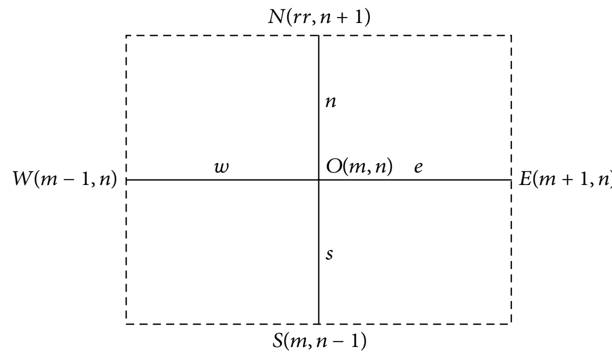

Figure 1: Gridding.

which shows that

$$\lim_{k \to \infty} V\left(\sigma_{\alpha_k}^{\delta_k}\right) = v, \tag{39}$$

$$\limsup_{k \to \infty} J\left(\sigma_{\alpha_k}^{\delta_k}\right) \le J(\sigma^+). \tag{40}$$

This combined with (15) implies that $\{\sigma_{\alpha_k}^{\delta_k}\}$ is bounded. Hence, there exist an element $\dot{\sigma} \in \mathscr{D}$ and a subsequence again denoted by $\{\sigma_{\alpha_k}^{\delta_k}\}$ such that

$$\sigma_{\alpha_k}^{\delta_k} \rightharpoonup \dot{\sigma} \quad \text{as } k \longrightarrow \infty. \tag{41}$$

Using the assumption that $V(\sigma)$ is continuous with respect to σ and that the norm convergence on $L^2(\Omega)$ is stronger, it follows from (40) that $V(\dot{\sigma}) = v$.

From the lower semicontinuity of $J(\sigma)$, it follows that

$$J(\dot{\sigma}) \le \liminf_{k \to \infty} J\left(\sigma_{\alpha_k}^{\delta_k}\right) \le \limsup_{k \to \infty} J\left(\sigma_{\alpha_k}^{\delta_k}\right) \le J(\sigma^+) \le J(\overline{\sigma}), \tag{42}$$

for all $\overline{\sigma} \in \mathscr{D}$ satisfying $V(\overline{\sigma}) = v$. Taking $\overline{\sigma} = \dot{\sigma}$ shows that $J(\dot{\sigma}) = J(\sigma^+)$. That means $\dot{\sigma}$ is a minimizing solution of the total variation regularization problem.

Using this and (42), it follows that $J(\sigma_{\alpha_k}^{\delta_k}) \to J(\sigma^+)$.

If the minimizing solution of (25) is unique denoted by σ^+, it follows that every sequence $\{\sigma_{\alpha_k}^{\delta_k}\}$ has a subsequence, and the limit of any subsequence of $\{\sigma_{\alpha_k}^{\delta_k}\}$ has to be equal to σ^+. This completes the proof. □

4. Discretization and Algorithm

Next we will discretize the term $\nabla \cdot (\nabla\sigma / \sqrt{|\nabla\sigma|^2 + \beta^2})$. Let $\{\Delta S, \Delta t\}$ denote the grid size and construct an approximation for $\sigma(S, t)$ at a set of points $(m\Delta S, n\Delta t)$ on Ω.

As in Figure 1, at a given target pixel $O(m, n)$ (we denote $O(m\Delta S, m\Delta t)$ by $O(m, n)$ for convenience sake), let E, N, W, and S denote its four adjoint pixels, and let e, n, w, and s be the corresponding four midway points (not directly available from the gridding).

