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Research Article

Periodic Solutions and *S*-Asymptotically Periodic Solutions to Fractional Evolution Equations

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This paper deals with the existence and uniqueness of periodic solutions, S-asymptotically periodic solutions, and other types of bounded solutions for some fractional evolution equations with the Weyl-Liouville fractional derivative defined for periodic functions. Applying Fourier transform we give reasonable definitions of mild solutions. Then we accurately estimate the spectral radius of resolvent operator and obtain some existence and uniqueness results.

1. Introduction

It is well known that fractional order differential equations provide an excellent setting for capturing in a model framework real-world problems in many disciplines, such as chemistry, physics, engineering, and biology [1–4]. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives; see the monographs of Podlubny [2], Kilbas et al. [1], Zhou [4], and the recent papers [5–8] and the references therein.

As a matter of fact, periodic motion is a very important and special phenomenon not only in natural science but also in social science such as climate, food supplement, insecticide population, and sustainable development.

In our previous paper [9], we discussed a class periodic boundary value problem to fractional evolution equations and obtained the existence and uniqueness results for positive mild solutions. However, since the fractional derivatives provide the description of memory property, the solution of periodic boundary value problems cannot be periodically extended to the time $t \in \mathbb{R}$.

On the other hand, several authors have showed that for some fractional order systems the solutions do not show any periodic behavior if the lower terminal of the derivative is finite; see [10–14]. Let $\beta \in (0,\infty) \setminus \mathbb{N}$ and $n = [\beta] + 1$. If $u : (0,\infty) \to \mathbb{R}$ is a nonconstant ω-periodic function of class C^n, [10, 11] tell us that $D^\beta u$ cannot be ω-periodic function, where D^β is understood as one of the fractional derivatives (Caputo, Riemann-Liouville or Grunwald-Letnikov) with the lower terminal finite. Nevertheless, the authors also point out in [11] that fractional order derivative of other form, such as Weyl-Liouville fractional derivatives defined for periodic function [15], perhaps preserves periodicity. As indicated in [2, 15], the Weyl-Liouville derivative coincides with the Caputo, Riemann-Liouville or Grunwald-Letnikov derivative with lower limit $-\infty$, which is denoted by D_+^β. There is essential difference between finity and infinity. As in, for instance, [2, 11], ${}^C D_0^\beta \sin s = s^{1-\beta} E_{2,2-\beta}(-s^2)$ for $\beta \in (0,1)$, while $D_+^\beta \sin s = \sin(s + (\pi/2)\beta)$ for $\beta \in (-1,\infty)$, where ${}^C D_0^\beta$ is the Caputo fractional derivative with order β and the lower terminal 0, and $E_{a,b}(z)$ denotes the two parameters Mittag-Leffler functions. It is obvious that the Weyl-Liouville fractional derivative is suitable for the study of periodic solutions to differential equations.

In real life, many phenomena are not strictly periodic; therefore many other generalized periodic cases need to be studied, such as almost periodic, asymptotically almost periodic, S-asymptotically periodic, asymptotically periodic, pseudoperiodic, and pseudo-almost periodic. As the advantages of fractional derivatives, such as the memorability and heredity, many papers concern these types of solutions for fractional differential equations. Since S-asymptotically periodic functions in Banach space were first studied by Henríquez et al. [16], there are some papers about S-asymptotically periodic solutions for fractional equations; one can refer to [17–19]. For almost periodic solutions, asymptotically almost periodic and other types of bounded solutions to fractional differential equations, one can refer to [13, 17, 20–22]. Ponce [22] studied the existence and uniqueness of bounded solutions for semilinear fractional integrodifferential equation

$$D_+^\alpha u(t) = Au(t) + \int_{-\infty}^{t} a(t-s) Au(s)\,ds + f(t,u(t)), \quad t \in \mathbb{R}, \tag{1}$$

where A is a closed linear operator defined on a Banach space X, D_+^α is Weyl fractional derivative of order $\alpha > 0$ with the lower limit $-\infty$, $a \in L^1(\mathbb{R}^+)$ is a scalar-valued kernel, and $f : \mathbb{R} \times X \to X$ satisfies some Lipschitz type conditions. Assume that A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\ge0}$ which is uniformly integrable. The mild solutions of (1) was given by

$$u(t) = \int_{-\infty}^{t} S_\alpha(t-s) f(s,u(s))\,ds, \quad t \in \mathbb{R}. \tag{2}$$

By Banach contraction principle, existence and uniqueness results of almost periodic, asymptotically almost periodic and other types of bounded solutions are established. In addition, Lizama and Poblete [21] gave some sufficient conditions ensuring the existence and uniqueness of bounded solutions to a fractional semilinear equation of order $1 < \alpha < 2$.

In this paper, we study the fractional evolution equations in an ordered Banach space X

$$D_+^\alpha u(t) + Au(t) = f(t,u(t)), \quad t \in \mathbb{R}, \tag{3}$$

where D_+^α is the Weyl-Liouville fractional derivative of order $\alpha \in (0,1)$ which is defined in Section 2 and $-A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\ge0}$. Applying Fourier transform, we give reasonable definitions of mild solutions of (3). Then the existence and uniqueness results for the corresponding linear fractional evolution equations are established, and the spectral radius of resolvent operator is accurately estimated. Finally, some sufficient conditions are established for the existence and uniqueness of periodic solutions, S-asymptotically periodic solutions, and other types of bounded solutions when $f : \mathbb{R} \times X \to X$ satisfies some ordered or Lipschitz conditions.

Compared with the earlier related existence results which appeared in [21–23], there are at least four essence differences: (i) the order of the fractional derivative is $\alpha \in (0,1)$; (ii) in many papers concerning bounded solutions for differential equations, such as [21, 23], the solution is defined by the limit of the solution for the limit equation, without strict proofs of the solution for the given equation possessing that form, but, in this paper, we apply the Fourier transform to (3) itself and obtain the expression of the solution; naturally, we may define expression as a mild solution under suitable conditions; besides, the mild solution in this paper is expressed by the semigroups generated by $-A$, which are more convenient than some resolvent families applied in [22]; (iii) the spectral radius of resolvent operator is accurately estimated; (iv) the conditions on f contain some ordered relations, and the associated method is monotone iterative technique.

The paper is organized as follows. Section 2 provides the definitions and preliminary results to be used in the article. In Section 3, the existence and uniqueness results for the linear equations are obtained. In Section 4, we consider the existence and uniqueness of the mild solutions for (3). In Section 5, we also give two examples to apply the abstract results.

2. Preliminaries

This section is devoted to some preliminary facts needed in the sequel. Let X be an ordered Banach space with norm $\|\cdot\|$ and partial order $\le$, whose positive cone $P := \{y \in X \mid y \ge \theta\}$ (θ is the zero element of X) is normal with normal constant N. Denote by $\mathscr{L}(X)$ the space of all linear bounded operator on Banach space X, with the norm $\|\cdot\|_{\mathscr{L}(X)}$. The notation $C_b(X)$ stands for the Banach space of all bounded continuous functions from $\mathbb{R}$ into X equipped with the sup norm $\|\cdot\|_\infty$; that is,

$$C_b(X) := \left\{u : \mathbb{R} \longrightarrow X \mid u \text{ is continuous, } \|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\| \right\}. \tag{4}$$

For $u, v \in C_b(X)$, $u \le v$ if $u(t) \le v(t)$ for all $t \in \mathbb{R}$. $P_\omega(X)$ stands for the subspace of $C_b(X)$ consisting of all X-valued continuous ω-periodic functions. Set

$$SAP_\omega(X) = \{f \in C_b(X) \mid \text{there exists } \omega > 0 \text{ such that } \|f(t+\omega) - f(t)\| \longrightarrow 0 \text{ as } t \longrightarrow \infty\}. \tag{5}$$

These functions in $SAP_\omega(X)$ are called S-asymptotically ω-periodic (see [16]). We note that $P_\omega(X)$ and $SAP_\omega(X)$ are Banach spaces (see [16]), and $P_\omega(X) \subset SAP_\omega(X)$.

