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Research Article

Lax Integrability and Soliton Solutions for a Nonisospectral Integrodifferential System

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Searching for integrable systems and constructing their exact solutions are of both theoretical and practical value. In this paper, Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem and its time evolution equation are first generalized by embedding a new spectral parameter. Based on the generalized AKNS spectral problem and its time evolution equation, Lax integrability of a nonisospectral integrodifferential system is then verified. Furthermore, exact solutions of the nonisospectral integrodifferential system are formulated through the inverse scattering transform (IST) method. Finally, in the case of reflectionless potentials, the obtained exact solutions are reduced to n-soliton solutions. When $n = 1$ and $n = 2$, the characteristics of soliton dynamics of one-soliton solutions and two-soliton solutions are analyzed with the help of figures.

1. Introduction

Nonlinear phenomena involved in many fields such as physics, biology, chemistry, and mechanics are often related to nonlinear partial differential equations (PDEs). The investigation of exact solutions of nonlinear PDEs plays an important role because of its direct connection with dynamical processes in these nonlinear phenomena. Since the initial-value problem of the Korteweg–de Vries (KdV) equation was exactly solved by the IST method [1], finding soliton solutions of nonlinear PDEs has become extremely active and some effective methods were proposed such as Hirota's bilinear method [2], Painlevé expansion [3], homogeneous balance method [4], and function expansion methods [5–10]. Among these methods, the IST [1] is a systematic method which has achieved considerable development and received a wide range of applications like those in [11–21] since it is put forward by Gardner, Greene, Kruskal, and Miura in 1967. One of the advantages of the IST is that it can solve a whole hierarchy of nonlinear PDEs associated with a certain spectral problem. As early as in 1976, the framework of IST with varying spectral parameter was introduced for the first time by Chen and Liu to the nonlinear Schrödinger (NLS) equation with a linear external potential [22] and by Hirota and Satsuma to the KdV equation in nonuniform media [23]. Serkin et al. [24–28] pointed out that the soliton dynamics of nonautonomous ones which interact elastically and generally move with varying amplitudes, speeds, and spectra adapted both to the external potentials and to the dispersion and nonlinearity variations can be described in the framework of the IST theory with varying in time spectral parameter.

In soliton theory, nonlinear PDEs associated with some linear spectral problems can be generally classified as the isospectral equations which often describe solitary waves in lossless and uniform media and the nonisospectral equations describing the solitary waves in a certain type of nonuniform media. Specifically, when the spectral parameter of the associated linear spectral problem is independent of time, one could construct isospectral equations. While starting from the spectral problem with a time-dependent spectral parameter, nonisospectral equations are usually derived. In 1974, Ablowitz, Kaup, Newell, and Segur [21] successfully constructed a hierarchy of isospectral nonlinear PDEs; here it is written as

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^n \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (n = 0, 1, 2, \ldots), \tag{1}$$

$$L = \sigma\partial + 2\begin{pmatrix} q \\ -r \end{pmatrix}\partial^{-1}(r,q),$$
$$\partial = \frac{\partial}{\partial x},$$
$$\partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} \mathrm{d}x - \int_{x}^{+\infty} \mathrm{d}x\right), \tag{2}$$
$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

from the compatibility condition, that is, the zero curvature equation

$$M_t - N_x + [M, N] = 0 \tag{3}$$

of the following spectral problem

$$\phi_x = M\phi, \quad M = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}, \ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{4}$$

and its evolution equation

$$\phi_t = N\phi, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{5}$$

where $q = q(x,t)$, $r = r(x,t)$, and their derivatives of any order with respect to x and t are smooth functions which vanish as x tends to infinity, the spectral parameter k is independent with x and t, and A, B, and C are undetermined functions of x, t, q, r, and k.

When $n = 2$, the isospectral AKNS hierarchy (1) gives

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} q_{xxx} - 6qrq_x \\ r_{xxx} - 6qrr_x \end{pmatrix}, \tag{6}$$

which includes the famous KdV equation $q_t = q_{xxx} + 6qq_x$ as a special case.

Subsequently, in the case of spectral parameter k being dependent on time t, Calogero and Degasperis [29–31] and Li [32] proposed effective methods to derive different hierarchies of nonisospectral equations. For example, the nonisospectral AKNS hierarchy [20]

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^n \begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad (n = 0, 1, 2, \ldots), \tag{7}$$

can be derived from (3)–(5) equipped with $ik_t = (2ik)^n/2$. It is easy to see that when $n = 0, 1, 2$, the nonisospectral AKNS hierarchy (1) gives the following nonisospectral systems:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -xq \\ xr \end{pmatrix},$$
$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} q + xq_x \\ r + xr_x \end{pmatrix}, \tag{8}$$
$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -2q_x - xq_{xx} + 2q\partial^{-1}qr + 2xq^2r \\ 2r_x + xr_{xx} - 2r\partial^{-1}qr - 2xq^2r \end{pmatrix}.$$

The aim of this paper is to generalize AKNS spectral problem (4) and its evolution equation (5) for testing the integrability of the following new and more general nonisospectral integrodifferential system:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -2q_x - xq_{xx} + 2q\partial^{-1}qr + 2xq^2r + q + xq_x - xq - tq \\ 2r_x + xr_{xx} - 2r\partial^{-1}qr - 2xq^2r + r + xr_x + xr + tr \end{pmatrix} \tag{9}$$

and extending the IST to system (9). With the help of (2), we can rewrite system (9) in the form

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^2\begin{pmatrix} -xq \\ xr \end{pmatrix} + L\begin{pmatrix} -xq \\ xr \end{pmatrix} + \begin{pmatrix} -xq \\ xr \end{pmatrix} + \begin{pmatrix} -tq \\ tr \end{pmatrix}, \tag{10}$$

from which we can see that the nonisospectral integrodifferential system (9) with time-dependent coefficient terms is different from that in [33]

