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Research Article

Robust H_∞ Fuzzy Control for Nonlinear Discrete-Time Stochastic Systems with Markovian Jump and Parametric Uncertainties

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The paper mainly investigates the H_∞ fuzzy control problem for a class of nonlinear discrete-time stochastic systems with Markovian jump and parametric uncertainties. The class of systems is modeled by a state space Takagi-Sugeno (T-S) fuzzy model that has linear nominal parts and norm-bounded parameter uncertainties in the state and output equations. An H_∞ control design method is developed by using the Lyapunov function. The decoupling technique makes the Lyapunov matrices and the system matrices separated, which makes the control design feasible. Then, some strict linear matrix inequalities are derived on robust H_∞ norm conditions in which both robust stability and H_∞ performance are required to be achieved. Finally, a computer-simulated truck-trailer example is given to verify the feasibility and effectiveness of the proposed design method.

1. Introduction

Over the past decade, there has been a rapidly growing interest in control and filtering of nonlinear systems, and there have been many successful applications [1–4]. In [1], based on the sum of squares approach, sufficient conditions for the existence of a nonlinear state feedback controller for polynomial discrete-time systems are given in terms of solvability of polynomial matrix inequalities. Reference [3] presents a stochastic distribution control algorithm; an optimal control law is then obtained using the penalty function method. Despite the success, it has become evident that many basic issues remain to be addressed. In particular, the control technique based on the so-called Takagi-Sugeno (T-S) fuzzy model [5] has attracted a great deal of attention (see [6–12]). This is because it is regarded as a powerful solution to bridging the gap between the fruitful linear control and the fuzzy logic control targeting complex nonlinear systems. The common practice of the technique is as follows. First, the nonlinear plant is represented by the T-S fuzzy model. This fuzzy model is described by a set of fuzzy IF-THEN rules which correspond to local linear input-output relations of the system, respectively. The overall model of the system is achieved by fuzzy "blending" of these fuzzy models. Then, based on this fuzzy model, a control design is carried out based on the fuzzy models via the so-called parallel distributed compensation (PDC) scheme. The idea is that a linear feedback control is designed for each local linear model. The resulting overall controller, which is, in general, nonlinear, is again a fuzzy blending of each individual linear controller. Their works introduce the model-based analysis methods into fuzzy logic control.

In recent years, the stability issue and robust performance of T-S fuzzy control systems have been discussed in an extensive literature. Since the pioneering work on the so-called H_∞ optimal control theory, there has been a dramatic progress in the H_∞ control theory. Recently, the problem of nonlinear H_∞ control was intensively studied (see [5, 7, 8, 13]). The design of H_∞ controller for fuzzy dynamic systems was presented in the paper [5]. Reference [4] introduces a new class of discrete-time networked nonlinear systems with mixed random delays and packet dropouts, and sufficient conditions for the existence of an admissible filter are established, which ensure the asymptotical stability as well as a prescribed H_∞ performance. The authors in [13] analyzed the H_∞ control design problem systematically for a class of nonlinear stochastic active fault-tolerant control systems with Markovian parameters. H_∞ controller design which was considered in [7] noted that most of the aforementioned research efforts focused on the use of single quadratic Lyapunov function, which tended to give more conservative conditions. This is because with the use of parameter-dependent or basis-dependent Lyapunov function less conservative control results can be obtained than with the use of single Lyapunov quadratic function. More recently, there were many results on stability analysis and control synthesis of discrete-time uncertain systems based on parameter-dependent Lyapunov functions [9] or basis-dependent Lyapunov function (see [8, 14]). It is shown that with the use of Lyapunov function well-pleasing control results can be obtained.

In practice, a lot of physical systems have variable structures subject to random changes. These changes may result from abrupt phenomena such as random failures and repairs of the components, changes in the interconnections of subsystems, and sudden environment changes. The so-called Markovian jump systems (MJSS) are special class of hybrid systems. As one of the most basic dynamics models, Markov jump nonlinear system can be used to represent these random failure processes in manufacturing and some investment portfolio models. In the MJSS, the random jumps in system parameters are governed by a Markov process which takes values in a finite set. The class of systems may represent a large variety of processes including those in the production systems, fault-tolerant systems, communication systems, and economic systems [15]. In the past two decades, many important issues of MJSS were researched extensively, such as the controllability and observability [16], stability and stabilization (see [17–22]), H_2 [23] and H_∞ performance [24–27], and robustness [28, 29]. To the best of our knowledge, despite these efforts, very few results are available for the control design for nonlinear MJSS. In this study, a mode-dependent control design is proposed to achieve robust stability of uncertain discrete-time nonlinear MJSS via fuzzy control.

In the paper, robust H_∞ control is investigated for nonlinear discrete-time stochastic MJSS with parametric uncertainties. Under our proposed fuzzy rules, the nonlinear discrete-time stochastic MJSS could be translated into a class of equivalent T-S fuzzy models with parametric uncertainties. The uncertainty could be translated into a linear fractional form, which includes the norm-bounded uncertainty as a special case and can describe a kind of rational nonlinearities. With the PDC scheme, the control law is designed to make the closed-loop system with H_∞ norm bound γ stable. Besides, some matrix variables and the Lyapunov function for robust stabilization with l_2-norm bound for the fuzzy discrete-time stochastic MJSS are given. It is shown that the solution of the control design problem can be obtained by solving a class of LMIs.

The paper is organized as follows. Section 2 discusses the T-S fuzzy models. And definitions and preliminary results are given for uncertain nonlinear discrete-time MJSS. Section 3 gives the analysis results of robust stability with H_∞ performance, and the results are employed in the following to develop an H_∞ control design. In Section 4, a numerical simulation example is proposed to illustrate the effectiveness of the approach. Finally, the paper is concluded in Section 5.

For convenience, the following basic notations are adopted throughout the paper. $\mathscr{R}^n$ denotes the n-dimensional real space, and $\mathscr{R}^{n\times m}$ denotes the set of all real $n\times m$ matrices. U' indicates the transpose of matrix U and $U \geq 0$ ($U > 0$) represents a nonnegative definite (positive definite) matrix. Similarly, $U \leq 0$ ($U < 0$) represents a nonpositive definite matrix (negative definite). $l_2[0,\infty)$ refers to the space of square summable infinite vector sequences. $\|\cdot\|_2$ stands for the usual $l_2[0,\infty)$ norm. $E(\cdot)$ represents the mathematical expectation.

2. Problem Formulation and Preliminaries

Consider the following nonlinear MJSS:

$$\begin{aligned} x(k+1) &= f(x(k),\theta_k) + g(x(k),\theta_k)u(k) \\ &\quad + k(x(k),\theta_k)v(k) + l(x(k),\theta_k)w(k), \\ y(k) &= m(x(k),\theta_k) + t(x(k),\theta_k)u(k) \\ &\quad + n(x(k),\theta_k)v(k). \end{aligned} \tag{1}$$

Assume that $f(x(k),\theta_k)$, $g(x(k),\theta_k)$, $k(x(k),\theta_k)$, $l(x(k),\theta_k)$, $m(x(k),\theta_k)$, $t(x(k),\theta_k)$, and $n(x(k),\theta_k)$ are Borel measurable on $\mathscr{R}^n$. And $x(k) \in \mathscr{R}^n$ is the system state, $u(k) \in \mathscr{R}^m$ and $v(k) \in \mathscr{R}^p$ represent the system control inputs and disturbance signal, and $y(k) \in \mathscr{R}^q$ is system output. $w(k)$ is a sequence of real random variables defined on a complete probability space $(\Omega, \mathscr{F}_k, P)$, which is wide sense stationary, second-order processes with $E[w(k)] = 0$ and $E[w(i)w(j)] = \delta_{ij}$, where δ_{ij} refers to a Kronecker function; that is, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. $\{\theta_k; k \geq 0\}$ is a measurable Markov chain taking values in a finite set $\mathscr{X} = \{1, 2, \ldots, N\}$, with transition probability matrix $\mathbb{P} = [p_{\alpha\beta}]$, where

$$p_{\alpha\beta} := P(\theta_{k+1} = \beta \mid \theta_k = \alpha), \quad \forall \alpha, \beta \in \mathscr{X},\ k \geq 0. \tag{2}$$

The paper considers the nonlinear discrete-time MJSS which can be described by the following T-S fuzzy model with uncertainties.

