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Some identities on Bernoulli and Euler polynomials arising from orthogonality of Legendre polynomials

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Abstract

The purpose of this paper is to investigate some interesting identities on the Bernoulli and Euler polynomials arising from the orthogonality of Legendre polynomials in the inner product space $\mathbb{P}_n$.

1 Introduction

As is well known, the Legendre polynomial $P_n(x)$ is a solutions of the following differential equation:

$$(1-x^2)u''-2xu'+n(n+1)u=0 \quad (\text{see [1–7]}),$$

where $n=0,1,2,\ldots$.

It is a polynomial of degree n. If n is even or odd, then $P_n(x)$ is accordingly even or odd. They are determined up to constant and normalized so that $P_n(1)=1$.

Rodrigues' formula is given by

$$P_n(x)=\frac{1}{2^n n!}\left\{\left(\frac{d}{dx}\right)^n(x^2-1)^n\right\}, \quad n\in\mathbb{Z}_+. \tag{1.1}$$

Integrating by parts, we can derive

$$\int_{-1}^{1}P_m(x)P_n(x)\,dx=\frac{2}{2n+1}\delta_{m,n} \quad (\text{see [1–7]}), \tag{1.2}$$

where $\delta_{m,n}$ is the Kronecker symbol.

By (1.1), we get

$$P_n(x)=\frac{1}{2^n}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}x^{n-2k}. \tag{1.3}$$

The generating function is given by

$$(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)t^n. \tag{1.4}$$

The Bernoulli polynomial is defined by a generating function to be

$$\frac{t}{e^t-1}e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{(see [8–13])} \tag{1.5}$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$.

In the special case, $x = 0$, $B_n(0) = B_n$ are called the *Bernoulli numbers*.

From (1.5), we have

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l}x^l \quad \text{(see [10–26])}. \tag{1.6}$$

As is well known, the *Euler numbers* are defined by

$$E_0 = 1, \qquad (E+1)^n + E_n = 2\delta_{0,n} \quad \text{(see [10–13])} \tag{1.7}$$

with the usual convention about replacing E^n by E_n.

The Euler polynomials are defined as

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l}x^l \quad \text{(see [27–31])}. \tag{1.8}$$

Let $\mathbb{P}_n = \{p(x) \in \mathbb{O}[x] \mid \deg p(x) \le n\}$. Then $\mathbb{P}_n$ is an inner product space with respect to the inner product $\langle \cdot, \cdot \rangle$ with

$$\langle q_1(x), q_2(x)\rangle = \int_{-1}^{1} q_1(x)q_2(x)\,dx,$$

where $q_1(x), q_2(x) \in \mathbb{P}_n$.

From (1.2), we can show that $\{P_0(x), P_1(x), \ldots, P_n(x)\}$ is an orthogonal basis for $\mathbb{P}_n$. In this paper, we derive some interesting identities on the Bernoulli and Euler polynomials from the orthogonality of Legendre polynomials in $\mathbb{P}_n$.

2 Some identities on the Bernoulli and Euler polynomials

For $q(x) \in \mathbb{P}_n$, let

$$q(x) = \sum_{k=0}^{n} C_k P_k(x). \tag{2.1}$$

Then, from (1.2), we have

$$\begin{aligned} \langle q(x), P_k(x)\rangle &= C_k\langle P_k(x), P_k(x)\rangle \\ &= C_k \int_{-1}^{1} \{P_k(x)\}^2\,dx \\ &= \frac{2}{2k+1}C_k. \end{aligned} \tag{2.2}$$

By (2.2), we get

$$\begin{aligned} C_k &= \frac{2k+1}{2}\langle q(x), P_k(x)\rangle = \frac{2k+1}{2}\int_{-1}^{1} P_k(x)q(x)\,dx \\ &= \left(\frac{2k+1}{2}\right)\frac{1}{2^k k!}\int_{-1}^{1}\left(\frac{d^k}{dx^k}\left(x^2-1\right)^k\right)q(x)\,dx \\ &= \left(\frac{2k+1}{2^{k+1}k!}\right)\int_{-1}^{1}\left(\frac{d^k}{dx^k}\left(x^2-1\right)^k\right)q(x)\,dx. \end{aligned} \tag{2.3}$$

Therefore, by (2.1) and (2.3), we obtain the following proposition.