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^{2m+1}\begin{pmatrix} -xq \\ xr \end{pmatrix} + L^{2m}\begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad (m = 0, 1, 2, \ldots). \tag{11}$$

In order to construct system (9), in this paper we shall employ a new and more general spectral parameter k which satisfies

$$ik_t = \frac{1}{2}\sum_{n=0}^{2}(2ik)^n. \tag{12}$$

It is easy to see that the nonisospectral parameter $\eta_t = (2\eta)^n/2$ in [33] is a special case of (12). Here ik in (12) is equivalent to η in [33]. On the other hand, we shall generalize the matrix N in [33]

$$N|_{(q,r)=(0,0)} = \begin{pmatrix} -\frac{1}{2}(2ik)x & 0 \\ 0 & \frac{1}{2}(2ik)x \end{pmatrix} \tag{13}$$

to the following form:

$$N|_{(q,r)=(0,0)} = \begin{pmatrix} -\frac{1}{2}\left[\sum_{n=0}^{2}(2ik)^n\right]x - \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\left[\sum_{n=0}^{2}(2ik)^n\right]x + \frac{1}{2} \end{pmatrix}. \tag{14}$$

In the very recent work [34], we let the parameter k satisfy

$$ik_t = \frac{1}{2}\sin 2ik = \sum_{j=0}^{+\infty}(-1)^j\frac{1}{(2j+1)!}(2ik)^{2j+1} \tag{15}$$

and employed

$$N|_{(q,r)=(0,0)}\begin{pmatrix} -\frac{1}{2}x\sin 2ik & 0 \\ 0 & \frac{1}{2}x\sin 2ik \end{pmatrix}, \tag{16}$$

then a general nonisospectral integrodifferential system of the form

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \sum_{j=0}^{+\infty}(-1)^j\frac{1}{(2j+1)!}L^{2j+1}\begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{17}$$

is constructed. Equation (17) can be rewritten as

$$\begin{aligned}\begin{pmatrix} q \\ r \end{pmatrix}_t &= L\begin{pmatrix} -xq \\ xr \end{pmatrix} - \frac{1}{3!}L^3\begin{pmatrix} -xq \\ xr \end{pmatrix} + \frac{1}{5!}L^5\begin{pmatrix} -xq \\ xr \end{pmatrix} \\ &\quad + \sum_{j=3}^{+\infty}(-1)^j\frac{1}{(2j+1)!}L^{2j+1}\begin{pmatrix} -xq \\ xr \end{pmatrix}\end{aligned} \tag{18}$$

which has the expansion in part

$$\begin{aligned}\begin{pmatrix} q \\ r \end{pmatrix}_t &= \begin{pmatrix} q + xq_x - \frac{1}{2}q_{xx} - \frac{1}{6}xq_{xxx} + \frac{1}{3}q_x\partial^{-1}(qr) + \frac{2}{3}xqrq_x + \frac{1}{3}xq^2r_x + \frac{4}{3}q\partial^{-1}(q_xr) + \frac{1}{3}xqrq_{xx} - \frac{1}{3}xq^2r_{xx} \\ r + xr_x - \frac{1}{2}r_{xx} - \frac{1}{6}xr_{xxx} + \frac{1}{3}r_x\partial^{-1}(qr) + \frac{2}{3}xqrr_x + \frac{1}{3}xr^2q_x + \frac{4}{3}r\partial^{-1}(qr_x) + \frac{1}{3}xqrr_{xx} - \frac{1}{3}xr^2q_{xx} \end{pmatrix} \\ &\quad + \sum_{j=2}^{+\infty}(-1)^j\frac{1}{(2j+1)!}L^{2j+1}\begin{pmatrix} -xq \\ xr \end{pmatrix}.\end{aligned} \tag{19}$$

Obviously, there is substantial difference between system (9) and (17) in [34]. It is because, except for the term

$$L\begin{pmatrix} -xq \\ xr \end{pmatrix} = \begin{pmatrix} q + xq_x \\ r + xr_x \end{pmatrix}, \tag{20}$$

the other three terms of (10)

$$\begin{aligned} &L^2\begin{pmatrix} -xq \\ xr \end{pmatrix} = \begin{pmatrix} -2q_x - xq_{xx} + 2q\partial^{-1}qr + 2xq^2r \\ 2r_x + xr_{xx} - 2r\partial^{-1}qr - 2xq^2r \end{pmatrix}, \\ &\begin{pmatrix} -xq \\ xr \end{pmatrix}, \begin{pmatrix} -tq \\ tr \end{pmatrix} \end{aligned} \tag{21}$$

cannot be contained in (18). Due to appearance of the third term of (21), system (9) is a variable-coefficient system with not only space-dependent coefficients but also time-dependent coefficients. However, (17) has not such a term with time-dependent coefficients. In fact, the different selections for (12) and (14) lead to the difference between system (9) and (17).

The rest of the paper is organized as follows. In Section 2, we prove the Lax integrability of system (9) by generalizing AKNS spectral problem (4) and its evolution equation (5). In Section 3, system (9) is solved via the IST. As a result, the uniform formulae of exact solutions are obtained. In the special case of reflectionless potentials, the obtained exact solutions are reduced to n-soliton solutions. In Section 4, we conclude this paper.

2. Lax Integrability

Theorem 1. *Suppose that the function A in (5) has the form*

$$A = \partial^{-1}(r,q)\begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}\left[\sum_{n=0}^{2}(2ik)^n\right]x - \frac{1}{2}, \tag{22}$$

then the nonisospectral integrodifferential system (9) can be derived from (3) and thus system (9) is Lax integrable.