A function $f \in C_b(\mathbb{X})$, if for any $\epsilon > 0$ there is a real number $\omega = \omega(\epsilon) > 0$ and $\tau = \tau(\varepsilon)$ in arbitrary interval of length $\omega(\varepsilon)$ such that $\|f(t+\tau) - f(t)\|_\infty < \epsilon$ for all $t \in \mathbb{R}$ is said to be almost periodic (in the sense of Bohr). We denote by $AP(X)$ the set of all these functions. The space of almost automorphic functions (resp., compact almost automorphic functions) will be written as $AA(X)$ (resp., $AA_c(X)$). The bounded continuous function $f \in AA(X)$ (resp., $f \in AA_c(X)$) $\Leftrightarrow$ for every sequence $\{s_n'\}_{n\in\mathbb{N}}$ there is a subsequence $\{s_n\}_{n\in\mathbb{N}} \subset \{s_n'\}_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\to\infty} f(t+s_n)$ and $f(t) = \lim_{n\to\infty} g(t-s_n)$ for each $t \in \mathbb{R}$ (resp., uniformly on compact subsets of $\mathbb{R}$). Clearly the compact almost automorphic function f above is continuous on $\mathbb{R}$.

For convenience, we set $C_0(X) := \{g \in C_b(X) \mid \lim_{|t|\to\infty}\|g(t)\| = 0\}$. We regard the direct sum of $P_\omega(X)$ and $C_0(X)$ as the space of asymptotically periodic functions $AP_\omega(X)$, the direct sum of $AP(X)$ and $C_0(X)$ as the space of asymptotically almost periodic functions $AAP(X)$, the direct sum of $AA_c(X)$ and $C_0(X)$ as the space of asymptotically compact almost automorphic functions, and the direct sum of $AA(X)$ and $C_0(X)$ as the space of asymptotically almost automorphic functions.

Then we set

$$P_0(X) := \left\{ f \in C_b(X) \mid \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|f(s)\|\, ds = 0 \right\} \quad (6)$$

and regard the direct sum of $P_\omega(X)$ and $P_0(X)$ as the space of pseudoperiodic functions, the direct sum of $AP(X)$ and $P_0(X)$ as the space of pseudo-almost periodic functions, the direct sum of $AA_c(X)$ and $P_0(X)$ as the space of pseudocompact almost automorphic functions, and the direct sum of $AA(X)$ and $P_0(X)$ as the space of pseudo-almost automorphic functions.

The relationship of the above different classes of subspaces is presented in [21, p. 805]:

$$\begin{array}{ccccc} AA(X) & \subset & AAA(X) & \subset & PAA(X) \\ \cup & & \cup & & \cup \\ AA_c(X) & \subset & AAA_c(X) & \subset & PAA_c(X) \\ \cup & & \cup & & \cup \\ AP(X) & \subset & AAP(X) & \subset & PAP(X) \\ \cup & & \cup & & \cup \\ P_\omega(X) & \subset & AP_\omega(X) & \subset & PP_\omega(X) \\ & & \cap & & \\ & & SAP_\omega(X). & & \end{array} \quad (7)$$

Denote by $\mathscr{M}(\mathbb{R}, X)$ or simply $\mathscr{M}(X)$ the following function spaces:

$$\begin{aligned} \mathscr{M}(X) := \{ & P_\omega(X), AP(X), AA_c(X), AA(X), AP_\omega(X), \\ & AAP(X), AAA_c(X), AAA(X), PP_\omega(X), PAP(X), \\ & PAA_c(X), PAA(X), SAP_\omega(X) \} \end{aligned} \quad (8)$$

and $\mathscr{M}(\mathbb{R} \times X, X)$ the space of all functions $g : \mathbb{R} \times X \to X$ satisfying $g(\cdot, v) \in \mathscr{M}(X)$ uniformly for each v in any bounded subset V of X. For fixed $\Omega(X) \in \mathscr{M}(X) \cup \{C_0(X), P_0(X)\}$, then for $u \in \Omega(X)$ and $f \in C_b(\mathbb{R}\times X, X)$, the following conditions could ensure that $f(\cdot, u(\cdot)) \in \Omega(X)$:

(A_1) $f(t, \cdot)$ is uniformly continuous with respect to t on $\mathbb{R}$ for each bounded subset of X. More precisely, given $\epsilon > 0$ and $K \subset X$, there exists $\delta > 0$ such that $u_1, u_2 \in K$ and $\|u_1 - u_2\| < \delta$ imply that $\|f(t, u_1) - f(t, u_2)\| < \epsilon$;

(A_2) for all bounded subset $U \subset X$, $\{f(t, u) \mid t \in \mathbb{R},\ u \in U\}$ is bounded;

(A_3) if $f = f_1 + f_2$, where $f_1 \in \Phi(X) = \{AA_c(X), AA(X), P_\omega(X), AP(X)\}$ and $f_2 \in \{C_0(X), P_0\} \setminus \{0\}$, $f_1(t, \cdot)$ is uniformly continuous with respect to t on $\mathbb{R}$ for each bounded subset of X.

Table 1

$\Omega(X)$	(A_1)	(A_2)	(A_3)
P_w			
AP			
AA_c	•		
AA	•		
AP_ω	•		
AAP	•		
AAA_c	•		•
AAA	•		•
PAP		•	•
PP_ω		•	•
PAA	•	•	•
PAA_c	•	•	•
SAP_ω	•		
C_0	•		
P_0	•		

Then the results shown in Table 1 hold; see [21, Remark 3.4 and p. 810].

Corollary 1 (see [21, Corollary 3.9]). *Let $\Omega(X) \in \mathscr{M}(X)$ and $f \in \Omega(\mathbb{R} \times X, X) \in \mathscr{M}(\mathbb{R} \times X, X)$ be given and fixed. Assume that there exists a constant $L_f > 0$ such that*

$$\|f(t, u_1) - f(t, u_2)\| \le L_f \|u_1 - u_2\| \quad (9)$$

for all $t \in \mathbb{R}$ and $u_1, u_2 \in X$. Let $u \in \Omega(X)$; then $f(\cdot, u(\cdot)) \in \Omega(X)$.

Now we recall the definitions of some fractional derivatives and integrals which are used in this paper (see [15]).

Definition 2. Let $f \in L^p(\mathbb{R}/2\pi\mathbb{Z})$ $(1 \le p < \infty)$ be periodic with period 2π and such that its integral over a period vanishes. The Weyl fractional integral of order α is defined as

$$(I_{\pm}^{\alpha} f)(t) = \frac{1}{2\pi} \int_0^{2\pi} \Psi_{\pm}^{\alpha}(t - s) f(s)\, dy, \quad (10)$$

where

$$\Psi_{\pm}^{\alpha}(t) = \sum_{k=-\infty, k\neq 0}^{\infty} \frac{e^{ikt}}{(\pm ik)^{\alpha}} \quad (11)$$

for $0 < \alpha < 1$.

The above Weyl definition is accordant with the Riemann-Liouville definition [1]

$$\left(I_{+}^{\alpha} f\right)(t)=\frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t}(t-s)^{\alpha-1} f(s) ds,$$
$$\left(I_{-}^{\alpha} f\right)(t)=\frac{1}{\Gamma(\alpha)} \int_{t}^{\infty}(s-t)^{\alpha-1} f(s) ds, \tag{12}$$

for 2π periodic functions whose integral over a periodic vanishes.

The Weyl-Liouville fractional derivative is defined as

$$\left(D_{\pm}^{\alpha} f\right)(t)=\pm \frac{d}{dt}\left(I_{\pm}^{1-\alpha} f\right)(t) \tag{13}$$

for $0<\alpha<1$.

It is shown that the Weyl-Liouville derivative ($0<\alpha<1$)

$$\left(D_{+}^{\alpha} f\right)(t)=\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{-\infty}^{t} \frac{f(s)}{(t-s)^{\alpha}} ds \tag{14}$$

coincides with the Caputo, Riemann-Liouville, or Grunwald-Letnikov derivative with lower limit $-\infty$ [2]. It is known that $D_{\pm}^{\alpha+\beta}=D_{\pm}^{\alpha}(D_{\pm}^{\beta})$ for any $\alpha, \beta \in \mathbb{R}$, where $D_{\pm}^{0}=\mathrm{Id}$ denotes the identity operator and $(-1)^{n} D_{-}^{n}=D_{+}^{n}=d^{n}/dt^{n}$ holds with $n \in \mathbb{N}$; see [3]. For example, for the function,

$$D_{+}^{-\alpha} e^{\lambda t}=\lambda^{-\alpha} e^{\lambda t},$$
$$D_{+}^{\alpha} e^{\lambda t}=\lambda^{\alpha} e^{\lambda t}, \tag{15}$$
$$\operatorname{Re} \lambda \geq 0.$$

Denote by $\mathscr{F} f$ the Fourier transform of f; that is,

$$(\mathscr{F} f)(\lambda)=\int_{-\infty}^{\infty} e^{i\lambda t} f(t) dt \tag{16}$$

for $\lambda \in \mathbb{R}$ and $f \in L^{1}(\mathbb{R}, X)$. Thus $(\mathscr{F} D_{+}^{\alpha} f)(\lambda)=(-i\lambda)^{\alpha}(\mathscr{F} f)(\lambda)$; see [1, Remark 2.11].