Proposition 2.1 *For $q(x) \in \mathbb{P}_n$, let*

$$q(x) = \sum_{k=0}^{n} C_k P_k(x).$$

Then

$$C_k = \frac{2k+1}{2^{k+1}k!}\int_{-1}^{1}\left(\frac{d^k}{dx^k}\left(x^2-1\right)^k\right)q(x)\,dx.$$

Let us assume that $q(x) = x^n \in \mathbb{P}_n$.

From Proposition 2.1, we have

$$\begin{aligned} C_k &= \frac{2k+1}{2^{k+1}k!}\int_{-1}^{1}\left(\frac{d^k}{dx^k}\left(x^2-1\right)^k\right)x^n\,dx \\ &= \frac{2k+1}{2^{k+1}}(-1)^k\binom{n}{k}\int_{-1}^{1}\left(x^2-1\right)^k x^{n-k}\,dx \\ &= \frac{2k+1}{2^{k+1}}\binom{n}{k}\left(1+(-1)^{n-k}\right)\int_{0}^{1}\left(1-x^2\right)^k x^{n-k}\,dx. \end{aligned} \tag{2.4}$$

For $n-k \equiv 0 \pmod 2$, by (2.4), we get

$$\begin{aligned} C_k &= \frac{2k+1}{2^{k+1}}\binom{n}{k}\int_{0}^{1}(1-y)^k y^{\frac{n-k-1}{2}}\,dy \\ &= \frac{2k+1}{2^{k+1}}\binom{n}{k}B\left(k+1, \frac{n-k+1}{2}\right) \\ &= \frac{2k+1}{2^{k+1}}\binom{n}{k}\frac{\Gamma(k+1)\Gamma(\frac{n-k+1}{2})}{\Gamma(\frac{n+k+1}{2}+1)} \\ &= \frac{2k+1}{2^{k+1}}\binom{n}{k}\frac{k!\Gamma(\frac{n-k+1}{2})}{(\frac{n+k+1}{2})(\frac{n+k-1}{2})\cdots(\frac{n-k+1}{2})\Gamma(\frac{n-k+1}{2})} \\ &= \frac{2k+1}{2^{k+1}}\binom{n}{k}k!2^{k+1}\frac{(n-k)!(n+k+2)(n+k)\cdots(n-k+2)}{(n+k+2)!} \\ &= \frac{(2k+1)2^{k+1}}{(n+k+2)!}\times\frac{n!(\frac{n+k+2}{2})!}{(\frac{n-k}{2})!}. \end{aligned} \tag{2.5}$$

Here the beta function $B(x, y)$ is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \quad \bigl(\mathrm{Re}(x), \mathrm{Re}(y) > 0\bigr),$$

and it is well known that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt$ ($\mathrm{Re}(s) > 0$) is the gamma function.

By Proposition 2.1 and (2.5), we get

$$x^n = \sum_{0\le k\le n, n-k\equiv 0 \pmod 2} \frac{(2k+1)n!2^{k+1}(\frac{n+k+2}{2})!}{(n+k+2)!(\frac{n-k}{2})!}P_k(x). \tag{2.6}$$

From (1.5), we can easily derive the following equation (2.7):

$$x^n = \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}B_l(x) \quad (n \in \mathbb{Z}_+). \tag{2.7}$$

Therefore, by (2.6) and (2.7), we obtain the following Proposition 2.2.