Proof. Firstly, by virtue of (3) equipped with the new spectral parameter k satisfying (12) we have

$$A_x = qC - rB - \frac{1}{2}\sum_{n=0}^{2}(2ik)^n, \tag{23}$$

$$q_t = B_x + 2ikB + 2qA, \tag{24}$$

$$r_t = C_x - 2ikC - 2rA. \tag{25}$$

Supposing that

$$A = \partial^{-1}(r,q)\begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}\left[\sum_{n=0}^{2}(2ik)^n\right]x - \frac{1}{2}, \tag{26}$$

from (24) and (25), we have

$$\begin{aligned}\begin{pmatrix} q \\ r \end{pmatrix}_t &= L\begin{pmatrix} -B \\ C \end{pmatrix} - 2ik\begin{pmatrix} -B \\ C \end{pmatrix} + \sum_{n=0}^{2}(2ik)^n\begin{pmatrix} -xq \\ xr \end{pmatrix} \\ &\quad + \begin{pmatrix} -tq \\ tr \end{pmatrix},\end{aligned} \tag{27}$$

by the use of (2).

We next suppose that

$$\begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{l=1}^{2}\begin{pmatrix} -b_l \\ c_l \end{pmatrix}(2ik)^{2-l} \tag{28}$$

and substitute (28) into (27). Then comparing the coefficients of $2ik$ in (27) yields

$$\begin{aligned}(2ik)^0:& \begin{pmatrix} q \\ r \end{pmatrix}_t = L\begin{pmatrix} -b_2 \\ c_2 \end{pmatrix} + \begin{pmatrix} -xq \\ xr \end{pmatrix} + \begin{pmatrix} -tq \\ tr \end{pmatrix},\\ (2ik)^1:& \begin{pmatrix} -b_2 \\ c_2 \end{pmatrix} = L\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} -xq \\ xr \end{pmatrix},\\ (2ik)^2:& \begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} -xq \\ xr \end{pmatrix},\end{aligned} \tag{29}$$

from which we derive (10). Finally, the substitution of (2) into (10), we arrive at the nonisospectral integrodifferential system (9). Thus, the proof is completed. □

3. Soliton Solutions

In this section, we first determine the time dependence of scattering data for the AKNS spectral problem (4) with the generalized time evolution equation (5) caused by (22). Based on the determined scattering data, we then construct exact solutions of nonisospectral integrodifferential system (9). We finally reduce the obtained exact solutions to soliton solutions and analyze the soliton dynamics.

3.1. The Time Dependence of Scattering Data

Theorem 2. *The scattering data* $\kappa_j(t)$, $\overline{\kappa}_m(t)$, $c_j(t)$, $\overline{c}_m(t)$, $R(t,k) = \beta(t,k)/\alpha(t,k)$, *and* $\overline{R}(t,k) = \overline{\beta}(t,k)/\overline{\alpha}(t,k)$ $(j = 1, 2, \ldots, n;\ m = 1, 2, \ldots, \overline{n})$ *for the generalized spectral problem (4) possess the following time dependence:*

$$\begin{aligned}\kappa_{jt}(t) &= -\frac{i}{2}\sum_{m=0}^{2}\left[2i\kappa_j(t)\right]^m,\\ c_j(t) &= c_j(0)\,e^{\int_0^t[1+2i\kappa_j(s)]\mathrm{d}s},\\ \alpha(t,k) &= \alpha(0,k),\\ \beta(t,k) &= \beta(0,k),\\ \overline{\kappa}_{mt}(t) &= -\frac{i}{2}\sum_{l=0}^{2}\left[2i\overline{\kappa}_j(t)\right]^l,\\ \overline{c}_m(t) &= \overline{c}_m(0)\,e^{-\int_0^t[1+2i\overline{\kappa}_m(s)]\mathrm{d}s},\\ \overline{\alpha}(t,k) &= \overline{\alpha}(0,k),\\ \overline{\beta}(t,k) &= \overline{\beta}(0,k),\end{aligned} \tag{30}$$

where $c_j(0)$, $\overline{c}_m(0)$, $R(0,k) = \beta(0,k)/\alpha(0,k)$, *and* $\overline{R}(0,k) = \overline{\beta}(0,k)/\overline{\alpha}(0,k)$ *are the scattering data of the generalized spectral problem (4) in the case of* $(q(0,x), r(0,x))^T$.

Proof. It is easy to see that if $\phi(x,k)$ is a solution of the generalized spectral problem (4) then $P(x,k) = \phi_t(x,k) - N\phi(x,k)$ is also a solution of generalized spectral problem (4). Therefore, $P(x,k)$ can be represented by $\phi(x,k)$ and $\widetilde{\phi}(x,k)$ which also satisfies the generalized spectral problem (4) but is independent of $\phi(x,k)$; that is, there exist two functions $\gamma(t,k)$ and $\tau(t,k)$ such that

$$\begin{aligned}\phi_t(x,k) - N\phi(x,k) = \gamma(t,k)\,\phi(x,k)\\ + \tau(t,k)\,\widetilde{\phi}(x,k).\end{aligned} \tag{31}$$

Firstly, we consider the discrete spectral $k = \kappa_j(\operatorname{Im}\kappa_j > 0)$. Since $\phi(x,\kappa_j)$ decays exponentially while $\widetilde{\phi}(x,k)$ must increase exponentially as $x \to +\infty$, we then have $\tau(t,k) = 0$. Thus, (31) is simplified as

$$\phi_t(x,\kappa_j) - N\phi(x,\kappa_j) = \gamma(t,\kappa_j)\,\phi(x,\kappa_j). \tag{32}$$

Left-multiplying (32) by the inner product $(\phi_2(x,\kappa_j), \phi_1(x,\kappa_j))$ yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\phi_1(x,\kappa_j)\,\phi_2(x,\kappa_j) - \left[C\phi_1^2(x,\kappa_j) + B\phi_2^2(x,\kappa_j)\right]\\ &\quad= 2\gamma(t,\kappa_j)\,\phi_1(x,\kappa_j)\,\phi_2(x,\kappa_j).\end{aligned} \tag{33}$$