Rule i. If $z_1(k)$ is F_1^i, $z_2(k)$ is $F_2^i, \ldots,$ and $z_n(k)$ is F_n^i, then

$$\begin{aligned} x(k+1) &= \mathscr{A}\mathscr{A}_{i,\theta_k}x(k) + \mathscr{B}_{i,\theta_k}u(k) + \mathscr{C}_{i,\theta_k}v(k) \\ &\quad + \mathscr{D}_{i,\theta_k}x(k)w(k), \\ y(k) &= \mathscr{M}_{i,\theta_k}x(k) + \mathscr{T}_{i,\theta_k}u(k) + \mathscr{N}_{i,\theta_k}v(k). \end{aligned} \tag{3}$$

The F_j^i are fuzzy sets; r is the number of IF-THEN rules. Moreover, $\mathscr{A}_{i,\theta_k}$, $\mathscr{B}_{i,\theta_k}$, $\mathscr{C}_{i,\theta_k}$, $\mathscr{D}_{i,\theta_k}$, $\mathscr{M}_{i,\theta_k}$, $\mathscr{T}_{i,\theta_k}$, and $\mathscr{N}_{i,\theta_k}$ are system matrices with parametric uncertainties. Besides, $z_1(k), z_2(k), \ldots, z_n(k)$ are the premise variables of the fuzzy modes and they are the functions of state variables.

Following the PDC scheme, we consider a state feedback fuzzy controller which shares the same structure of the above T-S fuzzy system, as follows.

Rule j. IF $z_1(k)$ is F_1^j, $z_2(k)$ is $F_2^j, \ldots,$ and $z_n(k)$ is F_n^j, then

$$u(k) = K_{j,\theta_k} x(k), \tag{4}$$

where K_{j,θ_k} are the local feedback gain matrices.

And the fuzzy basis functions are given by

$$h_i[z(k)] = \frac{\prod_{j=1}^{n} \mu_{ij}[z_j(k)]}{\sum_{l=1}^{r} \prod_{j=1}^{n} \mu_{ij}[z_j(k)]}, \quad i = 1, 2, \ldots, r, \tag{5}$$

where $\mu_{ij}[z_j(k)]$ is the grade of membership of $z_j(k)$ in F_j^i. By definition, the fuzzy basis functions satisfy

$$\begin{gathered} h_i[z(k)] \geq 0, \quad i = 1, 2, \ldots, r, \\ \sum_{i=1}^{r} h_i[z(k)] = 1. \end{gathered} \tag{6}$$

A more compact presentation of the discrete-time T-S fuzzy model is given by

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r} h_i(z(k)) \left(\mathscr{A}_{i,\theta_k} x(k) + \mathscr{B}_{i,\theta_k} u(k) + \mathscr{C}_{i,\theta_k} v(k) \right. \\ &\qquad \left. + \mathscr{D}_{i,\theta_k} x(k) w(k)\right), \end{aligned} \tag{7}$$

$$\begin{aligned} y(k) = \sum_{i=1}^{r} h_i(z(k)) \\ \times \left(\mathscr{M}_{i,\theta_k} x(k) + \mathscr{T}_{i,\theta_k} u(k) + \mathscr{N}_{i,\theta_k} v(k)\right), \end{aligned} \tag{8}$$

where

$$\begin{aligned} \mathscr{A}_{i,\theta_k} &= A_{i,\theta_k} + \Delta A_{i,\theta_k}, \\ \mathscr{B}_{i,\theta_k} &= B_{i,\theta_k} + \Delta B_{i,\theta_k}, \\ \mathscr{C}_{i,\theta_k} &= C_{i,\theta_k} + \Delta C_{i,\theta_k}, \\ \mathscr{D}_{i,\theta_k} &= D_{i,\theta_k}, \\ \mathscr{M}_{i,\theta_k} &= M_{i,\theta_k} + \Delta M_{i,\theta_k}, \\ \mathscr{T}_{i,\theta_k} &= T_{i,\theta_k} + \Delta T_{i,\theta_k}, \\ \mathscr{N}_{i,\theta_k} &= N_{i,\theta_k} + \Delta N_{i,\theta_k}. \end{aligned} \tag{9}$$

And $A_{i,\theta_k}, B_{i,\theta_k}, C_{i,\theta_k}, D_{i,\theta_k}, M_{i,\theta_k}, T_{i,\theta_k}$, and N_{i,θ_k} are assumed to be deterministic matrices with appropriate dimension; $\Delta A_{i,\theta_k}, \Delta B_{i,\theta_k}, \Delta C_{i,\theta_k}, \Delta M_{i,\theta_k}, \Delta T_{i,\theta_k}$, and $\Delta N_{i,\theta_k}$ are unknown matrices which represent the time-varying parameter uncertainties and are assumed to be of the form

$$\begin{gathered} \begin{bmatrix} \Delta A_{i,\theta_k} & \Delta B_{i,\theta_k} & \Delta C_{i,\theta_k} \\ \Delta M_{i,\theta_k} & \Delta T_{i,\theta_k} & \Delta N_{i,\theta_k} \end{bmatrix} = \begin{bmatrix} E_{1i} \\ E_{2i} \end{bmatrix} \Delta \begin{bmatrix} H_{1i} & H_{2i} & H_{3i} \end{bmatrix}, \\ \Delta = \left[I - F_{i,\theta_k} J\right]^{-1} F_{i,\theta_k}, \end{gathered} \tag{10}$$

where $E_{1i}, H_{1i}, E_{2i}, H_{2i}, H_{3i}$, and J are known real constant matrices with appropriate dimension and unknown nonlinear time-varying matrix $F_{i,\theta_k} \in \mathscr{R}^{m\times n}$ satisfying $F_{i,\theta_k} F'_{i,\theta_k} \leq I$. To guarantee that the matrix $I - F_{i,\theta_k} J$ is invertible for all admissible F_{i,θ_k}, it is necessary that $I - JJ' > 0$.

The following definition and lemmas will be used later.

Definition 1. The discrete-time unforced uncertain fuzzy system is said to be robust stable with H_∞ norm bound γ if it is stable with H_∞ norm bound γ for all uncertainly admissible F_{i,θ_k}. For a given control law (4) and a prescribed level of disturbance attenuation $\gamma > 0$ to be achieved, the discrete-time fuzzy system (3) is said to be stabilizable with H_∞ norm bound γ if for all $v(k) \in l_2[0, \infty)$, $v(k) \neq 0$, the closed-loop system (7)-(8) is asymptotically stable and the response $\{y(k)\}$ of the system under the zero initial condition $(x(0) = x_0 = 0)$ satisfies

$$\|y(k)\|_2 < \gamma \|v(k)\|_2. \tag{11}$$

Lemma 2. *It is supposed that* $u(k) = 0$, $v(k) = 0$, *and* $\Delta = 0$, *so the discrete-time MJSS becomes*

$$x(k+1) = \sum_{i=1}^{r} h_i(z(k)) \left(A_{i,\theta_k} x(k) + D_{i,\theta_k} x(k) w(k)\right). \tag{12}$$

The system (12) *is globally asymptotically stable if there exists a symmetric piecewise matrix* $P_{i,\theta_k} > 0$ *such that*

$$A'_{i,\theta_k} \widetilde{P}_{j,\theta_k} A_{i,\theta_k} + D'_{i,\theta_k} \widetilde{P}_{j,\theta_k} D_{i,\theta_k} - P_{i,\theta_k} < 0, \tag{13}$$

where $\widetilde{P}_{j,\theta_k} = P_{j,\theta_{k+1}} = \sum_{\theta_k=1}^{N} p_{\theta_k \theta_{k+1}} P_{i,\theta_k}$.