Proposition 2.2 *For $n \in \mathbb{Z}_+$, we have*

$$\sum_{l=0}^{n}\frac{B_l(x)}{(n+1-l)!l!} = \sum_{0\le k\le n, n-k\equiv 0 \pmod 2}\frac{(2k+1)2^{k+1}(\frac{n+k+2}{2})!}{(n+k+2)!(\frac{n-k}{2})!}P_k(x).$$

Let us take $q(x) = B_n(x) \in \mathbb{P}_n$. By Proposition 2.1, we get

$$\begin{aligned}
C_k &= \frac{2k+1}{2^{k+1}k!}\int_{-1}^{1}\left(\frac{d^k}{dx^k}(x^2-1)^k\right)B_n(x)\,dx \\
&= \frac{(-1)^k(2k+1)}{2^{k+1}}\binom{n}{k}\int_{-1}^{1}(x^2-1)^k B_{n-k}(x)\,dx \\
&= \frac{(-1)^k(2k+1)}{2^{k+1}}\binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}B_{n-k-l}\int_{-1}^{1}(x^2-1)^k x^l\,dx \\
&= \frac{2k+1}{2^{k+1}}\binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}B_{n-k-l}\bigl(1+(-1)^l\bigr)\int_0^1(1-x^2)^k x^l\,dx.
\end{aligned} \tag{2.8}$$

For $l \in \mathbb{Z}_+$ with $l \equiv 0 \pmod 2$, we have

$$\begin{aligned}
C_k &= \frac{2k+1}{2^{k+1}}\binom{n}{k}\sum_{0\le l\le n-k, l \text{ is even}}\binom{n-k}{l}B_{n-k-l}\int_0^1(1-y)^k y^{\frac{l-1}{2}}\,dy \\
&= \frac{2k+1}{2^{k+1}}\binom{n}{k}\sum_{0\le l\le n-k, l \text{ is even}}\binom{n-k}{l}B_{n-k-l}\frac{\Gamma(k+1)\Gamma(\frac{l+1}{2})}{\Gamma(\frac{2k+l+1}{2}+1)} \\
&= (2k+1)2^{k+1}n!\sum_{0\le l\le n-k, l\equiv 0 \pmod 2}\frac{B_{n-k-l}}{(n-k-l)!}\times\frac{(\frac{2k+l+2}{2})!}{(2k+l+2)!(\frac{l}{2})!}.
\end{aligned} \tag{2.9}$$

In [14], we showed that

$$B_n(x) = \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k} E_k(x) + E_n(x) = \sum_{k=0, k\neq n-1}^{n} \binom{n}{k} B_{n-k} E_k(x). \tag{2.10}$$

Therefore, by Proposition 2.1, (2.9) and (2.10), we obtain the following theorem.

Theorem 2.3 *For $n \in \mathbb{Z}_+$, we have*

$$\frac{1}{n!} \sum_{k=0,k\neq n-1}^{n} \binom{n}{k} B_{n-k} E_k(x) = \sum_{k=0}^{n} \left(\sum_{0\le l \le n-k, l\equiv 0 \pmod 2} \frac{(2k+1)2^{k+1}(\frac{l+2k+2}{2})! B_{n-k-l}}{(n-k-l)!(l+2k+2)!(\frac{l}{2})!} \right) P_k(x).$$

By the same method of Theorem 2.3, we easily see that

$$\frac{E_n(x)}{n!} = \sum_{k=0}^{n} \left(\sum_{0\le l \le n-k, l\equiv 0 \pmod 2} \frac{(2k+1)2^{k+1}(\frac{l+2k+2}{2})! B_{n-k-l}}{(n-k-l)!(l+2k+2)!(\frac{l}{2})!} \right) P_k(x). \tag{2.11}$$

Let us take $q(x) = \sum_{k=0}^{n} B_k(x) B_{n-k}(x) \in \mathbb{P}_n$. Then we see that

$$\begin{aligned} &\sum_{k=0}^{n} B_k(x) B_{n-k}(x) \\ &\quad = (n+1) \sum_{k=0}^{n} \frac{\binom{n}{k}}{n-k+1} \left\{ \sum_{l=k}^{n} B_{l-k} B_{n-l} + B_{n-1-k} \right\} E_k(x) + \frac{(n^2-1)n}{12} E_{n-2}(x). \end{aligned} \tag{2.12}$$

The equation (2.12) was proved in [14].