Let $v = (v^1, v^2) = \nabla\sigma / |\nabla\sigma|_\beta$; then

$$\nabla \cdot v = \frac{\partial v^1}{\partial S} + \frac{\partial v^2}{\partial t} \approx \frac{v_e^1 - v_w^1}{\Delta S} + \frac{v_n^2 - v_s^2}{\Delta t}. \tag{43}$$

Next, we give further approximations at the midway points:

$$\begin{aligned} v_e^1 &= \frac{1}{|\nabla\sigma_e|_\beta}\left[\frac{\partial\sigma}{\partial S}\right]_e \approx \frac{1}{|\nabla\sigma_e|_\beta}\frac{\sigma_E - \sigma_O}{\Delta S}; \\ |\nabla\sigma_e|_\beta &\approx \sqrt{\left(\frac{\sigma_E - \sigma_O}{\Delta S}\right)^2 + \left(\frac{\sigma_{NE} + \sigma_N - \sigma_S - \sigma_{SE}}{4\Delta t}\right)^2 + \beta^2}. \end{aligned} \tag{44}$$

Namely, we approximate $[\partial\sigma/\partial S]_e$ by the central difference scheme and $[\partial\sigma/\partial t]_e$ by the average of $(\sigma_{NE} - \sigma_{SE})/2\Delta t$ and $(\sigma_N - \sigma_S)/2\Delta t$. Similar discussion applies to the other three directions N, W, and S:

$$\begin{aligned} v_w^1 &= \frac{1}{|\nabla\sigma_w|_\beta}\left[\frac{\partial\sigma}{\partial S}\right]_w \approx \frac{1}{|\nabla\sigma_w|_\beta}\frac{\sigma_W - \sigma_O}{\Delta S}; \\ |\nabla\sigma_w|_\beta &\approx \sqrt{\left(\frac{\sigma_W - \sigma_O}{\Delta S}\right)^2 + \left(\frac{\sigma_{NW} + \sigma_N - \sigma_S - \sigma_{SW}}{4\Delta t}\right)^2 + \beta^2}; \\ v_n^2 &= \frac{1}{|\nabla\sigma_n|_\beta}\left[\frac{\partial\sigma}{\partial t}\right]_n \approx \frac{1}{|\nabla\sigma_n|_\beta}\frac{\sigma_N - \sigma_O}{\Delta t}; \\ |\nabla\sigma_n|_\beta &\approx \sqrt{\left(\frac{\sigma_N - \sigma_O}{\Delta t}\right)^2 + \left(\frac{\sigma_{NE} + \sigma_E - \sigma_W - \sigma_{WN}}{4\Delta S}\right)^2 + \beta^2}; \\ v_s^2 &= \frac{1}{|\nabla\sigma_s|_\beta}\left[\frac{\partial\sigma}{\partial t}\right]_s \approx \frac{1}{|\nabla\sigma_s|_\beta}\frac{\sigma_S - \sigma_O}{\Delta t}; \\ |\nabla\sigma_s|_\beta &\approx \sqrt{\left(\frac{\sigma_S - \sigma_O}{\Delta t}\right)^2 + \left(\frac{\sigma_{ES} + \sigma_E - \sigma_W - \sigma_{WS}}{4\Delta S}\right)^2 + \beta^2}, \end{aligned} \tag{45}$$

and then we have

$$\begin{aligned} -\nabla \cdot \left(\frac{\nabla\sigma}{|\nabla\sigma|_\beta}\right) &= -\nabla \cdot v = \left(\frac{v_w^1 - v_e^1}{\Delta S}\right) + \left(\frac{v_s^2 - v_n^2}{\Delta t}\right) \\ &= \sum_{p \in W,E} \frac{1}{|\nabla\sigma_p|_\beta}\left[\frac{\sigma(O) - \sigma(p)}{(\Delta S)^2}\right] \\ &\quad + \sum_{p \in N,S} \frac{1}{|\nabla\sigma_p|_\beta}\left[\frac{\sigma(O) - \sigma(p)}{(\Delta t)^2}\right]. \end{aligned} \tag{46}$$

At a pixel $O(m, n)$, (19) is discretized to

$$\begin{aligned} 0 = &\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{\partial V}{\partial\sigma}\left(m\Delta S, n\Delta t, K_i, T_j, \sigma(m\Delta S, n\Delta t)\right) \\ &\times \left[V\left(m\Delta S, n\Delta t, K_i, T_j, \sigma(m\Delta S, n\Delta t)\right) - v_{ij}\right] \\ &+ \alpha \sum_{p \in W,E}\frac{1}{|\nabla\sigma_p|_\beta}\left[\frac{\sigma(m\Delta S, n\Delta t) - \sigma(p)}{(\Delta S)^2}\right] \\ &+ \alpha \sum_{p \in N,S}\frac{1}{|\nabla\sigma_p|_\beta}\left[\frac{\sigma(m\Delta S, n\Delta t) - \sigma(p)}{(\Delta t)^2}\right]. \end{aligned} \tag{47}$$