Presuming $\phi(x,\kappa_j)$ to be the normalization eigenfunction and noting that $2\int_{-\infty}^{\infty} c_j^2\phi_1(x,\kappa_j)\phi_2(x,\kappa_j)\mathrm{d}x = 1$, we have

$$\gamma(t,\kappa_j) = -c_j^2\int_{-\infty}^{\infty}\left[C\phi_1^2(x,\kappa_j) + B\phi_2^2(x,\kappa_j)\right]\mathrm{d}x. \tag{34}$$

For convenience, we rewrite (34) as

$$\gamma(t,\kappa_j) = -c_j^2\left(\left(\phi_2^2(x,\kappa_j), \phi_1^2(x,\kappa_j)\right)^T, (B,C)^T\right), \tag{35}$$

where the following inner product had been used

$$\begin{aligned}&(f(x), g(x))\\ &\quad= \int_{-\infty}^{\infty}(f_1(x)\,g_1(x) + f_2(x)\,g_2(x))\,\mathrm{d}x\end{aligned} \tag{36}$$

for arbitrary two vectors $f(x) = (f_1(x), f_2(x))^T$ and $g(x) = (g_1(x), g_2(x))^T$.

Using (4), we have

$$\begin{aligned}\phi_{1x}(x,\kappa_j) + i\kappa_j\phi_1(x,\kappa_j) &= q(x)\,\phi_2(x,\kappa_j),\\ \phi_{2x}(x,\kappa_j) - i\kappa_j\phi_2(x,\kappa_j) &= r(x)\,\phi_1(x,\kappa_j)\end{aligned} \tag{37}$$

and hence obtain

$$\begin{aligned}&\left[\phi_1(x,\kappa_j)\,\phi_2(x,\kappa_j)\right]_x\\ &\quad= q(x)\,\phi_2^2(x,\kappa_j) + r(x)\,\phi_1^2(x,\kappa_j).\end{aligned} \tag{38}$$

Integrating (38) with respect to x from $-\infty$ to $+\infty$ yields

$$\begin{aligned}&\int_{-\infty}^{\infty}\left[q(x)\,\phi_2^2(x,\kappa_j) + r(x)\,\phi_1^2(x,\kappa_j)\right]\mathrm{d}x\\ &\quad= \int_{-\infty}^{\infty}\left[\phi_1(x,\kappa_j)\,\phi_2(x,\kappa_j)\right]_x\mathrm{d}x = 0.\end{aligned} \tag{39}$$

On the other hand, we rewrite (28) as

$$\begin{pmatrix} B \\ C \end{pmatrix} = \sum_{l=1}^{2}\sum_{s=1}^{l}\overline{L}^{s-1}\begin{pmatrix} xq \\ xr \end{pmatrix}(2ik)^{2-l}, \qquad \overline{L} = \sigma\partial - 2\begin{pmatrix} q \\ r \end{pmatrix}\partial^{-1}(-r,q) \tag{40}$$

and then obtain

$$\begin{aligned} \gamma(t,\kappa_j) &= -c_j^2\left(\left(\phi_2^2(x,\kappa_j),\phi_1^2(x,\kappa_j)\right)^T,(B,C)^T\right) \\ &= -c_j^2\left(\left(\phi_2^2(x,\kappa_j),\phi_1^2(x,\kappa_j)\right)^T,\sum_{l=1}^{2}\sum_{s=1}^{l}\overline{L}^{s-1}\begin{pmatrix} xq \\ xr \end{pmatrix} \cdot (2ik)^{2-l}\right) = -c_j^2\sum_{l=1}^{2}\sum_{s=1}^{l}\left(2i\kappa_j\right)^{2-l} \\ &\cdot\left(\left(\phi_2^2(x,\kappa_j),\phi_1^2(x,\kappa_j)\right)^T,\overline{L}^{l-1}\begin{pmatrix} q \\ r \end{pmatrix}\right) = \frac{1}{2} \cdot \sum_{s=1}^{2} s\left(2i\kappa_j\right)^{s-1}. \end{aligned} \tag{41}$$

Then (32) becomes

$$\begin{aligned} &\phi_t(x,\kappa_j) - N\phi(x,\kappa_j) \\ &= \frac{1}{2}\left[\sum_{s=1}^{2} s\left(2i\kappa_j\right)^{s-1}\right]\phi(x,\kappa_j). \end{aligned} \tag{42}$$

Noting that

$$\begin{aligned} &N \longrightarrow \begin{pmatrix} -\frac{1}{2}\left[\sum_{n=0}^{2}\left(2i\kappa_j\right)^n\right]x - \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\left[\sum_{n=0}^{2}\left(2i\kappa_j\right)^n\right]x + \frac{1}{2} \end{pmatrix}, \\ &\phi(x,\kappa_j) \longrightarrow c_j\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{i\kappa_j x}, \\ &\phi_t(x,\kappa_j) \longrightarrow c_{jt}\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{i\kappa_j x} + xc_j\kappa_{jt}\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{i\kappa_j x}, \end{aligned} \tag{43}$$

as $x \longrightarrow +\infty$, then from (42)-(43) we have

$$\begin{aligned} \kappa_{jt} &= -\frac{i}{2}\sum_{n=0}^{2}\left(2i\kappa_j\right)^n, \\ c_{jt} &= c_j\left[\frac{1}{2}\sum_{s=1}^{2} s\left(2i\kappa_j\right)^{s-1} + \frac{1}{2}\right]. \end{aligned} \tag{44}$$

In a similar way, we obtain

$$\begin{aligned} \overline{\kappa}_{jt} &= -\frac{i}{2}\sum_{n=0}^{2}\left(2i\overline{\kappa}_j\right)^n, \\ \overline{c}_{jt} &= -\overline{c}_j\left[\frac{1}{2}\sum_{s=1}^{2} s\left(2i\overline{\kappa}_j\right)^{s-1} + \frac{1}{2}\right]. \end{aligned} \tag{45}$$