By (2.12) and Proposition 2.2, we have

$$\begin{aligned} C_k = \frac{2k+1}{2^{k+1}k!} \Bigg\{ &(n+1) \sum_{l=0}^{n} \frac{\binom{n}{l}}{n-l+1} \left(\sum_{m=l}^{n} B_{m-l} B_{n-m} + B_{n-1-l} \right) \\ &\times \int_{-1}^{1} E_l(x) \left(\frac{d^k}{dx^k} (x^2-1)^k \right) dx \\ &+ \frac{(n^2-1)n}{12} \int_{-1}^{1} \left(\frac{d^k}{dx^k} (x^2-1)^k \right) E_{n-2}(x)\, dx \Bigg\}. \end{aligned} \tag{2.13}$$

Integrating by parts, we get

$$\begin{aligned} &\int_{-1}^{1} E_l(x) \left(\frac{d^k}{dx^k} (x^2-1)^k \right) dx \\ &\quad = \sum_{j=0}^{l} \binom{l}{j} E_{l-j} \int_{-1}^{1} x^j \frac{d^k}{dx^k} (x^2-1)^k \, dx \\ &\quad = \sum_{j=k}^{l} \binom{l}{j} E_{l-j} \frac{(-1)^k j!}{(j-k)!} \int_{-1}^{1} x^{j-k} (x^2-1)^k \, dx \\ &\quad = \sum_{j=k}^{l} \binom{l}{j} E_{l-j} \frac{j!}{(j-k)!} \left(1 + (-1)^{j-k}\right) \int_{0}^{1} x^{j-k} (-x^2+1)^k \, dx. \end{aligned} \tag{2.14}$$

Then we see that

$$\begin{aligned}
&\int_{-1}^{1} E_l(x)\left(\frac{d^k}{dx^k}\left(x^2-1\right)^k\right)dx \\
&= \sum_{j=k,j-k\equiv 0 \pmod 2}^{l} \binom{l}{j} E_{l-j}\frac{j!}{(j-k)!}\int_0^1 t^{\frac{j-k-1}{2}}(1-t)^k\,dt \\
&= \sum_{j=k,j-k\equiv 0 \pmod 2}^{l} \binom{l}{j} E_{l-j}\frac{j!}{(j-k)!}\frac{\Gamma(\frac{j-k+1}{2})\Gamma(k+1)}{\Gamma(\frac{j+k+1}{2}+1)} \\
&= \sum_{j=k,j-k\equiv 0 \pmod 2}^{l} \binom{l}{j} E_{l-j}\frac{j!k!}{(j-k)!}\times\frac{(j-k)!2^{2k+2}}{(j+k+2)!}\times\frac{(\frac{j+k+2}{2})!}{(\frac{j-k}{2})!} \\
&= \sum_{j=k,j-k\equiv 0 \pmod 2}^{l} \binom{l}{j} E_{l-j}\frac{j!k!2^{2k+2}}{(j+k+2)!}\times\frac{(\frac{j+k+2}{2})!}{(\frac{j-k}{2})!}.
\end{aligned} \tag{2.15}$$

It is easy to show that

$$\begin{aligned}
&\int_{-1}^{1}\left(\frac{d^k}{dx^k}\left(x^2-1\right)^k\right)E_{n-2}(x)\,dx \\
&= \sum_{j=0}^{n-2}\binom{n-2}{j}E_{n-2-j}\int_{-1}^{1}\left(\frac{d^k}{dx^k}\left(x^2-1\right)^k\right)x^j\,dx \\
&= \sum_{j=k}^{n-2}\binom{n-2}{j}E_{n-2-j}\left(1+(-1)^{j-k}\right)(-1)^k\frac{j!}{(j-k)!}\int_0^1\left(x^2-1\right)^k x^{j-k}\,dx \\
&= \sum_{k\le j\le n-2,j-k\equiv 0 \pmod 2}\binom{n-2}{j}E_{n-2-j}\frac{j!k!2^{2k+2}}{(j+k+2)!}\times\frac{(\frac{j+k+2}{2})!}{(\frac{j-k}{2})!}.
\end{aligned} \tag{2.16}$$

