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Research Article

Explicit Finite Difference Methods for the Delay Pseudoparabolic Equations

I. Amirali,[1] G. M. Amiraliyev,[1] M. Cakir,[2] and E. Cimen[3]

[1] *Department of Mathematics, Faculty of Art and Science, Sinop University, 57000 Sinop, Turkey*
[2] *Department of Mathematics, Faculty of Science, Yüzüncü Yil University, 65080 Van, Turkey*
[3] *Department of Mathematics, Faculty of Education, Yüzüncü Yil University, 65080 Van, Turkey*

Correspondence should be addressed to I. Amirali; ailhame@gmail.com

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Finite difference technique is applied to numerical solution of the initial-boundary value problem for the semilinear delay Sobolev or pseudoparabolic equation. By the method of integral identities two-level difference scheme is constructed. For the time integration the implicit rule is being used. Based on the method of energy estimates the fully discrete scheme is shown to be absolutely stable and convergent of order two in space and of order one in time. The error estimates are obtained in the discrete norm. Some numerical results confirming the expected behavior of the method are shown.

1. Introduction

We consider the initial-boundary value problem for pseudoparabolic equation with delay in the domain $\overline{Q} = \overline{\Omega} \cup [0, T]$; $\overline{\Omega} = [0, l]$, $Q = \Omega \cup (0, T]$, $\Omega = (0, l)$:

$$\begin{aligned} Lu &:= \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial t \partial x}\right) - \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial u}{\partial x}\right) \\ &= f(x,t,u(x,t),u(x,t-r)), \quad (x,t) \in Q, \\ u(x,t) &= \varphi(x,t), \quad (x,t) \in \overline{\Omega} \times [-r,0], \\ u(0,t) &= u(l,t) = 0, \quad t \in (0,T], \end{aligned} \tag{1}$$

where r represents the delay parameter (for simplicity we assume that T/r is an integer; i.e. , $T = mr$ for some integer $m > 0$), $a(x,t) \geq \alpha > 0$, $|b(x,t)| \leq b^*$, and $\varphi(x,t)$ and $f(x,t,u_1,u_2)$ are given sufficiently smooth functions satisfying certain regularity conditions in $\overline{Q}$ and $\overline{\Omega} \times [-r, 0]$ and $\overline{Q} \times \mathbb{R}^2$, respectively, to be specified, and furthermore

$$\left|\frac{\partial f}{\partial u_1}\right| \leq c^* < \infty, \qquad \left|\frac{\partial f}{\partial u_2}\right| \leq d^* < \infty. \tag{2}$$

Equations of this type arise in many areas of mechanics and physics. Such equations are encountered, for example, as a model for two-phase porous media flows when dynamic effects in the capillary pressure are included [1–3]. They are used also to study heat conduction [4], homogeneous fluid flow in fissured rocks [5], shear in second order fluids [6–8], and other physical models. For a discussion of existence and uniqueness results of pseudoparabolic equations see [1, 9–11]. Various numerical treatments of equations of this type without delay have been considered in [2, 12–19] (see also the references cited in them).

In the present paper finite difference technique is applied to numerical solution of the initial-boundary value problem for the semilinear delay Sobolev or pseudoparabolic equation. By the method of integral identities with use of the piecewise linear basis functions in space and interpolating quadrature rules with weight and remainder term in integral form, two-level difference scheme is constructed (see also [12–14]) for singular perturbation cases without delay. For the time integration we use the implicit rule. The finite difference discretization is shown to be absolutely stable and convergent of order two in space and of order one in time. Based on the method of energy estimates the error analysis for approximate solution is presented. The error estimates are obtained in the discrete norm. Some numerical results confirming the expected behavior of the method are shown.

