Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 736585, 12 pages
http://dx.doi.org/10.1155/2013/736585

Research Article

Adaptive Synchronization of Complex Dynamical Multilinks Networks with Similar Nodes

Weiping Wang,[1] Lixiang Li,[2] Haipeng Peng,[2] Jialiang Yuan,[2] Jinghua Xiao,[1] and Yixian Yang[2,3]

[1] *School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China*
[2] *Information Security Center, Beijing University of Posts and Telecommunications, Beijing 100876, China*
[3] *National Engineering Laboratory for Disaster Backup and Recovery, Beijing University of Posts and Telecommunications, Beijing 100876, China*

Correspondence should be addressed to Lixiang Li; li_lixiang2006@163.com

Received 2 May 2013; Accepted 21 May 2013

Academic Editor: Ming Li

This paper studies the synchronization of complex dynamical networks with multilinks and similar nodes. The dynamics of all the nodes in the networks are impossible to be completely identical due to the differences of parameters or the existence of perturbations. Networks with similar nodes are universal in the real world. In order to depict the similarity of the similar nodes, we give the definition of the minimal similarity of the nodes in the network for the first time. We find the threshold of the minimal similarity of the nodes in the network. If the minimal similarity of the nodes is bigger than the threshold, then the similar nodes can achieve synchronization without controllers. Otherwise, adaptive synchronization method is adopted to synchronize similar nodes in the network. Some new synchronization criteria are proposed based on the Lyapunov stability theory. Finally, numerical simulations are given to illustrate the feasibility and the effectiveness of the proposed theoretical results.

1. Introduction

Complex dynamical networks have attracted increasing attention in recent years, since they have been widely exploited to model many complex systems in the science, engineering, and society [1, 2]. Synchronization of complex network has been found to be a universal phenomenon in nature and it has important potential applications to real-word dynamical systems. As an important and interesting collective behavior, synchronization of complex network has been studied extensively [3–8], such as complete synchronization, projective synchronization [9], impulsive synchronization [10, 11], exponential synchronization [12], adaptive synchronization [13–15], and pinning synchronization [16–21].

Most previous research assumes that the dynamics of all nodes are identical. Consequently, the synchronization problem is significantly simplified. However, the assumption that the nodes are completely identical is not realistic in many real-world networks [22], such as in the neural networks, where the internal neurons in the nervous system are impossible to be completely identical due to the differences of the parameters. And the authors of [23, 24] studied synchronization of complex dynamical networks with nonidentical nodes. While, in normal circumstances, the neurons are not completely identical or completely nonidentical. They are similar to each other and they will achieve synchronization to transmit information which shows that the neural system has certain robustness. At this time, we want to know the answers of the following questions, which have a practical meaning for us to analyze and control many realistic networks with similar nodes. How to depict the similarity of the similar nodes? What is the condition that the similar nodes have to satisfy in the network in order to achieve synchronization without controllers? If there is a mutation or a pathological change, then some neurons may have many different characteristics, and they can not synchronize with other neurons. When the similarity of the similar nodes is broken, how to synchronize the nodes in the network?

Furthermore, enormous works have been done on the synchronization in complex networks with single-link, and a lot of meaningful conclusions have been obtained. The authors of [19] propose that single-link network is a special case of multilinks network. Therefore, research on multilinks networks are more representative. Multilinks means that there is more than one link between two nodes and each of them has its own property. For instance, there are relationship networks, transportation networks, World Wide Web, and so forth. The transportation network as an example of a network with multilinks, which is made up by combining the corresponding airline network, railway network, and highway network. We can split the multilinks networks into many subnetworks based on the property of the connections. For a transportation network, the transmission speed is different among airline network, railway network, and highway network. In our previous work [19], time-delay was introduced to split complex dynamical networks into subnetworks, upon which a model of complex networks with multilinks has been constructed. However, the important issue of synchronization for complex dynamical networks with similar nodes and multilinks has so far received little attention. The study of the synchronization problem with similar nodes in the complex multilinks network becomes an interesting and challenging topic.