To obtain the local optimal solution, we have to handle the problem of calculating the partial derivative $\partial V/\partial\sigma$ in the Euler-Lagrange equation. By the Black-Scholes formula, the option price V and partial derivative $\partial V/\partial\sigma$ can be approximated, respectively, as follows:

$$\begin{aligned} C(S, t) &= SN(d_1) - Ke^{-r(T-t)}N(d_2), \\ \frac{\partial C}{\partial\sigma} &= SN'(d_1)\frac{\partial d_1}{\partial\sigma} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial\sigma} \\ &= \frac{S\sqrt{T-t}e^{-d_1^2/2}}{\sqrt{2\pi}}. \end{aligned} \tag{48}$$

Let

$$\begin{aligned} &A := \{W, E\}, \qquad B := \{N, S\}, \\ &A_{p1} = \alpha\sum_{p \in A}\frac{1}{|\nabla_p|_\beta(\Delta S)^2}, \qquad B_{p2} = \alpha\sum_{p \in B}\frac{1}{|\nabla\sigma_p|_\beta(\Delta t)^2}, \\ &C = \sum_{i=1}^{I}\sum_{j=1}^{J}\frac{\partial V}{\partial\sigma}\left(m\Delta S, n\Delta t, K_i, T_j, \sigma(m\Delta S, n\Delta t)\right) \\ &\qquad \times \left[V\left(m\Delta S, n\Delta t, K_i, T_j, \sigma(m\Delta S, n\Delta t)\right) - v_{ij}\right], \end{aligned} \tag{49}$$

and then we have

$$\sigma(m\Delta S, n\Delta t) = \frac{A_{p1}\sigma(p1) + B_{p2}\sigma(p2) - C}{A_{p1} + B_{p2}}. \tag{50}$$

We adopt the Gauss-Jacobi iteration scheme. At each step k, we update σ^{k-1} to σ^k by

$$\sigma^k(m\Delta S, n\Delta t) = \frac{A_{p1}^{k-1}\sigma^{k-1}(p1) + B_{p2}^{k-1}\sigma^{k-1}(p2) - C^{k-1}}{A_{p1}^{k-1} + B_{p2}^{k-1}}. \tag{51}$$

An important issue in practice is the choice of the regularization parameter α, which determines the balance between accuracy and regularity in the method. In general, the smaller the α, the preciser the solution. When $\alpha \to 0$, the optimal control functional can reach the exact solution but is unstable. So regularization parameter α should not be too big so that the process of seeking σ_α is stable. There are two main approaches to set α. One is a priori methods, in which the choice of α only depends on δ, the level on noise on the data, such as the size of bid-ask spread; the other is a posteriori methods, in which α may depend on the data in a less specific way. In financial literatures the most commonly used method for choosing α is the a posteriori methods based on the so-called discrepancy principle (such as Morozov discrepancy principle [52]), which consists in choosing the greatest level of α for which the data fidelity item does not exceed the level of noise δ on the observations:

$$\alpha := \sup\left\{\alpha > 0, \left\| V(\sigma) - v^{\delta} \right\| < d\delta \right\}. \quad (52)$$

Algorithm 6. Total variation for solving the implied volatility.

(1) Choose a function $\sigma_0(S,t)$. This will be the initial approximation to the true volatility.

(2) Determine $\sigma_0(m\Delta S, n\Delta t)$.