2. Discretization and Mesh

Notation. Let a set of mesh nodes that discretises Q be given by

$$\omega = \omega_N \times \omega_{N_0}, \tag{3}$$

with

$$\begin{gathered}\omega_N = \left\{x_i = ih,\ i = 1,2,\ldots,N-1,\ h = \frac{l}{N}\right\},\\ \omega_N^+ = \omega_N \cup \{x_N = l\}, \qquad \overline{\omega}_N = \omega_N \cup \{x_0 = 0,\ x_N = l\},\\ \omega_{N_0} = \left\{t_j = j\tau,\ j = 1,2,\ldots,N_0,\ \tau = \frac{T}{N_0} = \frac{r}{n_0}\right\},\\ \overline{\omega}_{N_0} = \omega_{N_0} \cup \{t_0 = 0\},\\ \omega_{n_0^-} = \left\{t_j = j\tau, j = -n_0,\ldots,0\right\}.\end{gathered} \tag{4}$$

Define the following finite differences for any mesh function $v_i = v(x_i)$ given on $\overline{\omega}_N$ by

$$\begin{gathered}v_{\overline{x},i} = \frac{v_i - v_{i-1}}{h}, \qquad v_{x,i} = \frac{v_{i+1} - v_i}{h},\\ v_{\overset{\circ}{x},i} = \frac{v_{i+1} - v_{i-1}}{2h},\\ v_{\overline{x}x,i} = \frac{v_{x,i} - v_{\overline{x},i}}{h} = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}.\end{gathered} \tag{5}$$

Introduce the inner products for the mesh functions v_i and w_i defined on $\overline{\omega}_N$ as follows:

$$\begin{gathered}(v,w)_0 \equiv (v,w)_{\omega_N} := \sum_{i=1}^{N-1} h v_i w_i,\\ (v,w] \equiv (v,w)_{\omega_N^+} := \sum_{i=1}^{N} h v_i w_i.\end{gathered} \tag{6}$$

For any mesh function v_i, vanishing for $i = 0$ and $i = N$ we introduce the norms

$$\begin{gathered}\|v\|_0^2 \equiv \|v\|_{0,\omega_N}^2 := (v,v)_0,\\ \left\|v_{\overline{x}}\right\|_0^2 \equiv \left\|v_{\overline{x}}\right\|_{0,\omega_N^+}^2 := \left(v_{\overline{x}}, v_{\overline{x}}\right],\\ \|v\|_1^2 = \|v\|_0^2 + \left\|v_{\overline{x}}\right\|_0^2,\\ \|v\|_\infty \equiv \|v\|_{\infty,\omega_N} := \max_{1\le i\le N-1} \left|v_i\right|,\end{gathered} \tag{7}$$

and "negative" norm for any function w_i $(1 \le i \le N-1)$

$$\|w\|_{-1} := \sup_{v\ne 0} \frac{\left|(w,v)_0\right|}{\|v\|_1}. \tag{8}$$

Given a function $g \equiv g_i^j \equiv g(x_i,t_j)$; $\breve{g} = g_i^{(j-1)}$, defined on $\overline{\omega}$, we will also use the notation

$$g_{\overline{t},i}^j = \frac{g_i^j - g_i^{j-1}}{\tau_j}, \qquad g_{t,i}^j = \frac{g_i^{j+1} - g_i^j}{\tau_{j+1}}. \tag{9}$$

2.1. Difference Scheme. The approach of generating difference scheme is through the integral identity

$$\begin{aligned}\tau^{-1}h^{-1} &\int_{t_{j-1}}^{t_j}\int_{x_{i-1}}^{x_{i+1}} Lu\psi_i(x)\,dx\,dt\\ &= \tau^{-1}h^{-1}\int_{t_{j-1}}^{t_j}\int_{x_{i-1}}^{x_{i+1}} f(x,t,u(x,t),u(x,t-r))\\ &\qquad \times \psi_i(x)\,dx\,dt,\end{aligned} \tag{10}$$

with the usual piecewise linear basis functions for the space

$$\psi_i(x) = \begin{cases} h^{-1}(x - x_{i-1}), & x_{i-1} < x < x_i,\\ h^{-1}(x_{i+1} - x), & x_i < x < x_{i+1},\\ 0, & x \notin (x_{i-1}, x_{i+1}),\\ & i = 1,2,\ldots,N-1.\end{cases} \tag{11}$$