Therefore, by (2.13), (2.14), (2.15) and (2.16), we get

$$\begin{aligned}
C_k &= (2k+1)2^{k+1}\Bigg\{(n+1)\sum_{l=k}^{n}\frac{\binom{n}{k}}{n-l+1}\left(\sum_{m=l}^{n}B_{m-l}B_{n-m}+B_{n-1-l}\right) \\
&\quad\times\sum_{k\le j\le l,j-k\equiv 0 \pmod 2}\binom{l}{j}E_{l-j}\frac{j!}{(j+k+2)!}\times\frac{(\frac{j+k+2}{2})!}{(\frac{j-k}{2})!} \\
&\quad+\frac{(n^2-1)n}{12}\sum_{k\le j\le n-2,j-k\equiv 0 \pmod 2}\binom{n-2}{j}E_{n-2-j}\frac{j!}{(j+k+2)!}\times\frac{(\frac{j+k+2}{2})!}{(\frac{j-k}{2})!}\Bigg\}.
\end{aligned} \tag{2.17}$$

Therefore, by Proposition 2.1 and (2.17), we obtain the following theorem.

Theorem 2.4 *For $n \in \mathbb{Z}_+$, we have*

$$\begin{aligned}
&\sum_{k=0}^{n}B_k(x)B_{n-k}(x) \\
&= \sum_{k=0}^{n}(2k+1)2^{k+1}\Bigg\{(n+1)\sum_{l=k}^{n}\frac{\binom{n}{k}}{n-l+1}\left(\sum_{m=l}^{n}B_{m-l}B_{n-m}+B_{n-1-l}\right)
\end{aligned}$$

$$\times \sum_{k\le j\le l, j-k\equiv 0 \pmod 2} \binom{l}{j} E_{l-j} \frac{j!}{(j+k+2)!} \times \frac{(\frac{j+k+2}{2})!}{(\frac{j-k}{2})!}$$
$$\left. + \frac{(n^2-1)n}{12} \sum_{k\le j\le n-2, j-k\equiv 0 \pmod 2} \binom{n-2}{j} E_{n-2-j} \frac{j!}{(j+k+2)!} \times \frac{(\frac{j+k+2}{2})!}{(\frac{j-k}{2})!} \right\} P_k(x).$$

Remark 2.5 The extended Laguerre polynomials are given by

$$L_n^{\alpha}(x) = \sum_{r=0}^{n} \frac{(-1)^r}{r!} \binom{n+\alpha}{n-r} x^r \quad (\alpha > -1).$$

By the same method, we get

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \sum_{0\le l\le n-k, l\equiv 0 \pmod 2} \frac{(-1)^{k+l}(2k+1)2^{k+1}\binom{n+\alpha}{n-k-l}(\frac{l+2k+2}{2})!}{(l+2k+2)!(\frac{l}{2})!} P_k(x)$$

and

$$H_n(x) = \sum_{k=0}^{n} \sum_{0\le l\le n-k, l\equiv 0 \pmod 2} \frac{(2k+1)2^{2k+l+1} n! (\frac{l+2k+2}{2})! H_{n-k-l}}{(n-k-l)!(l+2k+2)!(\frac{l}{2})!} P_k(x),$$

where $H_n(x)$ is the Hermite polynomial of degree n (see [7]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this paper. They read and approved the final manuscript.

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References
1. Carlitz, L: Some integrals containing products of Legendre polynomials. Arch. Math. **12**, 334-340 (1961)
2. Carlitz, L: Some congruence properties of the Legendre polynomials. Math. Mag. **34**, 387-390 (1960/1961)
3. Carlitz, L: Some arithmetic properties of the Legendre polynomials. Acta Arith. **4**, 99-107 (1958)
4. Al-Salam, WA, Carlitz, L: Finite summation formulas and congruences for Legendre and Jacobi polynomials. Monatshefte Math. **62**, 108-118 (1958)
5. Carlitz, L: Some arithmetic properties of the Legendre polynomials. Proc. Camb. Philos. Soc. **53**, 265-268 (1957)
6. Carlitz, L: Note on Legendre polynomials. Bull. Calcutta Math. Soc. **46**, 93-95 (1954)
7. Carlitz, L: Congruence properties of the polynomials of Hermite, Laguerre and Legendre. Math. Z. **59**, 474-483 (1954)
8. Ding, D, Yang, J: Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials. Adv. Stud. Contemp. Math. **20**(1), 7-21 (2010)
9. Kim, G, Kim, B, Choi, J: The DC algorithm for computing sums of powers of consecutive integers and Bernoulli numbers. Adv. Stud. Contemp. Math. **17**(2), 137-145 (2008)
10. Kim, T: A note on q-Bernstein polynomials. Russ. J. Math. Phys. **18**(1), 73-82 (2011)
11. Kim, T: Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$. Russ. J. Math. Phys. **16**(4), 484-491 (2009)
12. Kim, T: Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$. Russ. J. Math. Phys. **16**(1), 93-96 (2009)