Secondly, we consider k as a real continuous spectral and take a solution $\varphi(x,k)$ of the generalized spectral problem (4), then the solution of the generalized spectral problem (4)

$$P_1(x,k) = \varphi_t(x,k) - N\varphi(x,k) \tag{46}$$

can be represented linearly by $\varphi(x,k)$ and $\overline{\varphi}(x,k)$ which also satisfies the generalized spectral problem (4) but is independent of $\varphi(x,k)$, that is, there exist two functions $\omega(t,k)$ and $\vartheta(t,k)$ such that

$$\begin{aligned} \varphi_t(x,k) - N\varphi(x,k) &= \omega(t,k)\varphi(x,k) \\ &\quad + \vartheta(t,k)\overline{\varphi}(x,k). \end{aligned} \tag{47}$$

Using the asymptotical properties

$$\begin{aligned} \varphi_t(x,k) &\longrightarrow -ik_t x\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-ikx}, \\ \varphi(x,k) &\longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-ikx}, \\ \overline{\varphi}(x,k) &\longrightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{ikx}, \end{aligned} \tag{48}$$

as $x \rightarrow -\infty$, from (47) we obtain

$$\begin{aligned} \vartheta(t,k) &= 0, \\ \omega(t,k) &= 0. \end{aligned} \tag{49}$$

Substituting the Jost relationship $\varphi(x,k) = \alpha(t,k)\overline{\phi}(x,k) + \beta(t,k)\phi(x,k)$ into (47) yields

$$\begin{aligned} &\left[\alpha(t,k)\overline{\phi}(x,k) + \beta(t,k)\phi(x,k)\right]_t \\ &\quad - N\left[\alpha(t,k)\overline{\phi}(x,k) + \beta(t,k)\phi(x,k)\right] = 0. \end{aligned} \tag{50}$$

Letting $x \rightarrow -\infty$ and using

$$\begin{aligned} \phi(x,k) &\longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{ikx}, \\ \overline{\phi}(x,k) &\longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-ikx}, \end{aligned} \tag{51}$$

from (50) we derive

$$\begin{aligned} \frac{\mathrm{d}\alpha(t,k)}{\mathrm{d}t} &= 0, \\ \frac{\mathrm{d}\beta(t,k)}{\mathrm{d}t} &= 0. \end{aligned} \tag{52}$$

Similarly, we have

$$\frac{d\overline{\alpha}(t,k)}{dt}=0,\quad \frac{d\overline{\beta}(t,k)}{dt}=0. \tag{53}$$

Finally, solving (44), (45), (52), and (53) yields (30). We therefore finish the proof. □

3.2. Exact Solutions and Soliton Solutions. According to Theorem 1 and the results in [20], we have the following Theorem 3.

Theorem 3. *Given the scattering data for the generalized spectral problem (4), the nonisospectral integrodifferential system (9) has exact solutions as follows:*

$$q(x,t)=-2K_1(t,x,x),\quad r(x,t)=\frac{K_{2x}(t,x,x)}{K_1(t,x,x)}, \tag{54}$$

where $K(t,x,y)=(K_1(t,x,y),K_2(t,x,y))^T$ *satisfies the Gel'fand-Levitan-Marchenko (GLM) integral equation:*

$$\begin{aligned}&K(t,x,y)-\begin{pmatrix}0\\1\end{pmatrix}\overline{F}(t,x+y)\\&\quad+\begin{pmatrix}0\\1\end{pmatrix}\int_x^\infty F(t,z+x)\overline{F}(t,z+y)\,dz\\&\quad+\int_x^\infty K(t,x,s)\int_x^\infty F(t,z+s)\overline{F}(t,z+y)\,dz\,ds\\&=0,\end{aligned} \tag{55}$$

with

$$\begin{aligned}F(t,x)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}R(t,k)e^{ikx}dk+\sum_{j=1}^{n}c_j^2e^{i\kappa_jx},\\ \overline{F}(t,x)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{R}(t,k)e^{-ikx}dk-\sum_{j=1}^{\overline{n}}\overline{c}_j^2e^{-i\overline{\kappa}_jx}.\end{aligned} \tag{56}$$

In order to give explicit form of solutions (54), we consider $R(t,k)=\overline{R}(t,k)=0$. In this reflectionless potentials case, the GLM integral equation (55) can be solved exactly. For convenience, we use $K(t,x,y)=(K_1(t,x,y),K_2(t,x,y))^T$ to rewrite (55) as

$$\begin{aligned}&K_1(t,x,y)-\overline{F}_d(t,x+y)+\int_x^\infty K_1(t,x,s)\\&\quad\cdot\int_x^\infty F_d(t,z+s)\overline{F}_d(t,z+y)\,dz\,ds=0,\\&K_2(t,x,y)+\int_x^\infty F_d(t,z+x)\overline{F}_d(t,z+y)\,dz\\&\quad+\int_x^\infty K_2(t,x,s)\\&\quad\cdot\int_x^\infty F_d(t,z+s)\overline{F}_d(t,z+y)\,dz\,ds=0.\end{aligned} \tag{57}$$

Using (56), we can get

$$\begin{aligned}&\int_x^\infty F_d(t,s+z)\overline{F}_d(t,z+y)\,dz\\&\quad=-\sum_{j=1}^{n}\sum_{m=1}^{\overline{n}}\frac{ic_j^2(t)\overline{c}_m^2(t)}{\kappa_j-\overline{\kappa}_m}e^{i\kappa_j(x+s)-i\overline{\kappa}_m(x+y)}.\end{aligned} \tag{58}$$