Using the appropriate interpolating quadrature rules with weight and remainder term in integral form, consistent with [12–14], we obtain the precise relation

$$\begin{aligned}\ell u &:= u_{\overline{t}} - \left(Au_{\overline{t}\overline{x}}\right)_x - \left(Bu_{\overline{x}}\right)_x + R\\ &= f(x,t-\tau,u(x,t-\tau),u(x,t-\tau-r)),\\ &\qquad (x,t) \in \omega_N \times \omega_{N_0},\end{aligned} \tag{12}$$

where

$$\begin{gathered}A = \tau^{-1}h^{-1}\int_{t_{j-1}}^{t_j}\int_{x_{i-1}}^{x_i} a(x,t)\,dx\,dt,\\ B = \tau^{-1}h^{-1}\int_{t_{j-1}}^{t_j}\int_{x_{i-1}}^{x_i} b(x,t)\,dx\,dt.\end{gathered} \tag{13}$$

The remainder term R has the form

$$R = \left(R^{(0)}\right)_x + R^{(1)}, \quad \text{for } (x,t) \in \omega_N \times \omega_{N_0} \tag{14}$$

with

$$\begin{aligned}R^{(0)} = &-\tau^{-1}h^{-1}\int_{t_{j-1}}^{t_j} dt \int_{x_{i-1}}^{x_i} dx \frac{\partial a}{\partial x}\\ &\times \int_{x_{i-1}}^{x_i} \frac{\partial^3 u}{\partial t\partial x^2}(\xi,t) K_1(x,\xi)\,d\xi\end{aligned}$$

$$-\tau^{-1}h^{-1}\int_{t_{j-1}}^{t_j} dt\left\{\int_{x_{i-1}}^{x_i}\frac{\partial a}{\partial t}(x,t)\,dx\right\}$$
$$\times\int_{t_{j-1}}^{t_j} K_0(\eta,t)\frac{\partial}{\partial t}u_{\overline{x},i}(x,\eta)\,d\eta$$
$$-\tau^{-1}h^{-1}\int_{t_{j-1}}^{t_j} dt\int_{x_{i-1}}^{x_i} dx\frac{\partial b}{\partial x}(x,t)$$
$$\times\int_{x_{i-1}}^{x_i}\frac{\partial^2 u}{\partial x^2}(\xi,t)K_1(x,\xi)\,d\xi$$
$$+\tau^{-1}h^{-1}\int_{t_{j-1}}^{t_j} dt\left\{\int_{x_{i-1}}^{x_i} b(x,t)\,dx\right\}$$
$$\times\int_{t_{j-1}}^{t_j} T_0(t-\xi)\frac{\partial}{\partial t}u_{\overline{x},i}(x,\xi)\,d\xi, \tag{15}$$

$$R^{(1)} = -\tau^{-1}h^{-1}\int_{t_{j-1}}^{t_j} dt\int_{x_{i-1}}^{x_{i+1}} dx\varphi_i(x)$$
$$\times\int_{x_{i-1}}^{x_{i+1}}\frac{\partial^3 u}{\partial t\partial x^2}(\xi,t)K_1^*(x,\xi)\,d\xi$$
$$+\tau^{-1}h^{-1}\int_{t_{j-1}}^{t_j} dt\int_{x_{i-1}}^{x_{i+1}} dx\varphi_i(x)$$
$$\times\int_{x_{i-1}}^{x_{i+1}}\frac{d^2}{dx^2}f(\xi,t,u(\xi,t),u(\xi,t-r))$$
$$\times K_1^*(x,\xi)\,d\xi$$
$$+\tau^{-1}\int_{t_{j-1}}^{t_j}(t_j-t)\frac{d}{dt}$$
$$\times f(x_i,t,u(x_i,t),u(x_i,t-r))\,dt,$$