Let us recall the definitions and properties of operator semigroups; for details see [24]. Assume that $-A$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. If there are $M \geq 0$ and $\nu \in \mathbb{R}$ such that $\|T(t)\|_{\mathscr{L}(X)} \leq M e^{\nu t}$, then

$$(\lambda I+A)^{-1} x=\int_{0}^{\infty} e^{-\lambda t} T(t) x dt, \quad \operatorname{Re} \lambda>\nu,\ x \in X. \tag{17}$$

A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called exponentially stable if there exist constants $M>0$ and $\delta>0$ such that

$$\|T(t)\|_{\mathscr{L}(X)} \leq M e^{-\delta t}, \quad t \geq 0. \tag{18}$$

The growth bound of the semigroup $\{S(t)\}_{t\geq 0}$ is defined as

$$\nu_{0}=\inf \left\{\rho \in \mathbb{R} \mid \exists M_{1}>0,\ \|S(t)\|_{\mathscr{L}(X)} \leq M_{1} e^{\rho t},\ \forall t \geq 0\right\}. \tag{19}$$

Furthermore, ν_0 could also be expressed by

$$\nu_{0}=\limsup_{t \rightarrow+\infty} \frac{\ln \|R(t)\|_{\mathscr{L}(X)}}{t}. \tag{20}$$

For C_0-semigroup $\{T(t)\}_{t\geq 0}$, if there exists a constant $M>0$ such that

$$\|T(t)\|_{\mathscr{L}(X)} \leq M, \quad t \geq 0, \tag{21}$$

then it is called uniformly bounded. A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called compact if $T(t)$ is compact for $t>0$. The positive C_0-semigroup $\{T(t)\}_{t\geq 0}$ is C_0-semigroup $\{T(t)\}_{t\geq 0}$ satisfying $T(t)y \geq \theta$ for all $y \geq \theta$ and $t \geq 0$. For the positive operators semigroup, one can refer to [25].

In the following part, we shall recall the definitions and properties of Mittag-Leffler functions (see [1]). Note

$$E_{\alpha}(t)=\sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(\alpha k+1)},$$
$$E_{\alpha,\alpha}(t)=\sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(\alpha(k+1))}, \tag{22}$$
$$t \in \mathbb{C}.$$

These functions have the following properties for $\alpha \in (0,1)$ and $t \in \mathbb{R}$.

Lemma 3.

(1) *[26, Lemma 2.2]* $E_{\alpha}(t), E_{\alpha,\alpha}(t)>0$.

(2) *[27, p. 2004]* $(E_{\alpha}(t))'=(1/\alpha) E_{\alpha,\alpha}(t)$.

(3) *([28, Lemma 2.2], [29, Eq. (14)])* $\lim_{t\rightarrow-\infty} E_{\alpha}(t)=\lim_{t\rightarrow-\infty} E_{\alpha,\alpha}(t)=0$.

Considering the probability density function

$$\zeta_{\alpha}(\theta)=\frac{1}{\alpha} \theta^{-1-1/\alpha} \rho_{\alpha}\left(\theta^{-1/\alpha}\right),$$
$$\rho_{\alpha}(\theta)=\frac{1}{\pi} \sum_{n=0}^{\infty}(-1)^{n-1} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \tag{23}$$
$$\theta \in (0,\infty),$$

the following results hold.

Remark 4.

(1) [30, p. 212] $\zeta_{\alpha}(\theta) \geq 0$ for $\theta \in (0,\infty)$, $\int_{0}^{\infty} \theta^{\nu} \zeta_{\alpha}(\theta) d\theta=\Gamma(1+\nu)/\Gamma(1+\alpha\nu)$ for $\nu \in (-1,+\infty)$.

(2) [31, p. 90, p. 112, p. 168] $\int_{0}^{\infty}(\alpha/t^{\alpha+1}) \zeta_{\alpha}(1/t^{\alpha}) e^{-\lambda t} dt=\int_{0}^{\infty} \rho_{\alpha}(\theta) e^{-\lambda t} dt=e^{-\lambda^{\alpha}}$ for $\operatorname{Re} \lambda \geq 0$.

(3) [30, p. 212] $\int_{0}^{\infty} e^{-z\theta} \zeta_{\alpha}(\theta) d\theta=E_{\alpha}(-z)$ for $z \in \mathbb{C}$.

(4) [30, p. 212] $\int_{0}^{\infty} e^{-z\theta} \theta \zeta_{\alpha}(\theta) d\theta=(1/\alpha) E_{\alpha,\alpha}(-z)$ for $z \in \mathbb{C}$.

Let

$$V(t)=\alpha \int_{0}^{\infty} \theta \zeta_{\alpha}(\theta) T(t^{\alpha}\theta) d\theta, \quad t \geq 0, \tag{24}$$

where $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup. We have the following results.

Lemma 5.

(1) *Assume that $\{T(t)\}_{t\geq 0}$ is a uniformly bounded C_0 semigroup and satisfies (21). Then, for any fixed $t \geq 0$, $V(t)$ is a linear and bounded operator; that is, for any $x \in X$, we get*

$$\|V(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|. \quad (25)$$

(2) *If $\{T(t)\}_{t\geq 0}$ is a C_0 semigroup, then $\{V(t)\}_{t\geq 0}$ is strongly continuous.*

(3) *If $\{T(t)\}_{t\geq 0}$ is a positive C_0-semigroup, then $V(t)$ is positive for $t \geq 0$.*

(4) *If $\{T(t)\}_{t\geq 0}$ is exponentially stable and satisfies (18), then*

$$\|V(t)\|_{\mathscr{L}(X)} \leq ME_{\alpha,\alpha}(-\delta t^{\alpha}), \quad \text{for } t \geq 0. \quad (26)$$

Proof. For the proof of (1)-(2), we can see [32, Lemma 2.9]. By Remark 4(1), we obtain (3). In view of Remark 4(4), we have

$$\begin{aligned}\|V(t)\|_{\mathscr{L}(X)} &= \left\|\alpha\int_0^{\infty}\theta\zeta_{\alpha}(\theta)T(t^{\alpha}\theta)\,d\theta\right\|_{\mathscr{L}(X)} \\ &\leq \alpha\int_0^{\infty}\theta\zeta_{\alpha}(\theta)Me^{-\delta t^{\alpha}\theta}d\theta \\ &= ME_{\alpha,\alpha}(-\delta t^{\alpha}).\end{aligned} \quad (27)$$

Then (4) holds. □

Next, we consider the following linear abstract fractional evolution equation:

$$D_+^{\alpha}u(t) + Au(t) = h(t), \quad t \in \mathbb{R}, \quad (28)$$

where $-A : D(A) \subset X \rightarrow X$ generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of operators on Banach space X and $h : \mathbb{R} \rightarrow X$ is continuous.

For convenience, we assume the following condition:

(H) $u \in C(\mathbb{R}, X)$, $\int_{-\infty}^{t} g_{1-\alpha}(t-s)x(s)ds \in C^1(\mathbb{R}, X)$, $u(t) \in D(A)$ for $t \in \mathbb{R}$, $Au \in L^1(\mathbb{R}, X)$ and u satisfies (28), where

$$g_{1-\alpha}(t) = \begin{cases}\dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}, & t > 0, \\ 0, & t \leq 0.\end{cases} \quad (29)$$

Lemma 6. *Assume that $-A$ generates an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$. If $u : \mathbb{R} \rightarrow X$ is a function satisfying equation (28) and assumption (H), then u satisfies the following integral equation:*

$$u(t) = \int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)h(s)\,ds, \quad t \in \mathbb{R}, \quad (30)$$

where V is defined by (24).