Lemma 3. *Given a matrix* A_{i,θ_k}, D_{i,θ_k}, *suppose* $P_{i,\theta_k} > 0$, $\widetilde{P}_{j,\theta_k} > 0$, $P_{l,\theta_k} > 0$. *If*

$$\begin{gathered} A'_{i,\theta_k} \widetilde{P}_{j,\theta_k} A_{i,\theta_k} + D'_{i,\theta_k} \widetilde{P}_{j,\theta_k} D_{i,\theta_k} - P_{i,\theta_k} < 0, \\ A'_{l,\theta_k} \widetilde{P}_{j,\theta_k} A_{l,\theta_k} + D'_{l,\theta_k} \widetilde{P}_{j,\theta_k} D_{l,\theta_k} - P_{l,\theta_k} < 0, \end{gathered} \tag{14}$$

then

$$\begin{aligned} A'_{i,\theta_k} \widetilde{P}_{j,\theta_k} A_{l,\theta_k} + A'_{l,\theta_k} \widetilde{P}_{j,\theta_k} A_{i,\theta_k} + D'_{i,\theta_k} \widetilde{P}_{j,\theta_k} D_{l,\theta_k} \\ + D'_{l,\theta_k} \widetilde{P}_{j,\theta_k} D_{i,\theta_k} - P_{i,\theta_k} - P_{l,\theta_k} < 0. \end{aligned} \tag{15}$$

Proof. It is noted that

$$\begin{aligned} \widetilde{P}_{j,\theta_k} > 0 \Longrightarrow &\left(A_{i,\theta_k} - A_{l,\theta_k}\right)' \widetilde{P}_{j,\theta_k} \left(A_{i,\theta_k} - A_{l,\theta_k}\right) \\ &+ \left(D_{i,\theta_k} - D_{l,\theta_k}\right)' \widetilde{P}_{j,\theta_k} \left(D_{i,\theta_k} - D_{l,\theta_k}\right) \geq 0, \end{aligned} \tag{16}$$

which gives

$$\begin{aligned} &A'_{i,\theta_k}\widetilde{P}_{j,\theta_k}A_{l,\theta_k} + A'_{l,\theta_k}\widetilde{P}_{j,\theta_k}A_{i,\theta_k} + D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{l,\theta_k} \\ &\quad + D'_{l,\theta_k}\widetilde{P}_{j,\theta_k}D_{i,\theta_k} \\ &\leq A'_{i,\theta_k}\widetilde{P}_{j,\theta_k}A_{i,\theta_k} + A'_{l,\theta_k}\widetilde{P}_{j,\theta_k}A_{l,\theta_k} \\ &\quad + D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{i,\theta_k} + D'_{l,\theta_k}\widetilde{P}_{j,\theta_k}D_{l,\theta_k}. \end{aligned} \tag{17}$$

Therefore, the result of Lemma 3 can be easily proven. □

Lemma 4 (see [14]). *Suppose Δ is given by* (10). *With matrices $M = M'$, S, and N with appropriate dimension, the inequality*

$$M + S\Delta N + N'\Delta' S' < 0 \tag{18}$$

holds for all F_{i,θ_k} such that $F_{i,\theta_k}F'_{i,\theta_k} \leq I$, if and only if, for some $\delta = \epsilon^2 > 0$,

$$\begin{bmatrix} \delta M & S & \delta N' \\ S' & -I & J' \\ \delta N & J & -I \end{bmatrix} < 0. \tag{19}$$

3. Robust Stability and H_∞ Performance Analysis

In this section, the stability and H_∞ performance for the nominal fuzzy system will be analyzed. Under control law (4), the closed-loop fuzzy system becomes

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r} h_i(z(k)) \left[\left(\mathscr{A}_{i,\theta_k} + \mathscr{B}_{i,\theta_k}K_{j,\theta_k}\right)x(k) + \mathscr{C}_{i,\theta_k}v(k) \right. \\ &\qquad \left. + \mathscr{D}_{i,\theta_k}x(k)w(k) \right], \end{aligned} \tag{20}$$

$$\begin{aligned} y(k) &= \sum_{i=1}^{r} h_i(z(k)) \\ &\quad \times \left[\left(\mathscr{M}_{i,\theta_k} + \mathscr{T}_{i,\theta_k}K_{j,\theta_k}\right)x(k) + \mathscr{N}_{i,\theta_k}v(k) \right]. \end{aligned} \tag{21}$$

When $\Delta = 0$, the nominal closed-loop system becomes

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(k))h_j(z(k)) \\ &\quad \times \left(A_{ij,\theta_k}x(k) + C_{i,\theta_k}v(k) + D_{i,\theta_k}x(k)w(k)\right), \end{aligned} \tag{22}$$

$$\begin{aligned} y(k) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(k))h_j(z(k)) \\ &\quad \times \left(M_{ij,\theta_k}x(k) + N_{i,\theta_k}v(k)\right), \end{aligned} \tag{23}$$

where

$$\begin{aligned} A_{ij,\theta_k} &= A_{i,\theta_k} + B_{i,\theta_k}K_{j,\theta_k}, \\ M_{ij,\theta_k} &= M_{i,\theta_k} + T_{i,\theta_k}K_{j,\theta_k}. \end{aligned} \tag{24}$$

Theorem 5. *The nominal closed-loop fuzzy system* (22) *is stable with H_∞ norm bound γ; that is, $\|y(k)\|_2 < \gamma\|v(k)\|_2$ for all nonzero $v(k) \in l_2[0,\infty)$ under the zero initial condition, if there exists matrices $\{P_{i,\theta_k} > 0\}_{i=1}^{r}$ for all $i, j, l \in \{1, 2, \ldots, r\}$ such that*

$$\begin{bmatrix} \Phi & \Gamma \\ \Lambda & \Psi \end{bmatrix} < 0, \tag{25}$$

where

$$\begin{aligned} \widetilde{P}_{l,\theta_k} &= \sum_{\theta_k=1}^{N} p_{\theta_k\theta_{k+1}}P_{i,\theta_k}, \\ \Phi &= A'_{ij,\theta_k}\widetilde{P}_{l,\theta_k}A_{ij,\theta_k} + D'_{i,\theta_k}\widetilde{P}_{l,\theta_k}D_{i,\theta_k} + M'_{ij,\theta_k}M_{ij,\theta_k} - P_{i,\theta_k}, \\ \Gamma &= A'_{ij,\theta_k}\widetilde{P}_{l,\theta_k}C_{i,\theta_k} + M'_{ij,\theta_k}N_{i,\theta_k}, \\ \Lambda &= C'_{i,\theta_k}\widetilde{P}_{l,\theta_k}A_{ij,\theta_k} + N'_{i,\theta_k}M_{ij,\theta_k}, \\ \Psi &= C'_{i,\theta_k}\widetilde{P}_{l,\theta_k}C_{i,\theta_k} + N'_{i,\theta_k}N_{i,\theta_k} - \gamma^2 I. \end{aligned} \tag{26}$$