$$K_1(x,\xi)=T_1(x-\xi)-h^{-1}(x-x_{i-1})(x_i-\xi),$$
$$K_0(t,\eta)=T_0(t-\eta)-\tau^{-1}(t-t_{j-1}),$$
$$K_1^*(x,\xi)=T_1(x-\xi)-(2h)^{-1}(x-x_{i-1})(x_{i+1}-\xi), \tag{16}$$
$$T_s(\lambda)=\lambda^s/s!,\quad \lambda>0,\qquad T_s(\lambda)=0,\quad \lambda<0.$$

Based on (12), we propose the following difference scheme for approximating (1):

$$\ell y = f(x,t-\tau,y(x,t-\tau),y(x,t-\tau-r)),$$
$$(x,t)\in\omega_N\times\omega_{N_0},$$
$$y(0,t)=y(l,t)=0,\quad t\in\overline{\omega}_{N_0}, \tag{17}$$
$$y(x,0)=\varphi(x,0),\quad x\in\overline{\omega}_N,$$

where ℓy is defined by (12).

3. The Error Estimates and Convergence

To estimate the convergence of this method, note that the error function $z = y-u$ is the solution of the discrete problem,

$$\ell z = f(x,t-\tau,y(x,t-\tau),y(x,t-\tau-r))$$
$$-f(x,t-\tau,u(x,t-\tau),u(x,t-\tau-r))+R,$$
$$(x,t)\in\omega_N\times\omega_{N_0}, \tag{18}$$
$$z(x,t)=0,\quad (x,t)\in\overline{\omega}_N\times\omega_{n_0^-},$$
$$z(0,t)=z(l,t)=0,\quad t\in\omega_{N_0}.$$

Before obtaining the estimate for the solution (18) we give the following Lemma.

Lemma 1. *Let the mesh function $\delta\ge 0$, defined on ω_{N_0}, satisfy*

$$\delta_j\le\alpha+\tau\sum_{k=1}^{j}\left\{c_1\delta_{k-1}+c_*\delta_{j-n_0-1}+\rho_k\right\},\quad j=1,2,\ldots,N_0,$$
$$\delta_j\le\varphi_j,\quad j=-n_0,\ldots,0,\qquad \varphi_0\le\alpha, \tag{19}$$

where $0\le\alpha$, $c_1, c_* = const$, $\rho_j\ge 0$, φ_j given, $n_0\ge 0$ is an integer. Then

$$\delta_j\le\left(\alpha+c_*\|\varphi\|_{L_1(\omega_{n_0^-})}\right)e^{c_1 t_j}+\tau\sum_{k=1}^{j}e^{(c_1+c_*)t_{j-k}}\rho_k, \tag{20}$$

where

$$\|\varphi\|_{L_1(\omega_{n_0^-})}=\sum_{j=-n_0}^{0}\tau\varphi_j. \tag{21}$$

Proof. For $1\le j\le n_0+1$, $-n_0\le k-1-n_0\le 0$ and inequality (19) reduces to

$$\delta_j\le\alpha+\tau\sum_{k=1}^{j}(c_1\delta_{k-1}+\rho_k)+c_*\|\varphi\|_{L_1(\omega_{n_0^-})}. \tag{22}$$

Applying now the difference analogue of the Gronwall's inequality we get

$$\delta_j\le\left(\alpha+c_*\|\varphi\|_{L_1(\omega_{n_0^-})}\right)e^{c_1 t_j}+\tau\sum_{k=1}^{j}e^{c_1 t_{j-k}}\rho_k,$$
$$1\le j\le n_0+1. \tag{23}$$