Proof. Applying Fourier transform to (28), we get $(-i\lambda)^{\alpha}(\mathscr{F}u)(\lambda) + A(\mathscr{F}u)(\lambda) = (\mathscr{F}h)(\lambda)$ for $\lambda \in \mathbb{R}$. In view of (17) and Remark 4(2), we have

$$\begin{aligned}(\mathscr{F}u)(\lambda) &= ((-i\lambda)^{\alpha}I + A)^{-1}(\mathscr{F}h)(\lambda) \\ &= \int_0^{\infty}e^{-(-i\lambda)^{\alpha}t}T(t)(\mathscr{F}h)(\lambda)\,dt \\ &= \int_0^{\infty}\int_{-\infty}^{\infty}\alpha t^{\alpha-1}e^{-(-i\lambda t)^{\alpha}}T(t^{\alpha})h(s)e^{i\lambda s}ds\,dt \\ &= \int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\alpha^2}{\tau^{2\alpha+1}}t^{\alpha-1}\zeta_{\alpha}(\tau^{-\alpha})e^{i\lambda t}T\left(\frac{t^{\alpha}}{\tau^{\alpha}}\right)h(s) \\ &\cdot e^{i\lambda s}d\tau\,ds\,dt = \int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\alpha\tau t^{\alpha-1}\zeta_{\alpha}(\tau) \\ &\cdot T(t^{\alpha}\tau)h(s)e^{i\lambda(t+s)}d\tau\,ds\,dt \\ &= \int_{-\infty}^{\infty}e^{i\lambda t}\int_{-\infty}^{t}(t-s)^{\alpha-1}h(s) \\ &\cdot\left(\int_0^{\infty}\alpha\tau\zeta_{\alpha}(\tau)T((t-s)^{\alpha}\tau)\,d\tau\right)ds\,dt \\ &= \int_{-\infty}^{\infty}e^{i\lambda t}\left(\int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)h(s)\,ds\right)dt,\end{aligned} \quad (31)$$

where $(-i\lambda)^{\alpha}I \in \rho(-A)$. By the uniqueness of Fourier transform, we deduce that the assertion of lemma holds. This completes the proof. □

Definition 7. A function $u : \mathbb{R} \rightarrow X$ is said to be a mild solution of problem (28) if

$$u(t) = \int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)h(s)\,ds, \quad t \in \mathbb{R}, \quad (32)$$

where V is given by (24).

3. Results for Linear Equations

Theorem 8. *If $\{T(t)\}_{t\geq 0}$ is exponentially stable and satisfies (18), h belongs to one of $\mathscr{M}(X)$, and*

$$(Rh)(t) = \int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)h(s)\,ds, \quad (33)$$

where V is defined by (24); then Rh and h belong to the same space.

Proof. $P_\omega(X)$: If $h \in P_\omega(X)$, then

$$\begin{aligned}(Rh)(s+\omega) &= \int_{-\infty}^{s+\omega} (s+\omega-\tau)^{\alpha-1} V(s+\omega-\tau) h(\tau)\, d\tau \\ &= \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) h(s+\omega)\, d\tau \\ &= \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) h(s)\, ds = (Rh)(t).\end{aligned} \tag{34}$$

Therefore, $Rh \in P_\omega(X)$.

$AP(X)$: by the hypotheses, for any $\epsilon > 0$, we can find a real number $l = l(\epsilon) > 0$ for any interval of length $l(\varepsilon)$ and there exists a number $m = m(\varepsilon)$ in this interval such that $\|h(t+m) - h(t)\|_\infty < \epsilon$ for all $t \in \mathbb{R}$. From Lemmas 3, 5(4), and $E_\alpha(0) = 1$, we have

$$\begin{aligned}\sup_{t\in\mathbb{R}} \|(Rh)(t+m) - (Rh)(t)\| &= \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} \cdot V(t-s)[h(s+m) - h(s)]\, ds \right\| \le \sup_{t\in\mathbb{R}} \|h(t+m) \\ &- h(m)\| \left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s)\, ds \right\|_{\mathscr{L}(X)} \\ &\le M\epsilon \int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t-s)^\alpha)\, ds \\ &= \frac{M\epsilon}{\delta} E_\alpha(-\delta(t-s)^\alpha)\Big|_{-\infty}^{t} = \frac{M\epsilon}{\delta},\end{aligned} \tag{35}$$

and Rh and h are all almost periodic.

$AA_c(X)$: since $h \in AA_c(X)$, there is $\{t_n\}_{n\in\mathbb{N}}$ and $v \in C_b(X)$ such that $h(t+t_n) \to v(t)$ and $v(t-s_n) \to h(t)$ as $t \to \infty$, uniformly on compact subsets of $\mathbb{R}$.

$$\begin{aligned}(Rh)(t+t_n) &= \int_{-\infty}^{t+s_n} (t+t_n-s)^{\alpha-1} V(t+t_n-s) h(s)\, ds \\ &= \int_{-\infty}^{t} (t-\tau)^{\alpha-1} V(t-\tau) h(\tau+t_n)\, ds;\end{aligned} \tag{36}$$

therefore by Lebesgue dominated convergence theorem, when $n \to \infty$, we get $(Rh)(t+s_n) \to z(t) = \int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)v(s)ds$ for all $t \in \mathbb{R}$.

Furthermore, for a compact set $K = [-a, a]$ and $\epsilon > 0$, by Lemmas 3 and 5, we choose $M_\epsilon > 0$ and $N_\epsilon \in \mathbb{N}$ such that

$$\begin{aligned}\int_{K_\epsilon}^{\infty} s^{\alpha-1} \|V(s)\|_{\mathscr{L}(X)}\, ds &\le -\frac{M}{\delta} E_\alpha(-\delta s^\alpha)\Big|_{M_\epsilon}^{\infty} \\ &= \frac{M}{\delta} E_\alpha(-\delta L_\epsilon^\alpha) \le \epsilon, \\ \|h(\tau+s_n) - v(\tau)\| &\le \epsilon, \quad n \ge \mathbb{N}_\epsilon,\ \tau \in [-M, M],\end{aligned} \tag{37}$$

where $M = M_\epsilon + a$. For $t \in K$, by Lemmas 3 and 5, we estimate

$$\begin{aligned}\|(Rh)(t+s_n) - z(t)\| &\le \int_{-\infty}^{t} (t-s)^{\alpha-1} \cdot \|V(t-s)\|_{\mathscr{L}(X)} \|h(s+s_n) - v(s)\|\, ds \\ &\le \int_{-\infty}^{-L} (t-s)^{\alpha-1} \|V(t-s)\|_{\mathscr{L}(X)} \cdot \|h(s+s_n) - v(s)\|\, ds + \int_{-L}^{t} (t-s)^{\alpha-1} \\ &\cdot \|V(t-s)\|_{\mathscr{L}(X)} \|h(s+s_n) - v(s)\|\, ds \le (\|h\|_\infty \\ &+ \|v\|_\infty) \int_{t+L}^{\infty} s^{\alpha-1} \|V(s)\|_{\mathscr{L}(X)}\, ds \\ &+ \epsilon \int_{0}^{\infty} s^{\alpha-1} \|V(s)\|_{\mathscr{L}(X)}\, ds \le \epsilon(\|h\|_\infty + \|v\|_\infty) \\ &+ \frac{\epsilon M}{\delta},\end{aligned} \tag{38}$$

which implies that that the convergence is irrelevant to $t \in K$. Similarly, we can prove that $z(t-s_n) \to (Rh)(t)$ if $n \to \infty$ uniformly for t on compact subsets of $\mathbb{R}$. The case of the space $AA_c(X)$ is similar too.

$AA(X)$: set $h \in AA(X)$. For all sequence $\{t'_n\} \subset \mathbb{R}$, there is $\{t_n\}$ of $\{t'_n\}$ such that

$$\begin{aligned}\lim_{n\to\infty} h(t+t_n) &= v(t), \quad t \in \mathbb{R}, \\ \lim_{n\to\infty} v(t-t_n) &= h(t), \quad t \in \mathbb{R}.\end{aligned} \tag{39}$$

Since

$$\|(Rh)(t+t_n)\| \le \frac{M}{\delta} \|h\|_\infty, \tag{40}$$

by (36) and Lemmas 3 and 5, for any $t \ge s \in \mathbb{R}$, we get $V(t-s)h(s+s_n) \to V(t-s)v(s)$ as $n \to \infty$. For any $t \in \mathbb{R}$ Lebesgue dominated convergence theorem implies that $(Rh)(t+s_n) \to z(t) = \int_{-\infty}^{t}(t-s)^{\alpha-1}V(t-s)v(s)ds$ as $n \to \infty$. It is similar that

$$z(t-s_n) \longrightarrow (Rh)(t), \quad n \longrightarrow \infty,\ t \in \mathbb{R}. \tag{41}$$