Supposing that

$$\begin{aligned}K_1(x,y,t)&=\sum_{p=1}^{\overline{n}}\overline{c}_p(t)\,g_p(t,x)\,e^{-i\overline{\kappa}_py},\\ K_2(x,y,t)&=\sum_{p=1}^{\overline{n}}\overline{c}_p(t)\,h_p(t,x)\,e^{-i\overline{\kappa}_py}\end{aligned} \tag{59}$$

and substituting (59) into (57) yield

$$\begin{aligned}&g_m(t,x)+\overline{c}_m(t)e^{-i\overline{\kappa}_mx}\\&\quad+\sum_{j=1}^{n}\sum_{p=1}^{\overline{n}}\frac{c_j^2(t)\overline{c}_m(t)\overline{c}_p(t)}{(\kappa_j-\overline{\kappa}_m)(\kappa_j-\overline{\kappa}_p)}e^{i(2\kappa_j-\overline{\kappa}_m-\overline{\kappa}_p)x}g_p(x,t)\\&\quad=0,\\&h_m(x,t)-\sum_{j=1}^{n}\frac{1}{(\kappa_j-\overline{\kappa}_m)}c_j^2(t)\overline{c}_m(t)e^{i(2\kappa_j-\overline{\kappa}_m)x}\\&\quad+\sum_{j=1}^{n}\sum_{p=1}^{\overline{n}}\frac{c_j^2(t)\overline{c}_m(t)\overline{c}_p(t)}{(\kappa_j-\overline{\kappa}_m)(\kappa_j-\overline{\kappa}_p)}e^{i(2\kappa_j-\overline{\kappa}_m-\overline{\kappa}_p)x}h_p(x,t)\\&\quad=0,\quad(m=1,2,\ldots,\overline{n}).\end{aligned} \tag{60}$$

Introducing the vectors

$$\begin{aligned}g(t,x)&=(g_1(t,x),g_2(t,x),\ldots,g_{\overline{n}}(t,x))^T,\\ h(t,x)&=(h_1(t,x),h_2(t,x),\ldots,h_{\overline{n}}(t,x))^T,\\ \Lambda&=\left(c_1(t)e^{i\kappa_1x},c_2(t)e^{i\kappa_2x},\ldots,c_n(t)e^{i\kappa_nx}\right)^T,\\ \overline{\Lambda}&=\left(\overline{c}_1(t)e^{-i\overline{\kappa}_1x},\overline{c}_2(t)e^{-i\overline{\kappa}_2x},\ldots,\overline{c}_n(t)e^{-i\overline{\kappa}_nx}\right)^T,\end{aligned} \tag{61}$$

we can write (60) in the matrix forms

$$W(t,x)\,g(t,x) = -\overline{\Lambda}(t,x), \qquad W(t,x)\,h(t,x) = iP(t,x)\,\Lambda(t,x), \tag{62}$$

where

$$W(t,x) = E + P(t,x)\,P^{T}(t,x), \qquad P(t,x) = \left(\frac{c_j(t)\,\overline{c}_m(t)}{\kappa_j - \overline{\kappa}_m} e^{i(\kappa_j - \overline{\kappa}_m)x}\right)_{\overline{n}\times n}, \tag{63}$$

and E is a $\overline{n}\times\overline{n}$ unit matrix.

Supposing $W^{-1}(t,x)$ exists, then we have

$$g(t,x) = -W^{-1}(t,x)\,\overline{\Lambda}(t,x), \qquad h(t,x) = iW^{-1}(t,x)\,P(t,x)\,\Lambda(t,x). \tag{64}$$

Substituting (64) into (59) we have

$$K_1(x,y,t) = -\operatorname{tr}\left(W^{-1}(t,x)\,\overline{\Lambda}(t,x)\,\overline{\Lambda}^{T}(t,y)\right), \qquad K_2(x,y,t) = i\operatorname{tr}\left(W^{-1}(t,x)\,E(t,x)\,\Lambda(t,x)\,\overline{\Lambda}^{T}(t,y)\right), \tag{65}$$

where tr(·) means the trace of a given matrix.

Substituting (65) into (54), we obtain the following n-soliton solutions of the nonisospectral integrodifferential system (9):

$$q(x,t) = 2\operatorname{tr}\left(W^{-1}(t,x)\,\overline{\Lambda}(t,x)\,\overline{\Lambda}^{T}(t,x)\right), \qquad r(x,t) = -\frac{(\mathrm{d}/\mathrm{d}x)\operatorname{tr}\left(W^{-1}(t,x)\,E(t,x)\,(\mathrm{d}/\mathrm{d}x)\,E^{T}(t,x)\right)}{\operatorname{tr}\left(W^{-1}(t,x)\,\overline{\Lambda}(t,x)\,\overline{\Lambda}^{T}(t,x)\right)}. \tag{66}$$

Particularly, when $n = \overline{n} = 1$, (66) give the one-soliton solutions:

$$q = \frac{2\overline{c}_1^{2}(0)\,e^{-2i\overline{\kappa}_1 x - 2\int_0^t [1+2i\overline{\kappa}_1(s)]\mathrm{d}s}}{1 + \left(c_1^2(0)\,\overline{c}_1^2(0)/(\kappa_1 - \overline{\kappa}_1)^2\right) e^{2i(\kappa_1-\overline{\kappa}_1)x + 4i\int_0^t[\kappa_1(s)-\overline{\kappa}_1(s)]\mathrm{d}s}}, \tag{67}$$

$$r = \frac{2c_1^{2}(0)\,e^{2i\kappa_1 x + 2\int_0^t [1+2i\kappa_1(s)]\mathrm{d}s}}{1 + \left(c_1^2(0)\,\overline{c}_1^2(0)/(\kappa_1 - \overline{\kappa}_1)^2\right) e^{2i(\kappa_1-\overline{\kappa}_1)x + 4i\int_0^t[\kappa_1(s)-\overline{\kappa}_1(s)]\mathrm{d}s}}, \tag{68}$$

where κ_1 and $\overline{\kappa}_1$ are determined by the Riccati equations

$$\kappa_{1t} = -\frac{i}{2}\sum_{n=0}^{2}(2i\kappa_1)^n, \qquad \overline{\kappa}_{1t} = -\frac{i}{2}\sum_{n=0}^{2}(2i\overline{\kappa}_1)^n. \tag{69}$$