Proof. Obviously, inequality (25) implies the following inequality:

$$A'_{ij,\theta_k}\widetilde{P}_{l,\theta_k}A_{ij,\theta_k} + D'_{i,\theta_k}\widetilde{P}_{l,\theta_k}D_{i,\theta_k} - P_{i,\theta_k} < 0. \tag{27}$$

It can be checked from the result of Lemma 3 that for all $i, j, l, p, q \in \{1, 2, \ldots, r\}$,

$$\begin{aligned} &A'_{ip,\theta_k}\widetilde{P}_{j,\theta_k}A_{lq,\theta_k} + A'_{lq,\theta_k}\widetilde{P}_{j,\theta_k}A_{ip,\theta_k} + D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{l,\theta_k} \\ &\quad + D'_{l,\theta_k}\widetilde{P}_{j,\theta_k}D_{i,\theta_k} - P_{i,\theta_k} - P_{l,\theta_k} < 0. \end{aligned} \tag{28}$$

When $i = l$, the inequality (28) becomes

$$\begin{aligned} &A'_{ip,\theta_k}\widetilde{P}_{j,\theta_k}A_{iq,\theta_k} + A'_{iq,\theta_k}\widetilde{P}_{j,\theta_k}A_{ip,\theta_k} \\ &\quad + 2D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{i,\theta_k} - 2P_{i,\theta_k} < 0. \end{aligned} \tag{29}$$

Let

$$V(x(k), \theta_k) = x'(k)\left[\sum_{i=1}^{r} h_i(z(k))P_{i,\theta_k}\right]x(k). \tag{30}$$

When $v(k) = 0$, the system (22) becomes

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(k))h_j(z(k)) \\ &\quad \times \left(A_{ij,\theta_k}x(k) + D_{i,\theta_k}x(k)w(k)\right). \end{aligned} \tag{31}$$

In what follows, we will drop the argument of $h_i(z(k))$ for clarity. By some algebraic manipulation, with $h_j^+ = h_j(z(k+1))$, the difference of Lyapunov function $V(x(k),\theta_k)$ given by $\Delta V(x(k),\theta_k) = V(x(k+1),\theta_{k+1}) - V(x(k),\theta_k)$ along the solution of system (31) is

$$\begin{aligned}
&E\left[\Delta V(x(k),\theta_k)\big|_{(31)}\right] \\
&\quad = E\left[V(x(k+1),\theta_{k+1}) - V(x(k),\theta_k)\right] \\
&\quad = E\left\{x'(k+1)\left[\sum_{j=1}^{r} h_j(z(k))P_{j,\theta_{k+1}}\right]x(k+1)\right. \\
&\qquad\qquad \left. - x'(k)\left[\sum_{i=1}^{r} h_i(z(k))P_{i,\theta_k}\right]x(k)\right\} \\
&\quad = x'(k)\left[\sum_{j=1}^{r}h_j^+\sum_{i=1}^{r}\sum_{l=1}^{r}\sum_{p=1}^{r}\sum_{q=1}^{r}h_ih_lh_ph_q\right. \\
&\qquad\qquad \times\left(A'_{ip,\theta_k}\widetilde{P}_{j,\theta_k}A_{lq,\theta_k}\right. \\
&\qquad\qquad\quad \left.\left. + D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{l,\theta_k} - P_{i,\theta_k}\right)\right]x(k) \\
&\quad = x'(k) \\
&\qquad \times\left\{\sum_{j=1}^{r}h_j^+\left[\sum_{i=1}^{r}\sum_{p=1}^{r}h_i^2h_p^2\right.\right. \\
&\qquad\qquad \times\left(A'_{ip,\theta_k}\widetilde{P}_{j,\theta_k}A_{ip,\theta_k}\right. \\
&\qquad\qquad\quad \left. + D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{i,\theta_k} - P_{i,\theta_k}\right) \\
&\qquad\qquad + \sum_{i=1}^{r}\sum_{l>i}^{r}\sum_{p=1}^{r}\sum_{q=1}^{r}h_ih_lh_ph_q \\
&\qquad\qquad\quad \times\left(A'_{ip,\theta_k}\widetilde{P}_{j,\theta_k}A_{lq,\theta_k}\right. \\
&\qquad\qquad\qquad + A'_{lq,\theta_k}\widetilde{P}_{j,\theta_k}A_{ip,\theta_k} + D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{l,\theta_k} \\
&\qquad\qquad\qquad \left. + D'_{l,\theta_k}\widetilde{P}_{j,\theta_k}D_{i,\theta_k} - P_{i,\theta_k} - P_{l,\theta_k}\right) \\
&\qquad\qquad + \sum_{i=1}^{r}\sum_{p=1}^{r}\sum_{q>p}^{r}h_i^2h_ph_q \\
&\qquad\qquad\quad \times\left(A'_{iq,\theta_k}\widetilde{P}_{j,\theta_k}A_{iq,\theta_k}\right. \\
&\qquad\qquad\qquad + A'_{iq,\theta_k}\widetilde{P}_{j,\theta_k}A_{ip,\theta_k} \\
&\qquad\qquad\qquad \left.\left.\left. + 2D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{i,\theta_k} - 2P_{i,\theta_k}\right)\right]\right\}x(k),
\end{aligned} \tag{32}$$

where

$$\widetilde{P}_{j,\theta_k} = P_{j,\theta_{k+1}} = \sum_{\theta_k=1}^{N} p_{\theta_k\theta_{k+1}}P_{i,\theta_k}. \tag{33}$$

It follows from (13), (28), and (29) that

$$\Delta V(x(k),\theta_k)\big|_{(31)} < 0, \tag{34}$$

which proves the stability of system (31).

It follows from (25) that for all $i, j, l \in \{1, 2, \ldots, r\}$ exists

$$\begin{aligned}
&\begin{bmatrix} A_{ij,\theta_k} & C_{i,\theta_k} \\ M_{ij,\theta_k} & N_{i,\theta_k}\end{bmatrix}'\begin{bmatrix}\widetilde{P}_{l,\theta_k} & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix} A_{ij,\theta_k} & C_{i,\theta_k} \\ M_{ij,\theta_k} & N_{i,\theta_k}\end{bmatrix} \\
&\quad - \begin{bmatrix} P_{i,\theta_k} - D'_{i,\theta_k}\widetilde{P}_{l,\theta_k}D_{i,\theta_k} & 0 \\ 0 & \gamma^2 I\end{bmatrix} < 0.
\end{aligned} \tag{35}$$

Let

$$J_N = \sum_{k=0}^{N-1} E\left[y'(k)y(k) - \gamma^2 v'(k)v(k)\right]. \tag{36}$$

For zero initial condition $x(0) = x_0 = 0$, one has

$$\begin{aligned}
\sum_{k=0}^{N-1} E\left[\Delta V(x(k),\theta_k)\big|_{(22)}\right] &= V(x(N),\theta_N) - V(x(0),\theta_0) \\
&= V(x(N),\theta_N).
\end{aligned} \tag{37}$$

