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Research Article

On the Long-Range Dependence of Fractional Brownian Motion

Ming Li

School of Information Science & Technology, East China Normal University, No. 500 Dong-Chuan Road, Shanghai 200241, China

Correspondence should be addressed to Ming Li; ming_lihk@yahoo.com

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This paper clarifies that the fractional Brownian motion, $B_H(t)$, is of long-range dependence (LRD) for the Hurst parameter $0 < H < 1$ except $H = 1/2$. In addition, we note that the fractional Brownian motion is positively correlated for $0 < H < 1$ except $H = 1/2$. Moreover, we present a theorem to state that the differential or integral of a random function, $X(t)$, may substantially change the statistical dependence of $X(t)$. One example is that the differential of $B_H(t)$, in the domain of generalized functions, changes the LRD of $B_H(t)$ to be of short-range dependence (SRD) when $0 < H < 0.5$.

1. Introduction

Fractional Brownian motion (fBm) is widely used [1–10]. Its theory and applications attract the interests of researchers in various fields, ranging from telecommunications to biomedical engineering; see, for example, [11–44], simply citing a few.

There is a set of statistical properties of fBm, such as nonstationarity and being nondifferentiable in the domain of ordinary functions [45]. Two properties, namely, nonstationarity and nondifferentiable property, are the basic properties of standard Brownian motion (Bm) [46–52], which is well known in the fields of time series as well as stochastic processes [53, 54]. As the substantial generalization of Bm, fBm has a property that Bm lacks, that is, its statistical dependence [1–4, 45]. The measure of the statistical dependence of fBm is characterized by the Hurst parameter $H \in (0, 1)$.

Note that the fBm for the Hurst parameter $H \in (0, 1)$ and $H \neq 1/2$ is of LRD [11, 12, 45, 55, 56]. In addition, fBm is positively correlated for $H \in (0, 1)$ but $H \neq 1/2$ [57]. However, the LRD property of fBm may be sometimes conservatively expressed. For example, the LRD property of fBm was restricted by $H \in (0.5, 1)$ as can be seen from [58, page 2341] and [59, page 708]. For this reason, it may be meaningful to clarify, which this paper aims at.

The remaining paper is organized as follows. In Section 2, we describe that the range of H for fBm to be of LRD is $H \in (0, 1)$ and $H \neq 1/2$. Discussions are in Section 3, which is followed by conclusions.

2. FBm Is LRD for $0 < H < 1$ except $H = 0.5$

In what follows, a random function in general is denoted by $X(t)$ for $t \in (0, \infty)$. We denote $B_H(t)$ for $t \in (0, \infty)$ as fBm with $H \in (0, 1)$.

Without generality losing, we assume that $X(t)$ is a random function with mean zero. The autocorrelation function (ACF) of $X(t)$ is, for $t, s \in (0, \infty)$, denoted by

$$C_{XX}(t, s) = E[X(t) X(s)]. \quad (1)$$

By LRD [1, 2], we mean that

$$\int_0^\infty C_{XX}(t, s)\, dt = \infty. \quad (2)$$

If

$$\int_0^\infty C_{XX}(t, s)\, dt < \infty, \quad (3)$$

$X(t)$ is of short-range dependence (SRD).

Denote by $S_{XX}(\omega, t)$ the power spectrum density function (PSD) of $X(t)$. Denote by F the operator of the Fourier transform. Then [60–64],

$$S_{XX}(\omega, t) = F[C_{XX}(t, s)]. \quad (4)$$

The LRD condition described in the frequency domain is expressed by

$$\lim_{\omega \to 0} S_{XX}(\omega, s) = \infty. \quad (5)$$

The above expression implies the property of $1/f$ noise regarding random functions with LRD [1–4, 65–70]. On the other hand, $X(t)$ is of SRD if

$$\lim_{\omega \to 0} S_{XX}(\omega, s) < \infty. \tag{6}$$

Let W^{-v} be the Weyl integral of order $v > 0$. Then, for random function $X(t)$; see, for example, [71–75], one has

$$W^{-v} X(t) = \frac{1}{\Gamma(v)} \int_t^{\infty} (u-t)^{v-1} X(u)\, du. \tag{7}$$