(3) Compute $V(m\Delta S, n\Delta t, K_i, T_j, \sigma^k(m\Delta S, n\Delta t))$ and $(\partial V/\partial\sigma)(m\Delta S, n\Delta t, K_i, T_j, \sigma^k(m\Delta S, n\Delta t))$ by using the Black-Scholes formula:

$$\begin{aligned} C(S,t) &= SN(d_1) - Ke^{-r(T-t)}N(d_2), \\ \frac{\partial C}{\partial \sigma} &= SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma} \\ &= \frac{S\sqrt{T-t}\,e^{-d_1^2/2}}{\sqrt{2\pi}}. \end{aligned} \quad (53)$$

(4) Compute $A^k_{p1}, B^k_{p2}, C^k, \sigma^k(p1)$, and $\sigma^k(p2)$.

(5) Adopt the Gauss-Jacobi iteration scheme:

$$\sigma^{k+1}(m\Delta S, n\Delta t) = \frac{A^k_{p1}\sigma^k(p1) + B^k_{p2}\sigma^k(p2) - C^k}{A^k_{p1} + B^k_{p2}}. \quad (54)$$

(6) If $\|\sigma^{k+1} - \sigma^k\|_\infty < \text{tol}$, the iteration is stopped; otherwise $k = k+1$ and go to step 3.

5. Numerical Experiments

In this section, we present numerical experiments to illustrate the theory and algorithm presented in above sections. First we assume that the true volatility function, $\sigma_{ex}(S,t)$, is defined as

$$\sigma_{ex}(S,t) = \begin{cases} 0.2 + 0.01e^{-0.01S} + \dfrac{\cos(3t)}{100}; \\ 0.19 + 0.01e^{-0.01S} + \dfrac{\cos(3t)}{100}. \end{cases} \quad (55)$$

In numerical experiments, the interest rate $r = 0.05$, $S_{max} = 100$, we consider only one time to option maturity $T = 1$. We take $\Delta S = 1$, $\Delta t = 0.01$ and $K_1 = 40$, $K_2 = 41, \ldots K_{21} = 60$. Figure 2 displays the true volatility function.

We solve the volatility by using Algorithm 6, and Figure 3 shows the error between $\sigma_{ex}(S,t)$ and the estimated $\sigma^{50}_{TV}(S,t)$,

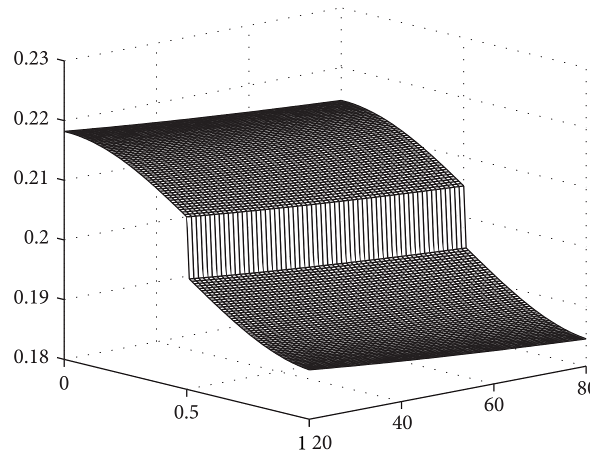

Figure 2: Volatility function $\sigma_{ex}(S,t)$.

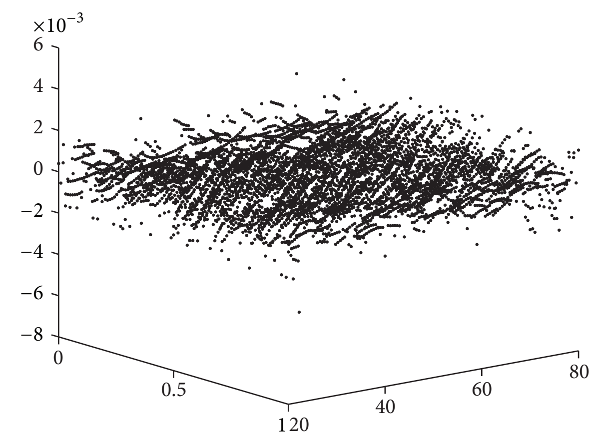

Figure 3: The error between $\sigma_{ex}(S,t)$ and $\sigma^{50}_{TV}(S,t)$.

where $\sigma^{50}_{TV}(S,t)$ denotes the 50 iterations of total variation algorithm. Almost all errors fell in the region $[-0.002, 0.002]$ and $\|\sigma^{50}_{TV}(S,t) - \sigma_{ex}(S,t)\|_\infty = 0.0069$.