In this paper, we give a model of complex multilinks networks with similar nodes. A definition of similar nodes is given and the minimal similarity of similar nodes in the network is analyzed for the first time. We find a threshold of the minimal similarity of similar nodes. If the minimal similarity of similar nodes is bigger than the threshold in the network, then the similar nodes can achieve synchronization. Otherwise, we should add some controllers to the nodes in order to get synchronization. Then some new adaptive synchronization criteria are proposed. Finally, numerical simulations of dynamical networks with similar nodes are presented to demonstrate the feasibility and the effectiveness of the results.

2. Model and Preliminaries

The model of complex multilinks network consisting of N similar nodes with m kinds of properties can be described by

$$\begin{aligned}\dot{x}_i(t) &= A_i x_i(t) + f(x_i(t)) + c\sum_{j=1}^{N} b_{(0)ij}\Gamma_0 x_j(t)\\ &\quad + c\sum_{l=1}^{m-1}\sum_{j=1}^{N} b_{(l)ij}\Gamma_l x_j(t-\tau_l), \quad i = 1, 2, \ldots, N,\end{aligned} \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{iN}(t))^T$ is the state vector of node i, A_i is a matrix, $f(x_i(t))$ is a smooth nonlinear vector function, $c > 0$ is the coupling strength, Γ_0 and Γ_l are the inner-coupling matrices, $B_l = (b_{(l)ij})_{N\times N}$, $0 \le l \le m-1$ represents the topological structure of the lth subnetwork, and τ_l is the time-delay of the lth subnetwork compared with the basic network ($\tau_0 = 0$). We define $b_{(l)ij} = b_{(l)ji} = 1$ if there is a connection between node i and node j ($j \ne i$) in the lth subnetwork, otherwise $b_{(l)ij} = b_{(l)ji} = 0$. And we define $b_{(l)ii} = -\sum_{j=1, j\ne i}^{N} b_{(l)ij}$.

Network (1) is in a state of asymptotical synchronization, if

$$x_1(t) = x_2(t) = \cdots = x_N(t) \longrightarrow s(t) \tag{2}$$

as $t \to \infty$ $(1 \le i \le N)$, where $s(t)\epsilon R^n$ is a synchronous solution of the node system $\dot{x}_i(t) = A_i x_i(t) + f(x_i(t))$. We define the error vectors as

$$e_i(t) = x_i(t) - s_i(t). \tag{3}$$

Hereafter, the definitions of similar nodes and the minimal similarity of the similar nodes are given, and a useful assumption and two lemmas are introduced.

Definition 1. If A_i and A_j are matrices with the similar element values, then the node i and j are similar nodes. In the network (1), we define A as the matrix of basic node and A_i $(1 \le i \le N-1)$ as matrices of other nodes. Because A and A_i are matrices with similar element values, we define $\delta_i = 1 - (\|A - A_i\|_F / \|A\|_F)$, $\nu = \min(\delta_i)$, $(1 \le i \le N-1)$, and ν $(0 < \nu < 1)$ represents the minimal similarity of the nodes in the network. The norm of matrix A is $\|A\|_F = (\sum_{i,j=1}^{N} a_{ij}^2)^{1/2}$, and a_{ij} $(1 \le i, j \le N)$ are elements of matrix A.

Remark 2. From Definition 1, we know ν is an important parameter. When ν approaches to 1, the nodes in the network are similar. If ν satisfies a certain condition, then the similar nodes of the network can achieve synchronization without controllers. On the contrary, when ν is far away from 1, the nodes in the network become not similar, so the nodes cannot achieve synchronization without controllers. That is to say, there exists a threshold, if ν is bigger than the threshold; then the similar nodes in the network can get synchronization without controllers. And the threshold is what we tried to find in the following.

Assumption 3. The smooth nonlinear function $f(\cdot)$ satisfies the following Lipschitz condition:

$$\|f(x_i(t)) - f(s(t))\| \le L\|x_i(t) - s(t)\|, \tag{4}$$

where L is a positive constant.