Therefore,

$$\begin{aligned}
J_N &= \sum_{k=0}^{N-1} E\left[y'(k)y(k) - \gamma^2 v'(k)v(k)\right] \\
&= \sum_{k=0}^{N-1} E\left[y(k)'y(k) - \gamma^2 v'(k)v(k) + \Delta V(x(k),\theta_k)\big|_{(22)}\right] \\
&\quad - V(x(N),\theta_N),
\end{aligned} \tag{38}$$

where $\Delta V(x(k),\theta_k)|_{(22)}$ defines the difference of $V(x(k),\theta_k)$ along system (22). It is noted that, with $\zeta(k)$ defined by

$$\zeta(k) = \begin{bmatrix} x(k) \\ v(k)\end{bmatrix}, \tag{39}$$

we have

$$\begin{aligned}
&E\left[y'(k)\,y(k)-\gamma^2 v'(k)\,v(k)\right] \\
&\quad=\zeta'(k) \\
&\qquad\times\left\{\sum_{i=1}^{r}\sum_{p=1}^{r}\sum_{l=1}^{r}\sum_{q=1}^{r}h_ih_ph_lh_q\right. \\
&\qquad\qquad\left.\times\left(\begin{bmatrix}M'_{ip,\theta_k}\\ N'_{i,\theta_k}\end{bmatrix}\begin{bmatrix}M_{lq,\theta_k} & N_{l,\theta_k}\end{bmatrix}-\begin{bmatrix}0 & 0\\ 0 & \gamma^2 I\end{bmatrix}\right)\right\}\zeta(k), \\
&E\left[\Delta V\left(x(k),\theta_k\right)|_{(22)}\right] \\
&\quad=\zeta'(k) \\
&\qquad\times\left\{\sum_{i=1}^{r}\sum_{p=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r}\sum_{q=1}^{r}h_ih_j^+h_ph_lh_q\right. \\
&\qquad\qquad\times\left(\begin{bmatrix}A'_{ip,\theta_k}\\ C'_{i,\theta_k}\end{bmatrix}\widetilde{P}_{j,\theta_k}\begin{bmatrix}A_{lq,\theta_k} & C_{l,\theta_k}\end{bmatrix}\right. \\
&\qquad\qquad\qquad\left.\left.-\begin{bmatrix}P_{i,\theta_k}-D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{l,\theta_k} & 0\\ 0 & 0\end{bmatrix}\right)\right\}\zeta(k).
\end{aligned}\tag{40}$$

Then, it is obtained that

$$\begin{aligned}
J_N&\le\sum_{k=0}^{N-1}\zeta'(k) \\
&\quad\times\left\{\sum_{j=1}^{r}h_j^+\sum_{i=1}^{r}\sum_{p=1}^{r}\sum_{l=1}^{r}\sum_{q=1}^{r}h_ih_ph_lh_q\right. \\
&\qquad\times\left(\begin{bmatrix}M'_{ip,\theta_k}\\ N'_{i,\theta_k}\end{bmatrix}\begin{bmatrix}M_{lq,\theta_k} & N_{l,\theta_k}\end{bmatrix}\right. \\
&\qquad\qquad+\begin{bmatrix}A'_{ip,\theta_k}\\ C'_{i,\theta_k}\end{bmatrix}\widetilde{P}_{j,\theta_k}\begin{bmatrix}A_{lq,\theta_k} & C_{l,\theta_k}\end{bmatrix} \\
&\qquad\qquad\left.\left.-\begin{bmatrix}P_{i,\theta_k}-D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{l,\theta_k} & 0\\ 0 & \gamma^2 I\end{bmatrix}\right)\right\}\zeta(k) \\
&=\sum_{k=0}^{N-1}\zeta'(k) \\
&\quad\times\left\{\sum_{j=1}^{r}h_j^+\sum_{i=1}^{r}\sum_{p=1}^{r}\sum_{l=1}^{r}\sum_{q=1}^{r}h_ih_ph_lh_q\right. \\
&\qquad\times\left(\begin{bmatrix}A_{ip,\theta_k} & C_{i,\theta_k}\\ M_{ip,\theta_k} & N_{i,\theta_k}\end{bmatrix}'\begin{bmatrix}\widetilde{P}_{j,\theta_k} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}A_{lq,\theta_k} & C_{l,\theta_k}\\ M_{lq,\theta_k} & N_{l,\theta_k}\end{bmatrix}\right. \\
&\qquad\qquad\left.\left.-\begin{bmatrix}P_{i,\theta_k}-D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{l,\theta_k} & 0\\ 0 & \gamma^2 I\end{bmatrix}\right)\right\}\zeta(k),
\end{aligned}\tag{41}$$

which can be rewritten as

$$\begin{aligned}
J_N&\le\sum_{k=0}^{N-1}\zeta'(k) \\
&\quad\times\left\{\sum_{j=1}^{r}h_j^+\sum_{i=1}^{r}\sum_{p=1}^{r}h_i^2h_p^2\right. \\
&\qquad\times\left(\begin{bmatrix}A_{ip,\theta_k} & C_{i,\theta_k}\\ M_{ip,\theta_k} & N_{i,\theta_k}\end{bmatrix}'\right. \\
&\qquad\qquad\times\begin{bmatrix}\widetilde{P}_{j,\theta_k} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}A_{ip,\theta_k} & C_{i,\theta_k}\\ M_{ip,\theta_k} & N_{i,\theta_k}\end{bmatrix} \\
&\qquad\qquad\left.\left.-\begin{bmatrix}P_{i,\theta_k}-D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{i,\theta_k} & 0\\ 0 & \gamma^2 I\end{bmatrix}\right)\right\}\zeta(k) \\
&\quad+\sum_{k=0}^{N-1}\zeta'(k) \\
&\quad\times\left\{\sum_{j=1}^{r}h_j^+\sum_{i=1}^{r}\sum_{p=1}^{r}\sum_{l>i}^{r}\sum_{q=1}^{r}h_ih_ph_lh_q\right. \\
&\qquad\times\left(\begin{bmatrix}A_{ip,\theta_k} & C_{i,\theta_k}\\ M_{ip,\theta_k} & N_{i,\theta_k}\end{bmatrix}'\begin{bmatrix}\widetilde{P}_{j,\theta_k} & 0\\ 0 & I\end{bmatrix}\right. \\
&\qquad\qquad\times\begin{bmatrix}A_{lq,\theta_k} & C_{l,\theta_k}\\ M_{lq,\theta_k} & N_{l,\theta_k}\end{bmatrix} \\
&\qquad\qquad-\begin{bmatrix}P_{i,\theta_k}-D'_{i,\theta_k}\widetilde{P}_{j,\theta_k}D_{l,\theta_k} & 0\\ 0 & \gamma^2 I\end{bmatrix} \\
&\qquad\qquad+\begin{bmatrix}A_{lq,\theta_k} & C_{l,\theta_k}\\ M_{lq,\theta_k} & N_{l,\theta_k}\end{bmatrix}' \\
&\qquad\qquad\times\begin{bmatrix}\widetilde{P}_{j,\theta_k} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}A_{ip,\theta_k} & C_{i,\theta_k}\\ M_{ip,\theta_k} & N_{i,\theta_k}\end{bmatrix} \\
&\qquad\qquad\left.\left.-\begin{bmatrix}P_{l,\theta_k}-D'_{l,\theta_k}\widetilde{P}_{j,\theta_k}D_{i,\theta_k} & 0\\ 0 & \gamma^2 I\end{bmatrix}\right)\right\}\zeta(k) \\
&\quad+\sum_{k=0}^{N-1}\zeta'(k) \\
&\quad\times\left\{\sum_{j=1}^{r}h_j^+\sum_{i=1}^{r}\sum_{p=1}^{r}\sum_{q>p}^{r}h_i^2h_ph_q\right. \\
&\qquad\times\left(\begin{bmatrix}A_{ip,\theta_k} & C_{i,\theta_k}\\ M_{ip,\theta_k} & N_{i,\theta_k}\end{bmatrix}'\right. \\
&\qquad\qquad\times\begin{bmatrix}\widetilde{P}_{j,\theta_k} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}A_{iq,\theta_k} & C_{i,\theta_k}\\ M_{iq,\theta_k} & N_{i,\theta_k}\end{bmatrix}
\end{aligned}$$

$$+\begin{bmatrix} A_{iq,\theta_k} & C_{i,\theta_k} \\ M_{iq,\theta_k} & N_{i,\theta_k} \end{bmatrix}' \times \begin{bmatrix} \widetilde{P}_{j,\theta_k} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{ip,\theta_k} & C_{i,\theta_k} \\ M_{ip,\theta_k} & N_{i,\theta_k} \end{bmatrix} -2 \begin{bmatrix} P_{i,\theta_k} - D'_{i,\theta_k} \widetilde{P}_{j,\theta_k} D_{i,\theta_k} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \Bigg) \Bigg\} \zeta(k). \tag{42}$$