$SAP_\omega(X)$: assume that $h \in SAP_\omega(X)$. For any $\forall\epsilon > 0$, there exists $T_\epsilon > 0$ such that $\|h(t+\omega) - h(t)\| < \epsilon$ for $t \ge T_\epsilon$. Then, by Lemmas 3 and 5, we get

$$\begin{aligned}\|(Rh)(t+\omega) - (Rh)(t)\| &= \left\| \int_{-\infty}^{t+\omega} (t+\omega-s)^{\alpha-1} \cdot V(t+\omega-s) h(s)\, ds - \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) \cdot h(s)\, ds \right\| \le \int_{-\infty}^{t} \|(t-\tau)^{\alpha-1} V(t-\tau) \\ &\cdot [h(\tau+\omega) - h(\tau)]\|\, d\tau \le 2\|h\|_\infty \int_{-\infty}^{L_\epsilon} (t \\ &- \tau)^{\alpha-1} \|V(t-\tau)\|_{\mathscr{L}(X)}\, d\tau + \epsilon \int_{T_\epsilon}^{t} (t-\tau)^{\alpha-1}\end{aligned}$$

$$\cdot \|V(t-\tau)\|_{\mathscr{L}(X)}\,d\tau \le 2\|h\|_\infty M\int_{-\infty}^{L_\epsilon}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\delta(t-\tau)^\alpha\right)d\tau + \epsilon M\int_{T_\epsilon}^{t}(t-\tau)^{\alpha-1}\cdot E_{\alpha,\alpha}\left(-\delta(t-\tau)^\alpha\right)d\tau = \frac{2\|h\|_\infty M}{\delta}\cdot E_\alpha\left(-\delta(t-\tau)^\alpha\right)\Big|_{-\infty}^{T_\epsilon} + \frac{\epsilon M}{\delta}E_\alpha\left(-\delta(t-\tau)^\alpha\right)\Big|_{T_\epsilon}^{t} = \frac{2\|h\|_\infty M-\epsilon M}{\delta}E_\alpha\left(-\delta\left(t-T_\epsilon\right)^\alpha\right)+\frac{\epsilon M}{\delta}, \tag{42}$$

for $t \ge T_\epsilon$. It follows that $\|(Rh)(t+\omega)-(Rh)(t)\| \to 0$ as $t\to\infty$. Thus, $Rh \in SAP_\omega(X)$.

We next consider the asymptotic property of the solutions. For $w \in C_0(X)$ and $\epsilon > 0$, we have $\|w(s)\| < \epsilon$ for some $T>0$ and $|s|>T$. Then Lemmas 3 and 5 imply that

$$\begin{aligned}\|(Rw)(t)\| &\le \int_{-\infty}^{T}(t-s)^{\alpha-1}\|V(t-s)\|_{\mathscr{L}(X)}\|w(s)\|\,ds + \epsilon\int_{T}^{t}(t-s)^{\alpha-1}\|V(t-s)\|_{\mathscr{L}(X)}\,ds \\ &\le \frac{\|w\|_\infty M}{\delta}E_\alpha\left(-\delta(t-T)^\alpha\right) + \frac{\epsilon M}{\delta}\left(1-E_\alpha\left(-\delta(t-T)^\alpha\right)\right),\end{aligned} \tag{43}$$

which implies that $(Rw)(t) \to 0$ as $t\to\infty$. Naturally, we can also get the results for the spaces $AP_\omega(X)$, $AAP(X)$, $AAA_c(X)$, and $AAA(X)$.

Set $w \in P_0(X)$. For $L>0$ we get

$$\frac{1}{2L}\int_{-L}^{L}\|(Rw)(t)\|\,dt \le \frac{1}{2L}\int_{-L}^{L}\left(\int_{-\infty}^{t}(t-s)^{\alpha-1}\cdot\|V(t-s)\|_{\mathscr{L}(X)}\|w(s)\|\,ds\right)dt \le \frac{1}{2L}\cdot\int_{-L}^{L}\left(\int_{0}^{\infty}s^{\alpha-1}\|V(s)\|_{\mathscr{L}(X)}\|w(t-s)\|\,ds\right)dt = \int_{0}^{\infty}s^{\alpha-1}\|V(s)\|_{\mathscr{L}(X)}\left(\frac{1}{2L}\cdot\int_{-L}^{L}\|w(t-s)\|\,dt\right)ds. \tag{44}$$

We can find that the set $P_0(X)$ is translation-invariant and get

$$\frac{1}{2L}\int_{-L}^{L}\|(Rw)(t)\|\,dt \longrightarrow 0 \quad \text{as } L\longrightarrow\infty, \tag{45}$$

by Lebesgue dominated convergence theorem. Then $PP_\omega(X)$, $PAP(X)$, $PAA_c(X)$, and $PAA(X)$ have the maximal regularity property under the convolution defined by (33). □

Theorem 9. *Assume that $h \in \Omega(X) \in \mathscr{M}(X)$; $-A$ generates an exponentially stable C_0-semigroup $\{T(t)\}_{t\ge0}$ and satisfies (18). Then linear fractional evolution equation (28) possesses a unique mild solution $u := Rh \in \Omega(X)$, and*

$$\|Rh\|_\infty \le \frac{M}{\delta}\|h\|_\infty. \tag{46}$$

Proof. In view of Definition 7 and Theorem 8, Rh is a mild solution of (28) and $Rh \in \Omega(X)$. By Lemmas 3 and 5, we have

$$\begin{aligned}\|(Rh)(t)\| &\le \int_{-\infty}^{t}(t-s)^{\alpha-1}\|V(t-s)h(s)\|\,ds \\ &\le M\|h\|_\infty\int_{-\infty}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\delta(t-s)^\alpha\right)ds \\ &= \frac{M}{\delta}\|h\|_\infty E_\alpha\left(-\delta(t-s)^\alpha\right)\Big|_{-\infty}^{t} = \frac{M}{\delta}\|h\|_\infty,\end{aligned} \tag{47}$$

where V is defined by (24). Then we obtain

$$\|Rh\|_\infty \le \frac{M}{\delta}\|h\|_\infty. \tag{48}$$

□

Remark 10. Equation (46) is an optimal estimation. In fact, for $X=\mathbb{R}$, the periodic solution of equation $D_+^\alpha x+\gamma x=1$ is $x=1/\gamma$.

Corollary 11. *Let $h \in \Omega(X) \in \mathscr{M}(X)$. Assume that $-A$ generates a uniformly bounded C_0-semigroup $\{T(t)\}_{t\ge0}$ and satisfies (21). If $\operatorname{Re}\lambda > 0$, then linear fractional evolution equation*

$$D_+^\alpha u(t)+Au(t)+\lambda u(t)=h(t), \quad t\in\mathbb{R} \tag{49}$$

has a unique mild solution $u := R_\lambda h$, and

$$\left\|R_\lambda h\right\|_\infty \le \frac{M}{\operatorname{Re}\lambda}\|h\|_\infty. \tag{50}$$

Proof. $-(A+\lambda I)$ generates a C_0-semigroup $\{S(t)\}_{t\ge0}$, and $S(t)=e^{-\lambda t}T(t)$. Then $\|S(t)\|_{\mathscr{L}(X)} = e^{-\operatorname{Re}\lambda t}\|T(t)\|_{\mathscr{L}(X)} \le Me^{-\operatorname{Re}\lambda t}$, so $S(t)$ is exponentially stable for $\operatorname{Re}\lambda > 0$. The conclusion follows by Theorem 9. □

Theorem 12. *Let $h \in \Omega(X) \in \mathscr{M}(X)$. Assume that $-A$ generates an exponentially stable C_0-semigroup $\{T(t)\}_{t\ge0}$, that is, the growth bound*

$$\nu_0 = \limsup_{t\to+\infty}\frac{\ln\|T(t)\|_{\mathscr{L}(X)}}{t} < 0. \tag{51}$$

Then linear fractional evolution equation (28) has a unique mild solution $u := Rh \in \Omega(X)$, $R:\Omega(X)\to\Omega(X)$ is bounded linear, and spectral radius $r(R)\le 1/|\nu_0|$.

Proof. By Theorem 9 and Lemma 5(1), we obtain that (28) has a unique mild solution $u := Rh \in \Omega(X)$, and $R:\Omega(X)\to\Omega(X)$ is bounded linear.