Lemma 4. *For any two vectors x and y, and a matrix $Q > 0$ with compatible dimensions, one has*

$$2x^T y \le x^T Q x + y^T Q^{-1} y. \tag{5}$$

Lemma 5. *If $A\epsilon C^{N\times N}$, the eigenvalues of A are λ_i $(i = 1, 2, \ldots, N)$, then $\max(\lambda_i) \le \|A\|$, where $\|A\|$ is an arbitrary matrix norm.*

3. Synchronization Analysis

In this section, suppose there is not a control scheme to synchronize a delayed complex multilinks network with similar nodes. According to system (1), the error dynamical system can be derived as

$$\dot{e}_i(t) = A_i e_i(t) + f(x_i(t)) - f(s(t)) + c\sum_{j=1}^{N} b_{(0)ij}\Gamma_0 e_j(t) + c\sum_{l=1}^{m-1}\sum_{j=1}^{N} b_{(l)ij}\Gamma_l e_j(t-\tau_l), \quad (6)$$

where $A_i = A + \Delta A_i$, A_i is the matrix of node i, and A is the matrix of basic node. Because A and A_i are matrices with similar element values, $\Delta A_i = A_i - A$. It is easy to see that the synchronization of the complex network (1) is achieved if the zero solution of the error system (6) is globally asymptotically stable, which is ensured by the following theorem. And we find that the minimal similarity of the similar nodes satisfies an inequality for synchronization.

Theorem 6. *Consider network* (1), *if the minimal similarity of the nodes ν is bigger than the threshold, where the threshold of ν is*

$$\left(\lambda_{\max}(A) + L + c\lambda_{\max}(P_0) + \frac{c}{2}\sum_{l=1}^{l=m-1}\lambda_{\max}\left(P_l P_l^T\right) + \frac{c}{2}(m-1) + \|A\|_F\right)\left(\|A\|_F\right)^{-1}, \quad (7)$$

and it also satisfies the following inequality:

$$\frac{\lambda_{\max}(A) + L + c\lambda_{\max}(P_0) + (c/2)\sum_{l=1}^{l=m-1}\lambda_{\max}\left(P_l P_l^T\right)}{\|A\|_F} + \frac{(c/2)(m-1) + \|A\|_F}{\|A\|_F} < \nu < 1, \quad (8)$$

then the system (1) *is synchronized without controllers.*

Proof. Construct the following Lyapunov function:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i(t)^T e_i(t) + \frac{c}{2}\sum_{l=1}^{m-1}\int_{t-\tau_l}^{t}\sum_{i=1}^{N} e_i(\theta)^T e_i(\theta)\,d\theta. \quad (9)$$

We get

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{N} e_i(t)^T \dot{e}_i(t) + \frac{c}{2}\sum_{l=1}^{m-1}\sum_{i=1}^{N}\left[e_i(t)^T e_i(t) - e_i(t-\tau_l)^T e_i(t-\tau_l)\right] \\ &= \sum_{i=1}^{N} e_i(t)^T A e_i(t) + \sum_{i=1}^{N} e_i(t)^T \Delta A_i e_i(t) + \sum_{i=1}^{N} e_i(t)\left[f(x_i(t)) - f(s(t))\right] \\ &\quad + \sum_{i=1}^{N} c e_i(t)^T \sum_{j=1}^{N} b_{(0)ij}\Gamma_0 e_j(t) + \sum_{i=1}^{N} c e_i(t)^T \sum_{l=1}^{m-1}\sum_{j=1}^{N} b_{(l)ij}\Gamma_l e_j(t-\tau_l) \\ &\quad + \frac{c}{2}\sum_{i=1}^{N}\sum_{l=1}^{m-1} e_i(t)^T e_i(t) - \frac{c}{2}\sum_{i=1}^{N}\sum_{l=1}^{m-1} e_i(t-\tau_l)^T e_i(t-\tau_l).\end{aligned} \quad (10)$$