For $j > n_0 + 1$, after replacing in (19) $k - n_0 = p$, we have

$$\begin{aligned}\delta_j &\le \alpha + \tau\sum_{k=1}^{j}\{c_1\delta_{k-1} + \rho_k\} + \tau\sum_{p=1-n_0}^{j-n_0} c_*\delta_{k-1}\\ &= \alpha + \tau\sum_{k=1}^{j}\{c_1\delta_{k-1} + \rho_k\}\\ &\quad + \tau\sum_{p=1-n_0}^{1} c_*\delta_{k-1} + \tau\sum_{p=2}^{j-n_0} c_*\delta_{k-1}\\ &\le \alpha + \tau\sum_{k=1}^{j}\{(c_1 + c_*)\delta_{k-1} + \rho_k\}\\ &\quad + c_*\|\varphi\|_{L_1(\omega_{n_0}^-)},\end{aligned} \tag{24}$$

which by virtue of difference analogue of the Gronwall's inequality leads to (20), immediately. □

Theorem 2. *Let the derivatives* $\partial f/\partial t, \partial^s f/\partial x^s$, $\partial^{1+k} f/\partial u_s \partial x^k$ $(s = 1, 2;\ k = 0, 1)$, *and* $\partial^2 f/\partial u_1^k \partial u_2^{2-k}$ $(k = 0, 1, 2)$ *be continuous and bounded on* $\overline{Q} \times \mathbb{R}^2$, $a, b \in C(\overline{Q})$, $(\partial^2/\partial x^2)\varphi(x,t)$, $(\partial/\partial t)\varphi(x,t) \in C(\overline{\Omega} \times [-r, 0])$, *and* $\partial u/\partial t$, $\partial u/\partial x$, $\partial^2 u/\partial x^2$, *and* $\partial^3 u/\partial x^2 \partial t \in C(\overline{Q})$. *Then for the discrete problem* (17) *the following error estimate holds:*

$$\|y - u\|_1 \le C(h^2 + \tau), \quad t \in \omega_{N_0}. \tag{25}$$

Proof. Consider identity

$$\begin{aligned}(\ell z, z_{\bar t})_{\omega_N} &= \big(f(x, t-\tau, y(x, t-\tau-n_0\tau))\\ &\quad - f(x, t-\tau, u(x, t-\tau-n_0\tau)), z_{\bar t}\big)_{\omega_N}\\ &\quad + (R, z_{\bar t})_{\omega_N}.\end{aligned} \tag{26}$$

After some manipulations, we get

$$\begin{aligned}\|z_{\bar t}\|_0^2 + \alpha\|z_{\bar t\bar x}\|_0^2 &\le b^*\|\check z_{\bar x}\|_0\|z_{\bar t\bar x}\|_0 + c^*\|z_{\bar t}\|_0\|\check z\|_0\\ &\quad + d^*\|z_{\bar t}\|_0\|z(t-\tau-n_0\tau)\|_0\\ &\quad + \|z_{\bar t\bar x}\|_0\|R\|_{-1},\\ \frac{1}{2}\|z_{\bar t}\|_0^2 + \frac{\alpha}{2}\|z_{\bar t\bar x}\|_0^2 &\le \alpha^{-1}(b^*)^2\|\check z_{\bar x}\|_0^2 + (c^*)^2\|\check z\|_0^2\\ &\quad + (d^*)^2\|z(t-\tau-n_0\tau)\|_0^2\\ &\quad + \alpha^{-1}\|R\|_{-1}^2.\end{aligned} \tag{27}$$

Multiplying this inequality by τ and summing it up from $k = 1$ to $k = j$, also, using here the inequality

$$v_j^2 \le t_j\tau\sum_{k=1}^{j} v_{\bar t,k}^2 \le T\tau\sum_{k=1}^{j} v_{\bar t,k}^2, \quad (v_0 = 0), \tag{28}$$

we obtain

$$\begin{aligned}&\|z^j\|_0^2 + \alpha\|z_{\bar x}^j\|_0^2\\ &\quad\le \tau\sum_{k=1}^{j}\Big\{2\alpha^{-1}T(b^*)^2\|\check z_{\bar x}\|_0^2 + 2(c^*)^2T\|\check z\|_0^2\\ &\qquad + 2(d^*)^2T\|\check z(t-r)\|_0^2 + 2T\alpha^{-1}\|R\|_{-1}^2\Big\}.\end{aligned} \tag{29}$$