By following similar line as in the proof of (34) it can be checked that, for any N, $J_N < 0$, which gives for any nonzero $v(k) \in l_2[0,\infty)$, $y(k) \in l_2[0,\infty)$, and $\|y(k)\|_2 < \gamma \|v(k)\|_2$. □

Theorem 6. *For the nominal fuzzy system* (20)*, there exists a state feedback fuzzy control law* (4) *such that the closed-loop system is stable with H_∞ norm bound γ, if there exist matrices $\{X_{i,\theta_k} > 0\}_{i=1}^r$, $\{\Omega_{i,\theta_k}\}_{i=1}^r$, and $\{Y_{i,\theta_k}\}_{i=1}^r$, $i, j, l \in \{1, 2, \ldots, r\}$ satisfying the following LMIs:*

$$\begin{bmatrix} X_{i,\theta_k} - \left(\Omega_{j,\theta_k} + \Omega'_{j,\theta_k}\right) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_{i,\theta_k}\Omega_{j,\theta_k} + B_{i,\theta_k} Y_{j,\theta_k} & C_{i,\theta_k} & -X_{l,\theta_k} & * \\ M_{i,\theta_k}\Omega_{j,\theta_k} + T_{i,\theta_k} Y_{j,\theta_k} & N_{i,\theta_k} & 0 & -I \end{bmatrix} < 0, \tag{43}$$

where $*$ represents the transposed matrices that are readily inferred by symmetry for all i and j except the pairs (i, j) such that $h_i(z(k))h_j(z(k)) = 0$, for all k. A robust stabilizing controller gain can be given by

$$K_{j,\theta_k} = Y_{j,\theta_k} \Omega_{j,\theta_k}^{-1}. \tag{44}$$

Proof. By using (24) and (44), (43) becomes

$$\begin{bmatrix} X_{i,\theta_k} - \left(\Omega_{j,\theta_k} + \Omega'_{j,\theta_k}\right) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_{ij,\theta_k}\Omega_{j,\theta_k} & C_{i,\theta_k} & -X_{l,\theta_k} & * \\ M_{ij,\theta_k}\Omega_{j,\theta_k} & N_{i,\theta_k} & 0 & -I \end{bmatrix} < 0, \tag{45}$$

which gives $0 < X_{i,\theta_k} < \Omega_{j,\theta_k} + \Omega'_{j,\theta_k}$. $(X_{i,\theta_k} - \Omega_{j,\theta_k})' X_{i,\theta_k}^{-1} (X_{i,\theta_k} - \Omega_{j,\theta_k}) \geq 0$ implies that $\Omega'_{j,\theta_k} X_{i,\theta_k}^{-1} \Omega_{j,\theta_k} \geq \Omega_{j,\theta_k} + \Omega'_{j,\theta_k} - X_{i,\theta_k}$ yielding

$$\begin{bmatrix} -\Omega'_{j,\theta_k} X_{i,\theta_k}^{-1} \Omega_{j,\theta_k} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_{ij,\theta_k}\Omega_{j,\theta_k} & C_{i,\theta_k} & -X_{l,\theta_k} & * \\ M_{ij,\theta_k}\Omega_{j,\theta_k} & N_{i,\theta_k} & 0 & -I \end{bmatrix} < 0. \tag{46}$$

Note that Ω_{j,θ_k} is invertible. Premultiplying $\operatorname{diag}(\Omega'_{j,\theta_k}, I, I, I)^{-1}$ and postmultiplying $\operatorname{diag}(\Omega_{j,\theta_k}^{-1}, I, I, I)$ to (46) give

$$\begin{bmatrix} -X_{i,\theta_k}^{-1} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_{ij,\theta_k} & C_{i,\theta_k} & -X_{l,\theta_k} & * \\ M_{ij,\theta_k} & N_{i,\theta_k} & 0 & -I \end{bmatrix} < 0. \tag{47}$$

Set $X_{i,\theta_k}^{-1} = P_{i,\theta_k} - D'_{i,\theta_k} \widetilde{P}_{l,\theta_k} D_{i,\theta_k}$ and $X_{l,\theta_k}^{-1} = \widetilde{P}_{l,\theta_k}$; there is

$$\begin{bmatrix} -\left(P_{i,\theta_k} - D'_{i,\theta_k} \widetilde{P}_{l,\theta_k} D_{i,\theta_k}\right) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_{ij,\theta_k} & C_{i,\theta_k} & -\widetilde{P}_{l,\theta_k}^{-1} & * \\ M_{ij,\theta_k} & N_{i,\theta_k} & 0 & -I \end{bmatrix} < 0. \tag{48}$$

It follows from Schur complement equivalence that

$$\begin{bmatrix} A_{ij,\theta_k} & C_{i,\theta_k} \\ M_{ij,\theta_k} & N_{i,\theta_k} \end{bmatrix}' \begin{bmatrix} \widetilde{P}_{l,\theta_k} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{ij,\theta_k} & C_{i,\theta_k} \\ M_{ij,\theta_k} & N_{i,\theta_k} \end{bmatrix} - \begin{bmatrix} P_{i,\theta_k} - D'_{i,\theta_k} \widetilde{P}_{l,\theta_k} D_{i,\theta_k} & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0. \tag{49}$$

The result then follows from Theorem 5. □

Theorem 7. *For the uncertain discrete-time MJSS* (7)*, there exists a state feedback fuzzy control law* (4) *such that the closed-loop system is stable with H_∞ norm bound γ, if there exist matrices $\{X_{i,\theta_k} > 0\}_{i=1}^r$, $\{\Omega_{i,\theta_k}\}_{i=1}^r$, and $\{Y_{i,\theta_k}\}_{i=1}^r$, $i, j, l \in \{1, 2, \ldots, r\}$ satisfying the following LMIs:*

$$\begin{bmatrix} X_{i,\theta_k} - \left(\Omega_{j,\theta_k} + \Omega'_{j,\theta_k}\right) & * & * & * & * & * \\ 0 & -\epsilon\gamma^2 I & * & * & * & * \\ A_{i,\theta_k}\Omega_{j,\theta_k} + B_{i,\theta_k} Y_{j,\theta_k} & \epsilon C_{i,\theta_k} & -X_{l,\theta_k} & * & * & * \\ M_{i,\theta_k}\Omega_{j,\theta_k} + T_{i,\theta_k} Y_{j,\theta_k} & \epsilon N_{i,\theta_k} & 0 & -\epsilon I & * & * \\ 0 & 0 & E'_{1i} & E'_{2i} & -I & * \\ H'_{1i}\Omega_{j,\theta_k} + H'_{2i} Y_{j,\theta_k} & \epsilon H_{3i} & 0 & 0 & J & -I \end{bmatrix} < 0, \tag{50}$$