Thus, the fBm of the Weyl type is in the form:

$$\begin{aligned} B_H(t) - B_H(0) = {} & \frac{1}{\Gamma(H+1/2)} \\ & \times \left\{ \int_{-\infty}^{0} \left[(t-u)^{H-0.5} - (-u)^{H-0.5} \right] dB(u) \right. \\ & \left. + \int_0^t (t-u)^{H-0.5} dB(u) \right\}. \end{aligned} \tag{8}$$

Following [76], the PSD of the fBm of the Weyl type is expressed by

$$S_{B_H B_H}(\omega, t) = \frac{1}{|\omega|^{2H+1}} \left(1 - 2^{1-2H} \cos 2\omega t\right). \tag{9}$$

Therefore, we have the following theorem.

Theorem 1. *FBm is of LRD for $H \in (0,1)$ except $H = 1/2$.*

Proof. Because $\lim_{\omega \to 0} S_{B_H B_H}(\omega, t) = \infty$ for all $t > 0$ and for $H \in (0,1)$ except $H = 1/2$, the theorem holds. □

As a matter of fact, fBm reduces to the standard Bm if $H = 1/2$. The PSD of BM, see [11], is given by

$$S_{B_{1/2} B_{1/2}}(t, \omega) = \frac{1}{\omega^2} (1 - \cos 2\omega t). \tag{10}$$

Thus,

$$\lim_{\omega \to 0} S_{B_{1/2} B_{1/2}}(t, \omega) = 2t^2 \neq \infty. \tag{11}$$

From the theorem, we have the following corollary.

Corollary 2. *FBm is not SRD for $H \in (0,1)$.*

In passing, we mention that the ACF of $B_H(t)$ of the Weyl type is in the form:

$$\begin{aligned} C_{B_H B_H}(t, s) = {} & \frac{V_H}{(H+1/2)\Gamma(H+1/2)} \\ & \times \left[|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right], \end{aligned} \tag{12}$$

where V_H is the strength of $B_H(t)$. It is given by

$$V_H = \mathrm{Var}[B_H(1)] = \Gamma(1-2H) \frac{\cos \pi H}{\pi H}. \tag{13}$$

Following [57, page 4], we have the following remark.

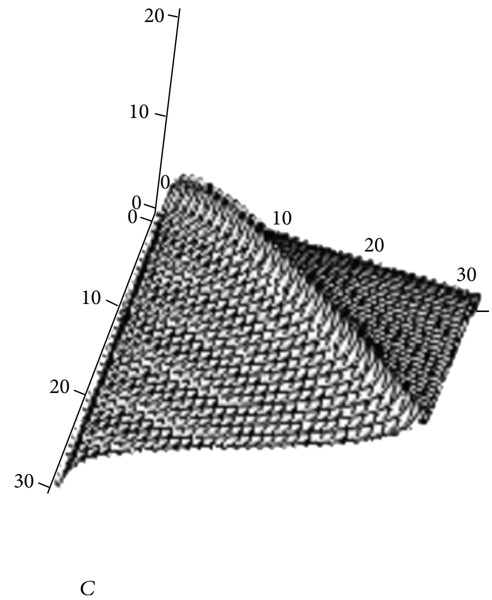

Figure 1: ACF of fBm for $t, s = 0, 1, \ldots, 30$ and $H = 0.2$.