Denoting

$$\delta_j = \|z^j\|_0^2 + \alpha\|z_{\bar x}^j\|_0^2, \tag{30}$$

we have

$$\delta_j \le \tau\sum_{k=1}^{j}\{c_1\delta_{k-1} + c_*\delta_{k-n_0-1} + \rho_k\}, \quad j \geqslant 1, \tag{31}$$

where

$$\begin{aligned}c_1 &= 2T\max\{\alpha^{-1}(b^*)^2, (c^*)^2\}, \qquad c_* = 2T(d^*)^2,\\ \rho &= 2T\alpha^{-1}\|R\|_{-1}^2.\end{aligned} \tag{32}$$

Applying now Lemma 1 we obtain

$$\|z^j\|_0^2 + \alpha\|z_{\bar x}^j\|_0^2 \leqslant 2T\alpha^{-1}\tau\sum_{k=1}^{j} e^{(c_1+c_*)t_{k-j}}\|R^k\|_{-1}^2. \tag{33}$$

Further, in view of the fact that

$$\begin{aligned}|(R, z)_0| &\le |(R_x^{(0)}, z)_0| + |(R^{(1)}, z)_0|\\ &= |(R^{(0)}, z_{\bar x}]| + |(R^{(1)}, z)_0|, \quad t \in \omega_{N_0},\end{aligned} \tag{34}$$

we obtain

$$\|R\|_{-1} \le \|R^{(0)}\|_{0,\omega_N^+} + \|R^{(1)}\|_{0,\omega_N}, \quad t \in \omega_{N_0}, \tag{35}$$

where $R^{(0)}$ and $R^{(1)}$are given by (15). From (35), under the assumed smoothness, we have

$$\tau\sum_{k=1}^{N_0}\|R^k\|_{-1}^2 = O(h^2 + \tau), \tag{36}$$

which together with (33) completes the proof of the theorem. □

Remark 3. Under sufficiently smoothness of $a(x,t)$ and $b(x,t)$ for calculations of A and B appropriate numerical quadrature formulae can be applied; for example, $A = a(x_i - h/2, t_j)$, $A = (1/2)[a(x_i, t_j) + a(x_{i-1}, t_j)]$, and so forth.

4. Numerical Results

In this section, we present numerical results obtained by applying the numerical method (17) to the particular problems.

Table 1: The numerical results on (0,1) × (0,1).

Nodes (x,t)	Exact solution	Numerical solution $h=0.02$ $\tau=0.02$	Pointwise error $\|y-u\|$	Numerical solution $h=0.02$ $\tau=0.01$	Pointwise error $\|y-u\|$
$(0.1, 0.1)$	$-1.570E-02$	$-1.571E-02$	$1.602E-05$	$-1.571E-02$	$8.060E-06$
$(0.2, 0.2)$	$-2.759E-02$	$-2.765E-02$	$5.784E-05$	$-2.762E-02$	$2.908E-05$
$(0.3, 0.3)$	$-3.558E-02$	$-3.570E-02$	$1.150E-04$	$-3.564E-02$	$5.779E-05$
$(0.4, 0.4)$	$-3.976E-02$	$-3.994E-02$	$1.760E-04$	$-3.985E-02$	$8.847E-05$
$(0.5, 0.5)$	$-4.033E-02$	$-4.056E-02$	$2.293E-04$	$-4.045E-02$	$1.152E-04$
$(0.6, 0.6)$	$-3.757E-02$	$-3.783E-02$	$2.633E-04$	$-3.770E-02$	$1.324E-04$
$(0.7, 0.7)$	$-3.180E-02$	$-3.207E-02$	$2.673E-04$	$-3.194E-02$	$1.343E-04$
$(0.8, 0.8)$	$-2.338E-02$	$-2.362E-02$	$2.308E-04$	$-2.350E-02$	$1.160E-04$
$(0.9, 0.9)$	$-1.267E-02$	$-1.281E-02$	$1.445E-04$	$-1.274E-02$	$7.261E-05$

Table 2: The numerical results on (0,1) × (1,2).