where $*$ represents the transposed matrices that are readily inferred by symmetry for all i and j except the pairs (i, j) such that $h_i(z(k))h_j(z(k)) = 0$, for all k. A robust stabilizing controller gain can be given by

$$K_{j,\theta_k} = Y_{j,\theta_k} \Omega_{j,\theta_k}^{-1}. \tag{51}$$

Proof. By using Lemma 4, it can be checked that the feasibility of inequality (50) is equivalent to

$$\begin{bmatrix} \widehat{X}_{i,\theta_k} - \left(\widehat{\Omega}_{j,\theta_k} + \widehat{\Omega}'_{j,\theta_k}\right) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \mathscr{A}_{i,\theta_k}\widehat{\Omega}_{j,\theta_k} + \mathscr{B}_{i,\theta_k} \widehat{Y}_{j,\theta_k} & \mathscr{C}_{i,\theta_k} & -\widehat{X}_{l,\theta_k} & * \\ \mathscr{M}_{i,\theta_k}\widehat{\Omega}_{j,\theta_k} + \mathscr{T}_{i,\theta_k} \widehat{Y}_{j,\theta_k} & \mathscr{N}_{i,\theta_k} & 0 & -I \end{bmatrix} < 0, \tag{52}$$

where

$$\widehat{X}_{i,\theta_k} = \epsilon^{-1} X_{i,\theta_k}, \qquad \widehat{\Omega}_{j,\theta_k} = \epsilon^{-1} \Omega_{j,\theta_k}, \qquad \widehat{Y}_{j,\theta_k} = \epsilon^{-1} Y_{j,\theta_k}. \tag{53}$$

It follows from (51)–(53) that

$$\begin{bmatrix} \widehat{X}_{i,\theta_k} - \left(\widehat{\Omega}_{j,\theta_k} + \widehat{\Omega}'_{j,\theta_k}\right) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \left(\mathscr{A}_{i,\theta_k} + \mathscr{B}_{i,\theta_k} K_{j,\theta_k}\right)\widehat{\Omega}_{j,\theta_k} & \mathscr{C}_{i,\theta_k} & -\widehat{X}_{l,\theta_k} & * \\ \left(\mathscr{M}_{i,\theta_k} + \mathscr{T}_{i,\theta_k} K_{j,\theta_k}\right)\widehat{\Omega}_{j,\theta_k} & \mathscr{N}_{i,\theta_k} & 0 & -I \end{bmatrix} < 0, \tag{54}$$

which implies that the closed-loop system (20) and (21) is robustly stable with H_∞ norm bound γ by Theorem 6. □

Corollary 8. *Consider the nominal unforced system*

$$x(k+1) = \sum_{i=1}^{r} h_i(z(k)) \times \left(A_{i,\theta_k} x(k) + C_{i,\theta_k} v(k) + D_{i,\theta_k} x(k) w(k)\right),$$
$$y(k) = \sum_{i=1}^{r} h_i(z(k)) \left(M_{i,\theta_k} x(k) + N_{i,\theta_k} v(k)\right). \tag{55}$$

The nominal unforced fuzzy system is stable with H_∞ norm bound γ; that is, $\|y(k)\|_2 < \gamma \|v(k)\|_2$ for all nonzero $v(k) \in l_2[0,\infty)$ under the zero initial condition, if there exist matrices $\{P_{i,\theta_k} > 0\}_{i=1}^{r}$, for all $i, j \in \{1, 2, \ldots, r\}$ such that

$$\begin{bmatrix} A_{i,\theta_k} & C_{i,\theta_k} \\ M_{i,\theta_k} & N_{i,\theta_k} \end{bmatrix}' \begin{bmatrix} \widetilde{P}_{j,\theta_k} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{i,\theta_k} & C_{i,\theta_k} \\ M_{i,\theta_k} & N_{i,\theta_k} \end{bmatrix} - \begin{bmatrix} P_{i,\theta_k} - D'_{i,\theta_k} \widetilde{P}_{j,\theta_k} D_{i,\theta_k} & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0, \tag{56}$$

where $\widetilde{P}_{j,\theta_k} = \sum_{\theta_k=1}^{N} p_{\theta_k \theta_{k+1}} P_{i,\theta_k}$.

Corollary 9. *The nominal unforced fuzzy system* (55) *is stable with H_∞ norm bound γ, if there exist matrices $\{X_{i,\theta_k} > 0\}_{i=1}^{r}$, $\{\Omega_{i,\theta_k}\}_{i=1}^{r}$, $i, j \in \{1, 2, \ldots, r\}$ satisfying the following LMIs:*

$$\begin{bmatrix} X_{i,\theta_k} - \left(\Omega_{j,\theta_k} + \Omega'_{j,\theta_k}\right) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_{i,\theta_k} \Omega_{j,\theta_k} & C_{i,\theta_k} & -X_{j,\theta_k} & * \\ M_{i,\theta_k} \Omega_{j,\theta_k} & N_{i,\theta_k} & 0 & -I \end{bmatrix} < 0. \tag{57}$$

Corollary 10. *When $u(k) = 0$, the unforced system* (7) *and* (8) *is robustly stable with H_∞ norm bound γ, if there exist matrices $\{X_{i,\theta_k} > 0\}_{i=1}^{r}$ and $\epsilon > 0$, $i, j \in \{1, 2, \ldots, r\}$ satisfying the following LMIs:*

$$\begin{bmatrix} -X_{i,\theta_k} & * & * & * & * & * \\ 0 & -\epsilon\gamma^2 I & * & * & * & * \\ A_{i,\theta_k} X_{i,\theta_k} & \epsilon C_{i,\theta_k} & -X_{j,\theta_k} & * & * & * \\ M_{i,\theta_k} X_{i,\theta_k} & \epsilon N_{i,\theta_k} & 0 & -\epsilon I & * & * \\ 0 & 0 & E'_{1i} & E'_{2i} & -I & * \\ H'_{1i} X_{i,\theta_k} & \epsilon H_{3i} & 0 & 0 & J & -I \end{bmatrix} < 0. \tag{58}$$

4. Numerical Simulation Example

In this section, to illustrate the proposed new H_∞ fuzzy control method, the backing-up control of a computer-simulated truck-trailer is considered [30]. For example, the truck-trailer model is given by

$$x_1(k+1) = \left(1 - \frac{vt}{L}\right) x_1(k) + \delta \frac{vt}{\ell} u(k), \tag{59}$$

$$x_2(k+1) = x_2(k) + \frac{vt}{L} x_1(k), \tag{60}$$

$$x_3(k+1) = x_3(k) + vt \sin\left(x_2(k) + \frac{vt}{2L} x_1(k)\right), \tag{61}$$

where $x_1(k)$ is the angle difference between truck and trailer, $x_2(k)$ is the angle of trailer, and $x_3(k)$ is the vertical position of rear end of trailer. The parameter $\delta \in [0, 1]$ is used to describe the actuator failure, where $\delta = 1$ implies no failure, $\delta = 0$ implies a total failure, and $0 < \delta < 1$ implies a partial failure. It is assumed that there is no actuator failure. Equations (59) and (60) are linear, but (61) is nonlinear. The model parameters are given as $L = 5.5$, $\ell = 2.8$, $v = -1.0$, $t = 2.0$, and $\delta = 0$.