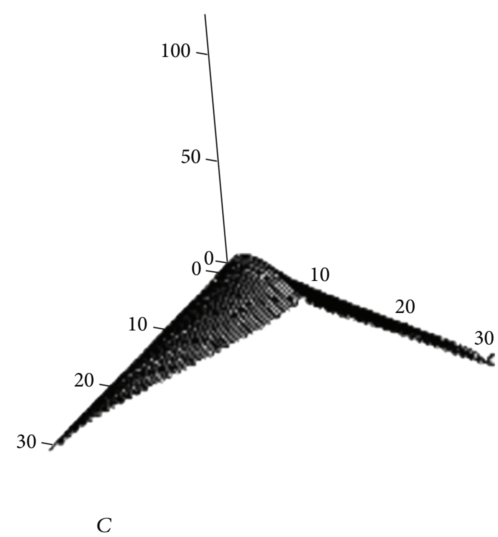

Figure 2: ACF of fBm for $t, s = 0, 1, \ldots, 30$ and $H = 0.4$.

Remark 3. The ACF of fBm is positively correlated for $H \in (0,1)$ except $H = 1/2$. That is, $C_{B_H B_H}(t, s) \geq 0$ for $t, s \in (0, \infty)$. Figures 1 and 2 indicate the plots of $C_{B_H B_H}(t, s)$ for $t, s = 0, 1, \ldots, 30$ with $H = 0.2$ and 0.4, respectively.

3. Discussions

Let $G_H(t)$ be the fractional Gaussian noise (fGn). Then, in the domain of generalized functions over the Schwartz space of test functions [45], we write

$$G_H(t) = \frac{dB_H(t)}{dt}. \tag{14}$$

Denote by $C_{G_H G_H}(\tau, s)$ the ACF of $G_H(t)$. Then, for $\varepsilon > 0$ [19, 45], one has

$$C_{G_H G_H}(\tau;\varepsilon) = \frac{V_H \varepsilon^{2H-2}}{2} \times \left[\left(\frac{|\tau|}{\varepsilon} + 1 \right)^{2H} + \left| \frac{|\tau|}{\varepsilon} - 1 \right|^{2H} - 2 \left| \frac{\tau}{\varepsilon} \right|^{2H} \right]. \tag{15}$$

From the contents in Section 2, we have the following theorem.

Theorem 4. *Let $X(t)$ be a random function. Then, the statistical dependence of $dX(t)/dt$ may substantially differ from that of $X(t)$, where the differential is in the domain of generalized functions.*

Proof. To prove the theorem, we only need an example to show it. Let $X(t) = B_H(t)$. Then, $dX(t)/dt = G_H(t)$. It is well known that fGn is LRD when $H \in (0.5, 1)$ as $C_{G_H G_H}$ is nonintegrable if $H \in (0.5, 1)$. On the other hand, for $H \in (0, 0.5)$, the integral of $C_{G_H G_H}$ is zero. Hence, fGn is SRD when $H \in (0, 0.5)$. In passing, we note that $C_{G_H G_H}(\tau;\varepsilon)$ changes its sign and becomes negative for some τ proportional to ε in this parameter domain [45, page 434]. Since $B_H(t)$ is LRD for $H \in (0, 1)$ except $H = 1/2$, the statistical dependence of $G_H(t)$ substantially differs from that of $B_H(t)$. This completes the proof. □

From Theorem 4, we immediately obtain the corollary below.

Corollary 5. *Let $X(t)$ be a random function. Then, the statistical dependence of $D^{-1}X(t)$ may substantially differ from that of $X(t)$, where D^{-1} is the integral operator of order one.*

Proof. Let $X(t) = G_H(t)$. Then, $D^{-1}X(t) = B_H(t)$. Since $B_H(t)$ is LRD for $H \in (0, 0.5)$ while $G_H(t)$ is SRD when $H \in (0, 0.5)$, one sees that the statistical dependence of $D^{-1}X(t)$ substantially differs from that of $X(t)$. Thus, Corollary 5 results. □

4. Conclusions

We have clarified that fBm is LRD and positively correlated for $H \in (0, 1)$ except $H = 1/2$. In addition, we have proved that the differential or integral of a random function may considerably change its statistical dependence.

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