Nodes (x,t)	Exact solution	Numerical solution $h=0.02$ $\tau=0.02$	Pointwise error $\|y-u\|$	Numerical solution $h=0.02$ $\tau=0.01$	Pointwise error $\|y-u\|$
$(0.1, 1.1)$	$-5.776E-03$	$-5.862E-03$	$8.565E-05$	$-5.819E-03$	$4.203E-05$
$(0.2, 1.2)$	$-1.015E-02$	$-1.032E-02$	$1.688E-04$	$-1.023E-02$	$8.481E-05$
$(0.3, 1.3)$	$-1.309E-02$	$-1.333E-02$	$2.422E-04$	$-1.321E-02$	$1.217E-04$
$(0.4, 1.4)$	$-1.463E-02$	$-1.492E-02$	$2.993E-04$	$-1.478E-02$	$1.504E-04$
$(0.5, 1.5)$	$-1.483E-02$	$-1.517E-02$	$3.339E-04$	$-1.500E-02$	$1.678E-04$
$(0.6, 1.6)$	$-1.382E-02$	$-1.416E-02$	$3.406E-04$	$-1.399E-02$	$1.711E-04$
$(0.7, 1.7)$	$-1.170E-02$	$-1.201E-02$	$3.145E-04$	$-1.186E-02$	$1.580E-04$
$(0.8, 1.8)$	$-8.604E-03$	$-8.856E-03$	$2.514E-04$	$-8.730E-03$	$1.263E-04$
$(0.9, 1.9)$	$-4.661E-03$	$-4.808E-03$	$1.476E-04$	$-4.735E-03$	$7.415E-05$

Example 1. Consider the following linear problems:

$$\begin{gathered}\frac{\partial u}{\partial t} - 2\frac{\partial^3 u}{\partial t \partial x^2} - \frac{\partial^2 u}{\partial x^2} + u(x, t-1) = f(x,t), \\ (x,t) \in [0,1] \times [0,2] \\ u(x,t) = e^{-t}(\sinh(x) - x\sinh(1)), \\ (x,t) \in [0,1] \times [-1,0] \\ u(0,t) = 0, \qquad u(1,t) = 0, \quad t \in (0,2], \end{gathered} \tag{37}$$

where

$$f(x,t) = e^{-t} x \sinh(1) + e^{1-t}(\sinh(x) - x\sinh(1)). \tag{38}$$

The exact solution of this problem is $u(x,t) = e^{-t}(\sinh(x) - x\sinh(1))$. The computational results are presented in Tables 1 and 2.

Example 2. Now consider the following nonlinear problem:

$$\begin{gathered}\frac{\partial u}{\partial t} - 3\frac{\partial^3 u}{\partial t \partial x^2} - 2\frac{\partial^2 u}{\partial x^2} + \tanh(u(x,t-1)) = f(x,t), \\ (x,t) \in [0,1] \times [0,2], \\ u(x,t) = e^{-t}(\cosh(x) - x\cosh(1) + x - 1), \\ (x,t) \in [0,1] \times [-1,0], \\ u(0,t) = 0, \qquad u(1,t) = 0, \quad t \in (0,2], \end{gathered} \tag{39}$$

where

$$\begin{aligned} f(x,t) &= e^{-t}(x\cosh(1) - x + 1) \\ &\quad + \tanh\left(e^{1-t}(\cosh(x) - x\cosh(1) + x - 1)\right). \end{aligned} \tag{40}$$

The exact solution of this problem is $u(x,t) = e^{-t}(\cosh(x) - x\cosh(1) + x - 1)$. The computational results are presented in Tables 3 and 4.