For all $\nu \in (0, |\nu_0|)$, there exists $M_1 \geq 1$ such that

$$\|T(t)\|_{\mathscr{L}(X)} \leq M_1 e^{-\nu t}, \quad t \geq 0. \tag{52}$$

Define a new norm $|\cdot|$ in X as

$$|u| = \sup_{t\geq 0} \left\| e^{\nu t} T(t) u \right\|. \tag{53}$$

Since $\|u\| \leq |u| \leq M_1 \|u\|$, then $|\cdot|$ is equivalent to $\|\cdot\|$. The norm of $T(t)$ in $X_0 := (X, |\cdot|)$ is denoted by $|T(t)|_{\mathscr{L}(X_0)}$. Then, for $t \geq 0$, we have

$$\begin{aligned} |T(t) u| &= \sup_{s\geq 0} \left\| e^{\nu s} T(s) T(t) u \right\| \\ &= e^{-\nu t} \sup_{s\geq 0} \left\| e^{\nu(s+t)} T(s+t) u \right\| \\ &= e^{-\nu t} \sup_{\eta \geq t} \left\| e^{\nu \eta} T(\eta) u \right\| \leq e^{-\nu t} |u|. \end{aligned} \tag{54}$$

This implies that $|T(t)|_{\mathscr{L}(X_0)} \leq e^{-\nu t}$. Remark 4(4) implies that

$$\begin{aligned} |V(t)|_{\mathscr{L}(X_0)} &= \left| \alpha \int_0^\infty \theta \zeta_\alpha(\theta) T(t^\alpha \theta) \, d\theta \right|_{\mathscr{L}(X_0)} \\ &\leq \alpha \int_0^\infty \theta \zeta_\alpha(\theta) e^{-\nu t^\alpha \theta} d\theta = E_{\alpha,\alpha}(-\nu t^\alpha), \end{aligned} \tag{55}$$

where V is defined by (24). Since $h \in \Omega(X)$, we have $|h|_\infty := \sup_{t\in\mathbb{R}} |h(t)| < \infty$. By (55) and Lemma 3, we have

$$\begin{aligned} |(Rh)(t)| &\leq \left| \int_{-\infty}^t (t-s)^{\alpha-1} V(t-s) h(s) \, ds \right| \\ &\leq |h|_\infty \int_{-\infty}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\nu(t-s)^\alpha) \, ds \\ &= \frac{|h|_\infty}{\nu} E_\alpha(-\nu(t-s)^\alpha) \Big|_{-\infty}^t = \frac{|h|_\infty}{\nu}, \\ &\qquad \text{for } t \geq 0, \end{aligned} \tag{56}$$

which implies that $|Rh|_\infty \leq |h|_\infty / \nu$. Then $|R|_{\mathscr{L}(X_0)} \leq 1/\nu$, and spectral radius $r(R) \leq 1/\nu$. Since ν is any number in $(0, |\nu_0|)$, we obtain $r(R) \leq 1/|\nu_0|$. □

Remark 13. Similarly to [9], if $\{T(t)\}_{t\geq 0}$ is a compact and positive analytic semigroup and P is a regeneration cone, then $\nu_0 = -\lambda_1$, where λ_1 is the first eigenvalue of A.

Corollary 14. *Assume that $h \in \Omega(X)$ and positive cone $P \in X$ is a regeneration cone, compact and positive C_0-semigroup $\{T(t)\}_{t\geq 0}$ is generated by $-A$, whose first eigenvalue*

$$\lambda_1 = \inf \{\operatorname{Re} \lambda \mid \lambda \in \sigma(A)\} > 0. \tag{57}$$

Then (28) possesses a unique mild solution $u := Rh \in \Omega(X)$, $R : \Omega(X) \to \Omega(X)$ is positive and bounded linear, and the corresponding spectral radius $r(R) = 1/\lambda_1$.

Proof. The proof is similar to that in [9]. □

4. Results for Nonlinear Equations

Theorem 15. *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N. Assume that $-A$ generates a positive C_0-semigroup $\{T(t)\}_{t\geq 0}$, $\Omega(X) \in \mathscr{M}(X)$, $f \in \Omega(\mathbb{R} \times X, X) \in \mathscr{M}(\mathbb{R} \times X, X)$ and satisfies the results shown in Table 1, $f(t, \theta) \geq \theta$ for $\forall t \in \mathbb{R}$, and the following assumptions hold:*

(H_1) *for $\kappa > 0$, there is $C = C(\kappa) > 0$ such that*

$$f(t, u_2) - f(t, u_1) \geq -C(u_2 - u_1), \tag{58}$$

where $t \in \mathbb{R}$, $\theta \leq u_1 \leq u_2$, $\|u_1\|$, $\|u_2\| \leq \kappa$;

(H_2) *there exists $L < -\nu_0$, such that*

$$f(t, u_2) - f(t, u_1) \leq L(u_2 - u_1), \tag{59}$$

where $t \in \mathbb{R}$, $\theta \leq u_1 \leq u_2$.

Then (3) has a unique positive mild solution $u \in \Omega(X)$.

Proof. Let $h_0(t) = f(t, \theta)$; then $h_0 \in \Omega(X)$, $h_0 \geq \theta$. Next we consider linear equation

$$D_+^\alpha u(t) + (A - LI) u(t) = h_0(t), \quad t \in \mathbb{R}. \tag{60}$$

We know that a positive C_0-semigroup $e^{Lt}T(t)$ could be generated by $-(A - LI)$, whose growth bound is $L + \nu_0 < 0$. From Theorem 12, linear equation (60) has a unique positive mild solution $w_0 \in \Omega(X)$.

If $\kappa_0 = N\|w_0\|_\infty + 1$ and C is the constant in (H_1), without loss of generality, we may assume that $C > \max\{\nu_0, -L\}$. In the following part, we consider linear equation

$$D_+^\alpha u(t) + (A + CI) u(t) = h(t), \quad t \in \mathbb{R}. \tag{61}$$

$-(A + CI)$ is the generator of a positive C_0-semigroup $T_1(t) = e^{-Ct}T(t)$ with growth bound $-C + \nu_0 < 0$. From Theorem 12, for $h \in \Omega(X)$, linear equation (61) has a unique mild solution $u := Q_1 h$, and $Q_1 : \Omega(X) \to \Omega(X)$ is a positive bounded linear operator, and spectral radius $r(Q_1) \leq 1/(C - \nu_0)$.

Set $F(u) = f(t, u) + Cu$; then by Lemma 5(2), Corollary 1, and Theorem 8, it follows that $F(\theta) = h_0 \geq \theta$ and $Q_1 F : \Omega(X) \to \Omega(X)$ is continuous. By (H_1), F is incremental on $[\theta, w_0]$. Set $v_0 \equiv \theta$; we can construct the sequences

$$\begin{aligned} v_n &= (Q_1 \circ F)(v_{n-1}), \\ w_n &= (Q_1 \circ F)(w_{n-1}), \\ &\qquad n = 1, 2, \ldots. \end{aligned} \tag{62}$$

By (61), we have that $Q_1(h_0 + Lw_0 + Cw_0)$ is another mild solution of (60). Since the mild solution of (60) is unique, we have

$$w_0 = Q_1(h_0 + Lw_0 + Cw_0). \tag{63}$$

Let $u_1 = \theta, u_2 = w_0(t)$ in (H_2); then

$$f(t, w_0) \le h_0(t) + Lw_0(t), \quad \theta \le F(\theta) \le F(w_0) \le h_0 + Lw_0 + Cw_0. \tag{64}$$

By (63) and (64) and the definition and the positivity of Q_1, we obtain

$$Q_1\theta = \theta = v_0 \le v_1 \le w_1 \le w_0. \tag{65}$$

Since $Q_1 \circ F$ is an increasing operator on $[\theta, w_0]$, in view of (62) we can show that

$$\theta \le v_1 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0. \tag{66}$$

Therefore,

$$\begin{aligned}\theta \le w_n - v_n &= Q_1(F(w_{n-1}) - F(v_{n-1})) \\ &= Q_1(f(\cdot, w_{n-1}) - f(\cdot, v_{n-1}) + C(w_{n-1} - v_{n-1})) \\ &\le (C+L) Q_1(w_{n-1} - v_{n-1}).\end{aligned} \tag{67}$$

By induction,

$$\begin{aligned}\theta \le w_n - v_n &\le (C+L)^n Q_1^n(w_0 - v_0) \\ &= (C+L)^n Q_1^n(w_0).\end{aligned} \tag{68}$$

Since the cone P is normal, then we get

$$\begin{aligned}\|w_n - v_n\|_\infty &\le N(C+L)^n \|Q_1^n(w_0)\|_\infty \\ &\le N(C+L)^n \|Q_1^n\|_{\mathscr{L}(X)} \|w_0\|_\infty.\end{aligned} \tag{69}$$

Moreover, $0 < C+L < C-v_0$ for some $\varepsilon > 0$, so it follows that $C+L+\varepsilon < C-v_0$. By the Gelfand formula, $\lim_{n\to\infty}\|Q_1^n\|_{\mathscr{L}(X)}^{1/n} = r(Q_1) \le 1/(C-v_0)$. Then there exists N_0 such that $\|Q_1^n\|_{\mathscr{L}(X)} \le 1/(C+L+\varepsilon)^n$ for $n \ge N_0$. Equation (69) implies that

$$\|w_n - v_n\|_\infty \le N\|w_0\|_\infty \left(\frac{C+L}{C+L+\varepsilon}\right)^n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{70}$$

Combining (66) and (70), by the nested interval method, there is a unique $u^* \in \bigcap_{n=1}^{\infty}[v_n, w_n]$ such that

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} w_n = u^*. \tag{71}$$

Since operator $Q_1 \circ F$ is continuous, by (62) we have

$$u^* = (Q_1 \circ F)(u^*). \tag{72}$$

It follows from the definition of Q_1 and (66) that u^* is a positive mild solution of (61) for $h(t) = f(t, u^*(t)) + Cu^*(t)$. Hence, u^* is a positive mild solution of (3).