If we fix $S = S^*$, for example, $S^* = 40$, Figure 4 shows the comparison between $\sigma_{ex}(40,t)$ (continuous line) and $\sigma^{50}_{TV}(40,t)$. Figure 5 shows the comparison between the $\sigma_{ex}(40,t)$ and $\sigma^{50}_{TIK}(40,t)$ (by using the classical Tikhonov regularization strategy) and $\|\sigma^{50}_{TIK}(S,t) - \sigma_{ex}(S,t)\|_\infty = 0.0161$.

According to Figures 4 and 5, the estimation of implied volatility using total variation regularization has two advantages compared with the classical Tikhonov regularization: one is that the total variation regularization maintains the singularities of the solution better (when $T = 0.5$) and the Tikhonov regularization oversmooths the discontinuity point; the other is that the error $(\|\sigma(S,t) - \sigma_{ex}(S,t)\|_\infty)$ obtained by total variation regularization is smaller.

6. Conclusion

A lot of research works have been made to determine the implied volatility by regularization strategies. Based on the

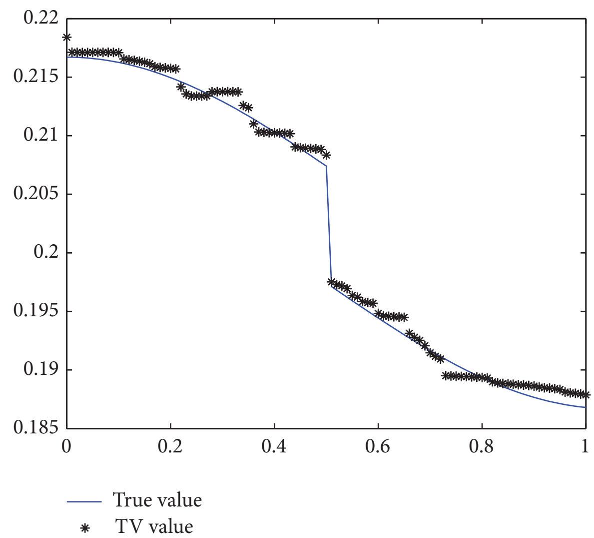

FIGURE 4: $\sigma_{ex}(40,t)$ and $\sigma_{\mathrm{TV}}^{50}(40,t)$.

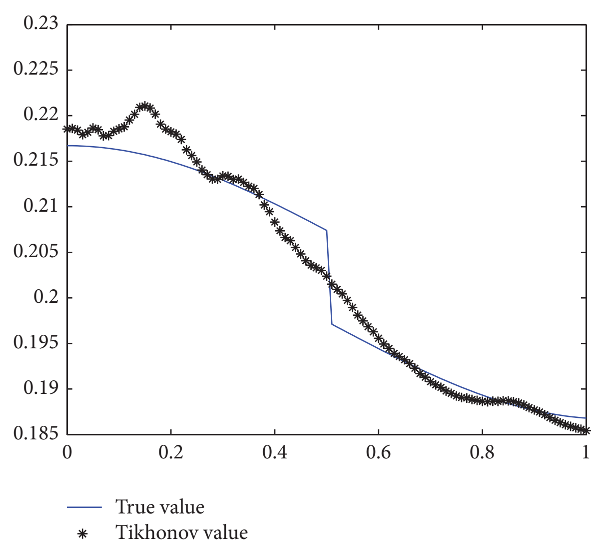

FIGURE 5: $\sigma_{ex}(40,t)$ and $\sigma_{\mathrm{TIK}}^{50}(40,t)$.

advantages and great success of the total variation regularization strategy in image processing, we propose the total variation regularization strategy to estimate the implied volatility under the framework of the Black-Scholes model. We identify the implied volatility by solving an optimal control problem and investigate a rigorous mathematical analysis. Not only the existence is discussed, but also the stability and convergence for this regularized approach are given. We also deduce the Euler-Lagrange equation. Furthermore, the results of numerical experiments are presented.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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