Table 3: The numerical results on (0,1) × (0,1).

Nodes (x,t)	Exact solution	Numerical solution $h = 0.02$ $\tau = 0.02$	Pointwise error $\lvert y-u\rvert$	Numerical solution $h = 0.02$ $\tau = 0.01$	Pointwise error $\lvert y-u\rvert$
(0.1, 0.1)	$-4.461E-02$	$-4.465E-02$	$4.503E-05$	$-4.463E-02$	$2.265E-05$
(0.2, 0.2)	$-7.249E-02$	$-7.264E-02$	$1.490E-04$	$-7.257E-02$	$7.494E-05$
(0.3, 0.3)	$-8.710E-02$	$-8.738E-02$	$2.736E-04$	$-8.724E-02$	$1.376E-04$
(0.4, 0.4)	$-9.127E-02$	$-9.166E-02$	$3.893E-04$	$-9.146E-02$	$1.957E-04$
(0.5, 0.5)	$-8.728E-02$	$-8.776E-02$	$4.741E-04$	$-8.752E-02$	$2.383E-04$
(0.6, 0.6)	$-7.704E-02$	$-7.755E-02$	$5.115E-04$	$-7.730E-02$	$2.571E-04$
(0.7, 0.7)	$-6.206E-02$	$-6.255E-02$	$4.896E-04$	$-6.231E-02$	$2.461E-04$
(0.8, 0.8)	$-4.359E-02$	$-4.399E-02$	$4.003E-04$	$-4.379E-02$	$2.012E-04$
(0.9, 0.9)	$-2.264E-02$	$-2.287E-02$	$2.381E-04$	$-2.276E-02$	$1.197E-04$

Table 4: The numerical results on (0,1) × (1,2).

Nodes (x,t)	Exact solution	Numerical solution $h = 0.02$ $\tau = 0.02$	Pointwise error $\lvert y-u\rvert$	Numerical solution $h = 0.02$ $\tau = 0.01$	Pointwise error $\lvert y-u\rvert$
(0.1, 1.1)	$-1.641E-02$	$-1.663E-02$	$2.192E-04$	$-1.652E-02$	$1.101E-04$
(0.2, 1.2)	$-2.667E-02$	$-2.706E-02$	$3.961E-04$	$-2.686E-02$	$1.990E-04$
(0.3, 1.3)	$-3.204E-02$	$-3.257E-02$	$5.251E-04$	$-3.230E-02$	$2.637E-04$
(0.4, 1.4)	$-3.357E-02$	$-3.417E-02$	$6.032E-04$	$-3.387E-02$	$3.029E-04$
(0.5, 1.5)	$-3.211E-02$	$-3.274E-02$	$6.289E-04$	$-3.242E-02$	$3.158E-04$
(0.6, 1.6)	$-2.834E-02$	$-2.894E-02$	$6.024E-04$	$-2.864E-02$	$3.024E-04$
(0.7, 1.7)	$-2.283E-02$	$-2.335E-02$	$5.244E-04$	$-2.309E-02$	$2.633E-04$
(0.8, 1.8)	$-1.603E-02$	$-1.643E-02$	$3.966E-04$	$-1.623E-02$	$1.991E-04$
(0.9, 1.9)	$-8.328E-03$	$-8.550E-03$	$2.211E-04$	$-8.439E-03$	$1.110E-04$

It can be observed that the obtained results are essentially in agreement with the theoretical analysis described above.

5. Conclusion

In this paper, we proposed an efficient numerical method for solving initial-boundary value problem for the semilinear pseudoparabolic equation. The proposed finite difference method was constructed, and based on the method of energy estimates the fully discrete scheme was shown to be absolutely stable and convergent of order two in space and of order one in time. The error estimates were obtained in discrete norm. Numerical results were presented, which numerically validate this theoretical result.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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