Finally, we prove the uniqueness. If u_1, u_2 are the positive mild solutions of (3), substitute u_1 and u_2 for w_0, respectively; then $w_n = (Q_1 \circ F)(u_i) = u_i$ $(i = 1, 2)$. Equation (70) implies that

$$\|u_i - v_n\|_\infty \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \ i = 1, 2. \tag{73}$$

Thus, $u_1 = u_2 = \lim_{n\to\infty} v_n$; (3) has a unique positive mild solution $u \in \Omega(X)$. □

From Theorem 15 and Remark 13, we obtain the following results.

Corollary 16. *Let X be an ordered Banach space, whose positive cone P is a regeneration cone. Assume that $-A$ generates a compact and positive C_0-semigroup $\{T(t)\}_{t\ge0}$, $\Omega(X) \in \mathscr{M}(X)$, $f \in \Omega(\mathbb{R} \times X, X) \in \mathscr{M}(\mathbb{R} \times X, X)$, and satisfies the results shown in Table 1, $f(t, \theta) \ge \theta$ for $\forall t \in \mathbb{R}$, (H_1) and the following conditions are satisfied:*

$(H_2)'$ *there exists $L < \lambda_1$ (where λ_1 is the first eigenvalue of A) such that*

$$f(t, u_2) - f(t, u_1) \le L(u_2 - u_1), \tag{74}$$

for any $t \in \mathbb{R}$, $\theta \le u_1 \le u_2$.

Then (3) has a unique positive mild solution $u \in \Omega(X)$.

Theorem 17. *Assume that $-A$ generates an exponentially stable C_0-semigroup $\{T(t)\}_{t\ge0}$ and satisfies (18), $\Omega(X) \in \mathscr{M}(X)$, $f \in \Omega(\mathbb{R} \times X, X) \in \mathscr{M}(\mathbb{R} \times X, X)$, and satisfies the results shown in Table 1. If the following condition is satisfied:*

(H_3) *$f(t, u)$ is Lipschitz continuous with respect to u; that is, there exists a constant $L \ge 0$ such that*

$$\|f(t, u_2) - f(t, u_1)\| \le L\|u_2 - u_1\|, \tag{75}$$

for all $t \in \mathbb{R}$ and $u_1, u_2 \in X$;

then (3) has a unique mild solution $u \in \Omega(X)$ for $\delta > ML$.

Proof. We define operator Q by

$$(Qu)(t) := \int_{-\infty}^{t} (t-s)^{\alpha-1} V(t-s) f(s, u(s))\, ds, \quad t \in \mathbb{R}. \tag{76}$$

Form Lemma 5(2), Corollary 1, and Theorem 8, it follows that $Q : \Omega(X) \to \Omega(X)$ is continuous.

By (H_3) and Lemmas 3 and 5, for all $t \in \mathbb{R}$, $u_1, u_2 \in \Omega(X)$, we get

$$\begin{aligned}\|(Qu_2)(t) - (Qu_1)(t)\| &\le \int_{-\infty}^{t} (t-s)^{\alpha-1} \\ &\quad \cdot \|V(t-s)[f(s, u_2(s)) - f(s, u_1(s))]\|\, ds \\ &\le ML\|u_2 - u_1\|_\infty \int_{-\infty}^{t} (t-s)^{\alpha-1} \\ &\quad \cdot E_{\alpha,\alpha}(-\delta(t-s)^\alpha)\, ds = \frac{ML}{\delta}\|u_2 - u_1\|_\infty \\ &\quad \cdot E_\alpha(-\delta(t-s)^\alpha)\Big|_{-\infty}^{t} = \frac{ML}{\delta}\|u_2 - u_1\|_\infty.\end{aligned} \tag{77}$$

Thus,

$$\|Qu_2 - Qu_1\|_\infty \le \frac{ML}{\delta}\|u_2 - u_1\|_\infty. \tag{78}$$

When $\delta > ML$, Q is a contraction in $\Omega(X)$. Using Banach fixed point theorem we have that Q has a unique fixed point in $\Omega(X)$. This completes the proof. □

Corollary 18. *Assume that $-A$ generates a uniformly bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ and satisfies (21), $\Omega(X) \in \mathscr{M}(X)$, $f \in \Omega(\mathbb{R}\times X, X) \in \mathscr{M}(\mathbb{R}\times X, X)$, and satisfies the results shown in Table 1. If f satisfies (H_3), then (3) has a unique mild solution $u \in \Omega(X)$ for $\operatorname{Re}\lambda > ML$.*

Proof. Similar to the proof in Corollary 11, we know that $-(A + \lambda I)$ generates an exponentially stable C_0-semigroup $\{S(t)\}_{t\geq 0}$. By Theorem 17, (3) possesses a unique mild solution for $\operatorname{Re}\lambda > ML$. □

Theorem 19. *Assume that $-A$ generates an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$ and satisfies (18), $\Omega(X) \in \mathscr{M}(X)$, $f \in \Omega(\mathbb{R} \times X, X) \in \mathscr{M}(\mathbb{R} \times X, X)$, and satisfies the results shown in Table 1. Assume that the following conditions are satisfied:*

(H_4) $f(t,u)$ is local Lipschitz continuous in u: for all $r > 0$, there exists $L(r) > 0$ such that

$$\left\|f\left(t,u_2\right) - f\left(t,u_1\right)\right\| \leq L\left(r\right)\left\|u_2 - u_1\right\|, \tag{79}$$

for $t \in \mathbb{R}$, $\|u_1\|$, $\|u_2\| \leq r$.

Denote $f_0 = f(t,\theta)$, then for $\delta > ML(r)+(M/r)\|f_0\|_\infty$ (3) has only one mild solution in $B(\theta,r) = \{u \in \Omega(X) \mid \|u\|_\infty < r\}$.

Proof. Let $F(u) = f(t,u)$, we know that the mild solution of (3) is the fixed point of $R \circ F$ in $B(\theta,r)$.

For any $u \in B(\theta,r)$, by (46) and (H_4), we have

$$\begin{aligned}\|R(F(u))\|_\infty &\leq \|R(F(\theta))\|_\infty + \|R(F(u) - F(\theta))\|_\infty \\ &\leq \frac{M\left\|f_0\right\|_\infty}{\delta} + \frac{M\left\|F(u) - F(\theta)\right\|_\infty}{\delta} \\ &\leq \frac{M\left\|f_0\right\|_\infty + ML(r)\,r}{\delta} < r,\end{aligned} \tag{80}$$

where $\delta > ML(r)+(M/r)\|f_0\|_\infty$. By Lemma 5(2), Corollary 1, and Theorem 8, we obtain that $R \circ F : B(\theta,r) \to B(\theta,r)$ is continuous. On the other hand, for $\forall u_1, u_2 \in B$, by (H_4), we get

$$\begin{aligned}&\left\|R\left(F\left(u_2\right)\right) - R\left(F\left(u_1\right)\right)\right\|_\infty \leq \left\|R\left(F\left(u_2\right) - F\left(u_1\right)\right)\right\|_\infty \\ &\quad\leq \frac{ML(r)}{\delta}\left\|u_2 - u_1\right\|_\infty.\end{aligned} \tag{81}$$

For $\delta > ML(r) + (M/r)\|f_0\|_\infty$, we have $ML(r)/\delta < 1$. Thus Q is a contraction in $\Omega(X)$.