13. Kim, T: q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients. Russ. J. Math. Phys. **15**(1), 51-57 (2008)
14. Kim, T, Kim, DS, Dolgy, DV, Rim, SH: Some identities on the Euler numbers arising from Euler basis polynomials. ARS Comb. **109** (2013, in press)
15. Kudo, A: A congruence of generalized Bernoulli number for the character of the first kind. Adv. Stud. Contemp. Math. **2**, 1-8 (2000)
16. Leyendekkers, JV, Shannon, AG, Wong, GCK: Integer structure analysis of the product of adjacent integers and Euler's extension of Fermat's last theorem. Adv. Stud. Contemp. Math. **17**(2), 221-229 (2008)
17. Ozden, H, Cangul, IN, Simsek, Y: Remarks on q-Bernoulli numbers associated with Daehee numbers. Adv. Stud. Contemp. Math. **18**(1), 41-48 (2009)
18. Rim, S-H, Lee, SJ: Some identities on the twisted (h,q)-Genocchi numbers and polynomials associated with q-Bernstein polynomials. Int. J. Math. Math. Sci. **2011**, Art. ID 482840 (2011)
19. Rim, SH, Jin, JH, Moon, EJ, Lee, SJ: Some identities on the q-Genocchi polynomials of higher-order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$. Int. J. Math. Math. Sci. **2010**, Art. ID 860280 (2010)
20. Ryoo, CS: A note on the Frobenius-Euler polynomials. Proc. Jangjeon Math. Soc. **14**(4), 495-501 (2011)
21. Ryoo, CS: Some identities of the twisted q-Euler numbers and polynomials associated with q-Bernstein polynomials. Proc. Jangjeon Math. Soc. **14**(2), 239-248 (2011)
22. Ryoo, CS: Some relations between twisted q-Euler numbers and Bernstein polynomials. Adv. Stud. Contemp. Math. **21**(2), 217-223 (2011)
23. Simsek, Y, Acikgoz, M: A new generating function of (q–) Bernstein-type polynomials and their interpolation function. Abstr. Appl. Anal. **2010**, Art. ID 769095 (2010)
24. Simsek, Y: Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions. Adv. Stud. Contemp. Math. **16**(2), 251-278 (2008)
25. Simsek, Y: Complete sum of products of (h,q)-extension of Euler polynomials and numbers. J. Differ. Equ. Appl. **16**(11), 1331-1348 (2010)
26. Zhang, Z, Yang, H: Some closed formulas for generalized Bernoulli-Euler numbers and polynomials. Proc. Jangjeon Math. Soc. **11**(2), 191-198 (2008)
27. Acikgoz, M, Erdal, D, Araci, S: A new approach to q-Bernoulli numbers and q-Bernoulli polynomials related to q-Bernstein polynomials. Adv. Differ. Equ. **2010**, Art. ID 951764 (2010)
28. Araci, S, Erdal, D, Seo, J: A study on the fermionic p-adic q-integral representation on $\mathbb{Z}_p$ associated with weighted q-Bernstein and q-Genocchi polynomials. Abstr. Appl. Anal. **2011**, Art. ID 649248 (2011)
29. Bayad, A: Modular properties of elliptic Bernoulli and Euler functions. Adv. Stud. Contemp. Math. **20**(3), 389-401 (2010)
30. Bayad, A, Kim, T: Identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials. Adv. Stud. Contemp. Math. **20**(2), 247-253 (2010)
31. Bayad, A, Kim, T: Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials. Russ. J. Math. Phys. **18**(2), 133-143 (2011)

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