The transition probability matrix that relates the three operation modes is given as follows:

$$\mathbb{P} = \begin{bmatrix} 0.48 & 0.29 & 0.23 \\ 0.6 & 0.1 & 0.3 \\ 0.1 & 0.65 & 0.25 \end{bmatrix}. \tag{62}$$

As in [30], we set $\omega = 0.01/\pi$ and the nonlinear term $\sin(z(k))$ as

$$\sin(z(k)) = h_1(z(k)) z(k) + h_2(z(k)) \omega z(k), \tag{63}$$

where $h_1(z(k)), h_2(z(k)) \in [0, 1]$, and $h_1(z(k)) + h_2(z(k)) = 1$. By solving the equations, it can be seen that the membership functions $h_1(z(k))$, $h_2(z(k))$ have the following relations. When $z(k)$ is about 0 rad, $h_1(z(k)) = 1$, $h_2(z(k)) = 0$, and when $z(k)$ is about $\pm\pi$ rad, $h_1(z(k)) = 0$, $h_2(z(k)) = 1$. Then the following fuzzy models can be used to design the fuzzy controller for the uncertain nonlinear MJSS.

Rule 1. If $z(k) = x_2(k) + (vt/2L) x_1(k)$ is about 0 rad, then

$$x(k+1) = \left(A_{1,\theta_k} + \Delta A_{1,\theta_k}\right) x(k) + \left(B_{1,\theta_k} + \Delta B_{1,\theta_k}\right) u(k) + \left(C_{1,\theta_k} + \Delta C_{1,\theta_k}\right) v(k) + D_{1,\theta_k} x_1(k) w(k). \tag{64}$$

Rule 2. If $z(k) = x_2(k) + (vt/2L) x_1(k)$ is about $\pm\pi$ rad, then

$$x(k+1) = \left(A_{2,\theta_k} + \Delta A_{2,\theta_k}\right) x(k) + \left(B_{2,\theta_k} + \Delta B_{2,\theta_k}\right) u(k) + \left(C_{2,\theta_k} + \Delta C_{2,\theta_k}\right) v(k) + D_{2,\theta_k}(k) w(k), \tag{65}$$

where

$$A_{1,\theta_k} = \begin{bmatrix} 1 - \dfrac{vt}{L} & 0 & 0 \\ \dfrac{vt}{L} & 1 & 0 \\ \dfrac{v^2 t^2}{2L} & vt & 1 \end{bmatrix},$$

$$A_{2,\theta_k} = \begin{bmatrix} 1 - \dfrac{vt}{L} & 0 & 0 \\ \dfrac{vt}{L} & 1 & 0 \\ \dfrac{\omega v^2 t^2}{2L} & \omega vt & 1 \end{bmatrix},$$

$$B_{1,\theta_k} = B_{2,\theta_k} = \left[\frac{vt}{L} \ \ 0 \ \ 0\right]',$$

$$C_{1,\theta_k} = C_{2,\theta_k} = \left[-0.1 \ \ 0.2 \ \ 0.15\right]',$$

$$D_{1,\theta_k} = D_{2,\theta_k} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{66}$$

In the above uncertain fuzzy models, the uncertainty to describe the modeling error is assumed to be in the following form:

$$\begin{gathered} \Delta A_{1,\theta_k} = E_{11}\Delta H_{11}, \qquad \Delta B_{1,\theta_k} = E_{11}\Delta H_{21}, \\ \Delta C_{1,\theta_k} = E_{11}\Delta H_{31}, \\ \Delta A_{2,\theta_k} = E_{12}\Delta H_{12}, \qquad \Delta B_{2,\theta_k} = E_{12}\Delta H_{22}, \\ \Delta C_{2,\theta_k} = E_{12}\Delta H_{32}, \end{gathered} \tag{67}$$

where

$$\begin{gathered} E_{11} = E_{12} = \begin{bmatrix} 0.06 & 0 & 0 \\ 0 & 0.06 & 0 \\ 0 & 0 & 0.06 \end{bmatrix}, \\ E_{21} = E_{22} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \\ H_{11} = H_{12} = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \\ H_{21} = H_{22} = 0, \qquad H_{31} = H_{32} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{gathered} \tag{68}$$

We choose the following matrices for the uncertain discrete-time MJSS:

$$\begin{gathered} M_{1,\theta_k} = M_{2,\theta_k} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \\ N_{1,\theta_k} = N_{2,\theta_k} = \left[0.5 \ \ 0 \ \ 0\right]', \\ T_{1,\theta_k} = T_{2,\theta_k} = \left[0.5 \ \ 0 \ \ 0\right]', \\ J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{gathered} \tag{69}$$

We set $i = 1$, $j = 1$, $l = 1$, $\theta_k = 1$ and $i = 2$, $j = 2$, $l = 2$, $\theta_k = 2$, respectively. Based on Theorem 7 and using LMI control toolbox in Matlab to solve LMIs (50), we can obtain the feasible set of solutions as follows:

$$\begin{gathered} X_{1,1} = \begin{bmatrix} 0.2389 & 0.1410 & 0.1770 \\ 0.1410 & 0.1387 & 0.3577 \\ 0.1770 & 0.3577 & 1.9030 \end{bmatrix}, \\ \Omega_{1,1} = \begin{bmatrix} 0.3259 & 0.1769 & 0.2347 \\ 0.1768 & 0.1730 & 0.4467 \\ 0.2286 & 0.4594 & 2.3108 \end{bmatrix}, \\ X_{2,2} = \begin{bmatrix} 0.2725 & 0.2913 & -0.0027 \\ 0.2913 & 0.8440 & 0.0210 \\ -0.0027 & 0.0210 & 0.0402 \end{bmatrix}, \\ \Omega_{2,2} = \begin{bmatrix} 0.3559 & 0.2937 & -0.0047 \\ 0.2937 & 0.9264 & 0.0186 \\ -0.0047 & 0.0186 & 0.0370 \end{bmatrix}, \\ Y_{1,1} = \left[0.8566 \ \ 0.2418 \ \ 0.0753\right], \\ Y_{2,2} = \left[0.8592 \ \ 0.2091 \ \ -0.0203\right]. \end{gathered} \tag{70}$$

By (51), there are the local state feedback gains given by

$$\begin{gathered} K_{1,1} = \left[5.0451 \ \ -5.1128 \ \ 0.5086\right], \\ K_{2,2} = \left[3.0255 \ \ -0.7375 \ \ 0.2056\right]. \end{gathered} \tag{71}$$

Then, we can get

$$\begin{gathered} P_{1,1} = \begin{bmatrix} 11.7492 & -15.7505 & 2.0651 \\ -15.7505 & 37.4327 & -4.7133 \\ 2.0651 & -4.7133 & 1.4101 \end{bmatrix}, \\ P_{2,2} = \begin{bmatrix} 107.2937 & -35.1096 & 10.0427 \\ -35.1096 & 29.0490 & -8.5669 \\ 10.0427 & -8.5669 & 3.3892 \end{bmatrix}. \end{gathered} \tag{72}$$

It can be found that the LMIs of Theorem 7 have some feasible solution for the uncertain discrete-time MJSS. Setting $\epsilon = 1$, $\gamma = 2.6$, the aforementioned simulation results are obtained. Employing the feedback gains K, the fuzzy controller can be obtained by (4). By using Theorem 7, with the fuzzy control applied, if the LMIs in (50) and (25) have a positive-definite solution for K_{j,θ_k} and P_{i,θ_k}, respectively, then the system (7) and (8) driven by the designed fuzzy controller is stable with satisfying the H_∞ performance constraint.

5. Conclusions

In the paper, the robust H_∞ control has been discussed for a class of nonlinear discrete-time stochastic MJSS. First, a new LMI characterization of stability with H_∞ norm bound for uncertain discrete-time stochastic MJSS has been given. Moreover, sufficient conditions on robust stabilization and H_∞ performance analysis and control have been presented on LMIs. Furthermore, there are some corollaries of the stability, and the nominal unforced system has been given. Finally, a numerical simulation example has been presented to show the effectiveness.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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