Banach fixed point theorem implies that there is only one $\tilde{u} \in B(\theta,r)$ such that $R(F(\tilde{u})) = \tilde{u}$. Therefore (3) has only one mild solution in $B(\theta,r)$. □

Corollary 20. *Assume that $-A$ generates a uniformly bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ and satisfies (21), $\Omega(X) \in \mathscr{M}(X)$, $f \in \Omega(\mathbb{R} \times X, X) \in \mathscr{M}(\mathbb{R} \times X, X)$, and satisfies the results shown in Table 1 and (H_4). Set $f_0 = f(t,\theta)$; then (3) has only one mild solution in $B(\theta,r) = \{u \in \Omega(X) \mid \|u\|_\infty < r\}$ for $\operatorname{Re}\lambda > ML(r) + (M/r)\|f_0\|_\infty$.*

Proof. Similar to the proof in Corollary 11, we know that $-(A+\lambda I)$ generates an exponentially stable C_0-semigroup $\{S(t)\}_{t\geq 0}$. By Theorem 19, equation (3) has only one mild solution in $B(\theta,r)$ for $\operatorname{Re}\lambda > ML(r) + (M/r)\|f_0\|_\infty$. □

5. Examples

Example 1. Let $X = C_0(\overline{\Omega})$; we discuss briefly the existence of mild solutions of the following fractional parabolic partial differential equation:

$$\begin{aligned}&D_+^{1/2}u - \Delta u = g\left(x,t,u\left(x,t\right)\right), \quad (x,t) \in \Omega \times \mathbb{R}^N, \\ &\qquad u|_{\partial\Omega} = 0,\end{aligned} \tag{82}$$

where $D_+^{1/2}$ is Liouville fractional partial derivative of order $1/2$ with the lower limit $-\infty$, $\Omega \subset \mathbb{R}^N$ is bounded and boundary $\partial\Omega$ is sufficiently smooth, and Δ is the Laplace operator.

Theorem 21. *Let $g(\cdot,t,u(\cdot,t)) \in \Omega(X) \in \mathscr{M}(X)$, and $g(x,t,0) \geq 0$. Assume that g has continuous partial derivatives for u in arbitrary bounded domain, and the supremum of $g_u(x,t,u)$ is smaller than λ_1, where λ_1 is the first eigenvalue of Laplace operator $-\Delta$ with $u|_{\partial\Omega} = 0$. Then (82) possesses only one positive mild solution $u(\cdot,t) \in \Omega(X)$.*

Proof. Let $K = C_0^+(\overline{\Omega}) := \{f \in C(\overline{\Omega}, \mathbb{R}_+) \mid f|_{\partial\Omega} = 0\}$; then K is a positive cone in X. Define operator A in X as follows:

$$D\left(A\right) = \left\{u \mid u \in X,\ \Delta u \in X\right\}, \quad Au = -\Delta u. \tag{83}$$

In view of [24], $-A$ generates a compact and analytic semigroup $\{T(t)\}_{t\geq 0}$. Thus, (82) can be formulated as the abstract fractional partial differential equation (3), where $f(t,u) := g(\cdot,t,u(\cdot,t))$. By the maximum principle of parabolic equations, $\{T(t)\}_{t\geq 0}$ is a positive C_0-semigroup. It is easy to see that f satisfies (H_1) and $(H_2)'$. By Corollary 16, (82) has only one positive mild solution $u(\cdot,t) \in \Omega(X)$. □

Example 2. Consider the problem

$$\begin{aligned}&D_+^{1/2}u\left(x,t\right) = \frac{\partial^2 u\left(x,t\right)}{\partial x^2} + b\left(t\right)\sin u\left(x,t\right), \\ &\qquad\qquad (x,t) \in [0,2\pi] \times \mathbb{R}, \\ &u\left(0,t\right) = u\left(2\pi,t\right) = 0,\end{aligned} \tag{84}$$

where $D_+^{1/2}$ is Liouville fractional partial derivative of order $1/2$ with the lower limit $-\infty$, $b \in C_b(\mathbb{R},\mathbb{R})$.

Proof. Set $X = L^2[0,2\pi]$, $A := \partial^2/\partial x^2$, and

$$D\left(A\right) = \left\{g \in H^2\left[0,2\pi\right] \mid g\left(0\right) = g\left(2\pi\right) = 0\right\}. \tag{85}$$

For $f(t,u) = b(t)\sin u$ for $u \in X$, $t \in \mathbb{R}$, it follows that $f \in \Omega(X) \in \mathscr{M}(X)$ for any $u \in X$, with

$$\begin{aligned}\|f(t,u_1) - f(t,u_2)\|_2^2 &\le \int_0^{\pi} |b(t)|^2 \left|\sin u_1(s) - \sin u_2(s)\right|^2 ds \\ &\le \|b\|_\infty^2 \|u_1 - u_2\|_2^2,\end{aligned} \tag{86}$$

for $u_1, u_2 \in X$. In consequence, (84) has only one mild solution $u(\cdot,t) \in \Omega(X)$ (by Theorem 17). □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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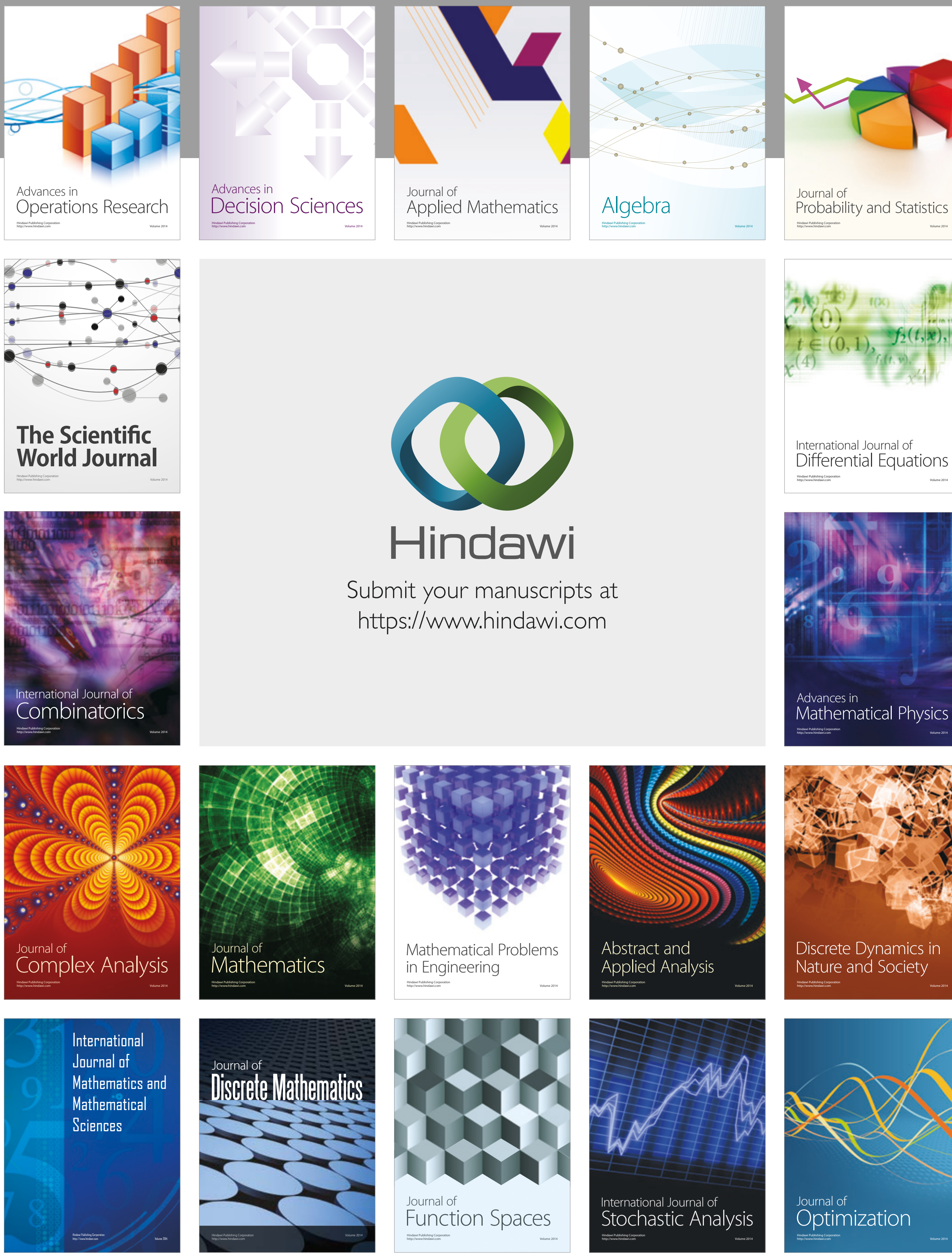
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