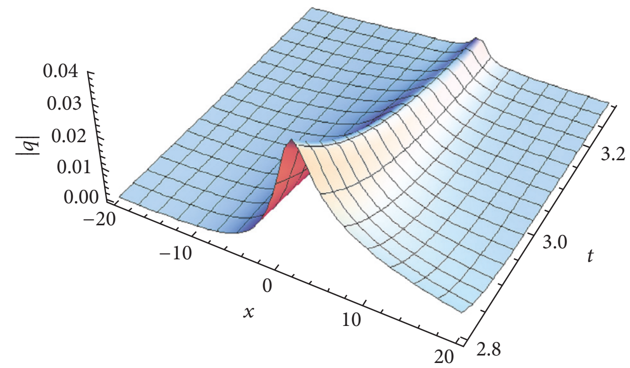

FIGURE 1: Local spatial structure of one-soliton solution (67) with κ_1 and $\overline{\kappa}_1$ satisfying (69).

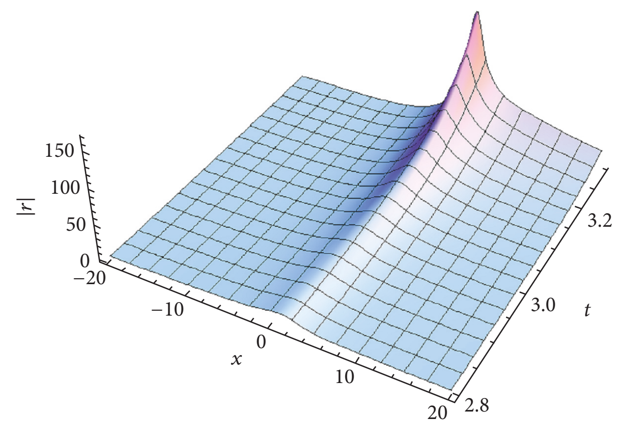

FIGURE 2: Local spatial structure of one-soliton solution (68).

3.3. Soliton Dynamics. In this part, we further investigate the soliton dynamics of system (9) by means of one-soliton solutions and two-soliton solutions. To determine κ_1 and $\overline{\kappa}_1$ of (69), we employ Zhang et al.'s direct algorithm [35] of exp-function method and gain two special solutions of (69):

$$\kappa_1 = \frac{i}{4}\left(1 + i\sqrt{3}\right) + \frac{\sqrt{3}}{2\left[1 + \left(\left(i + \sqrt{3} - 4\kappa_1(0)\right)/\left(-i + \sqrt{3} + 4\kappa_1(0)\right)\right)e^{i\sqrt{3}t}\right]}, \tag{70}$$

$$\overline{\kappa}_1 = \frac{i}{4}\left(1 + i\sqrt{3}\right) + \frac{\sqrt{3}}{2\left[1 + \left(\left(i + \sqrt{3} - 4\overline{\kappa}_1(0)\right)/\left(-i + \sqrt{3} + 4\overline{\kappa}_1(0)\right)\right)e^{i\sqrt{3}t}\right]}. \tag{71}$$

In Figures 1 and 2, two local spatial structures of one-soliton solutions (67) and (68) are shown by selecting the parameters as $\kappa_1(0) = 1$, $\overline{\kappa}_1(0) = -0.5$, $c_1(0) = 1$, and $\overline{c}_1(0) = 0.2$. We can see from Figures 1 and 2 that the local spatial structures of one-soliton solutions (67) and (68) possess the bell-shaped characteristics. The dynamical evolutions of two-soliton solutions determined by (66) are shown in Figures 3 and 4, where the parameters are selected as $\kappa_1(0) = 1$, $\overline{\kappa}_1(0) = 0.5$, $\kappa_2(0) = -0.3$, $\overline{\kappa}_2(0) = -1.5$, $c_1(0) = -0.01$,

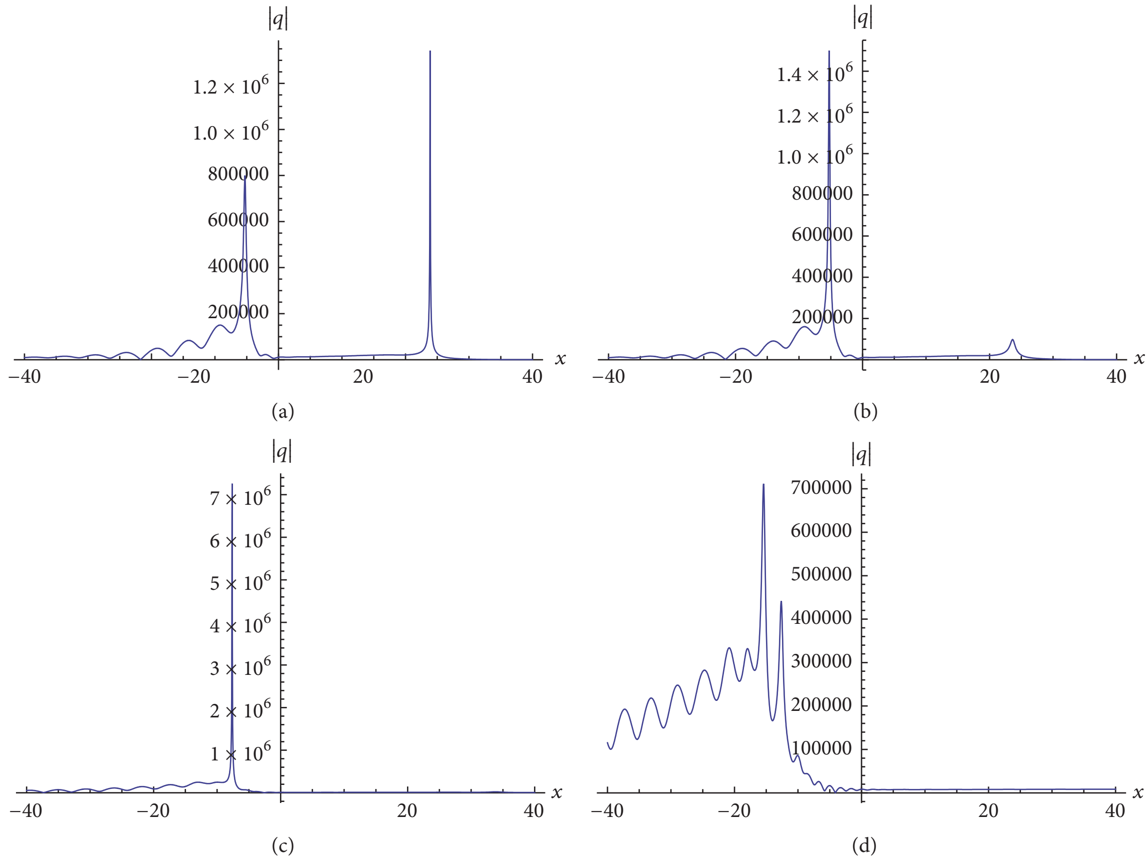

Figure 3: Dynamical evolutions of two-soliton solution determined by (66) at different times: (a) $t = -11.112$, (b) $t = -11.104$, (c) $t = -11$, and (d) $t = -10.94$.

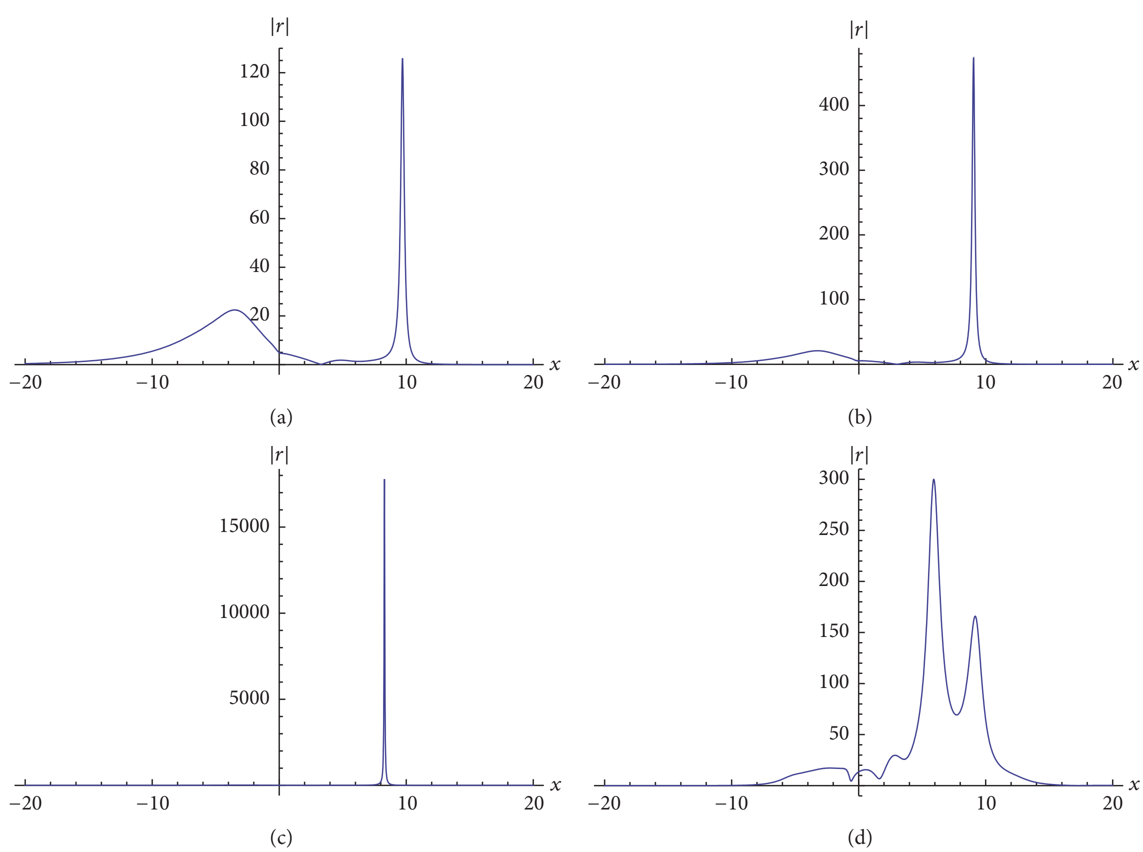

Figure 4: Dynamical evolutions of two-soliton solution determined by (66) at different times: (a) $t = 2.8$, (b) $t = 2.9$, (c) $t = 3$, and (d) $t = 3.3$.

$\bar{c}_1(0) = 0.25$, $c_2(0) = 0.5$, and $\bar{c}_2(0) = -0.1$, respectively. Figures 3 and 4 show that the inelastic scatterings can happen between two-soliton solutions determined by (66).

4. Conclusions and Discussions

In summary, we have verified Lax integrability of the new and more general nonisospectral integrodifferential system (9). This is due to the generalizations on AKNS spectral problem (4) and its time evolution equation (5) by embedding a new spectral parameter. To exactly solve the nonisospectral integrodifferential system (9), the IST is employed. As a result, exact solutions (54) are obtained. In the case of reflectionless potentials, the obtained exact solutions (54) are reduced to n-soliton solutions (66). When $n = 1$ and $n = 2$, the characteristics of soliton dynamics of one-soliton solutions and two-soliton solutions are analyzed with the help of figures. To the best of our knowledge, the nonisospectral integrodifferential system (9), the exact solutions (54), and the n-soliton solutions (66) have not been reported in literatures. How to construct other nonisospectral integrodifferential systems and their soliton solutions in the framework of IST method is worthy of study. This is our task in the future.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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