Let $e(t) = (e_1(t), e_2(t), \ldots, e_N(t))$, then we get

$$\begin{aligned}\dot{V}(t) \le &\left[e(t)^T A e(t) + e(t)^T \Delta A_i e(t) + L e(t)^T e(t)\right] \\ &+ c e(t)^T (B_0 \otimes \Gamma_0) e(t) \\ &+ c e(t)^T (B_1 \otimes \Gamma_1) e(t-\tau_1) \\ &+ c e(t)^T (B_2 \otimes \Gamma_2) e(t-\tau_2) \\ &+ \cdots + c e(t)^T (B_{m-1} \otimes \Gamma_{m-1}) e(t-\tau_{m-1}) \\ &+ \frac{c}{2}(m-1) e(t)^T e(t) \\ &- \frac{c}{2} e(t-\tau_1)^T e(t-\tau_1) - \frac{c}{2} e(t-\tau_2)^T e(t-\tau_2) \\ &- \cdots - \frac{c}{2} e(t-\tau_{m-1})^T e(t-\tau_{m-1}).\end{aligned} \quad (11)$$

Let

$$\begin{aligned}P_0 &= B_0 \otimes \Gamma_0, \\ P_1 &= B_1 \otimes \Gamma_1, \\ P_2 &= B_1 \otimes \Gamma_2, \\ &\vdots \\ P_{m-1} &= B_{m-1} \otimes \Gamma_{m-1},\end{aligned} \quad (12)$$

where $\otimes$ represents the Kronecker product. Then by Lemma 4, we have

$$\begin{aligned}\dot{V}(t) \le &\lambda_{\max}(A) e(t)^T e(t) + \lambda_{\max}(\Delta A_i) e(t)^T e(t) \\ &+ L e(t)^T e(t) + c\lambda_{\max}(P_0) e(t)^T e(t) \\ &+ \frac{c}{2}\lambda_{\max}\left(P_1 P_1^T\right) e(t)^T e(t) \\ &+ \frac{c}{2}\lambda_{\max}\left(P_2 P_2^T\right) e(t)^T e(t) \\ &+ \cdots + \frac{c}{2}\lambda_{\max}\left(P_{m-1} P_{m-1}^T\right) e(t)^T e(t)\end{aligned}$$

$$+\frac{c}{2}(m-1)e(t)^T e(t)$$
$$\leq \left[\lambda_{\max}(A)+\|\Delta A_i\|_F + L + c\lambda_{\max}(P_0) + \frac{c}{2}\sum_{l=1}^{m-1}\lambda_{\max}\left(P_l P_l^T\right)+\frac{c}{2}(m-1)\right]e(t)^T e(t)$$
$$\leq \left[\lambda_{\max}(A)+(1-\nu)\|A\|_F + L + c\lambda_{\max}(P_0) + \frac{c}{2}\sum_{l=1}^{m-1}\lambda_{\max}\left(P_l P_l^T\right)+\frac{c}{2}(m-1)\right]e(t)^T e(t). \tag{13}$$

Therefore, if we have

$$\lambda_{\max}(A)+(1-\nu)\|A\|_F + L + c\lambda_{\max}(P_0) + \frac{c}{2}\sum_{l=1}^{m-1}\lambda_{\max}\left(P_l P_l^T\right)+\frac{c}{2}(m-1)<0, \tag{14}$$

then $\dot{V}(t) \leq 0$. So we get the synchronization criterion as follows:

$$\frac{\lambda_{\max}(A)+L+c\lambda_{\max}(P_0)+(c/2)\sum_{l=1}^{l=m-1}\lambda_{\max}\left(P_l P_l^T\right)}{\|A\|_F} + \frac{(c/2)(m-1)+\|A\|_F}{\|A\|_F}<\nu<1. \tag{15}$$

If ν satisfies (15), the nodes are synchronized. Thus we complete the proof. □

Remark 7. The matrix of basic node can be chosen at random from A_i $(1 \leq i \leq N)$. No matter which one we choose, Theorem 6 also holds.

Furthermore, noise plays an important role in the process of synchronization. Here we consider the influence of the noise. If there is an additive noise in the system (1) in the form of

$$\dot{x}_i(t) = A_i x_i(t) + f(x_i(t)) + c\sum_{j=1}^{N} b_{(0)ij}\Gamma_0 x_j(t) + c\sum_{l=1}^{m-1}\sum_{j=1}^{N} b_{(l)ij}\Gamma_l x_j(t-\tau_l) + \eta_i(t), \quad i = 1,2,\ldots,N, \tag{16}$$

where $\eta_i(t)\epsilon R^n$ is the zero mean bounded noise. Using system (16), we can easily get the following error system:

$$\dot{e}_i(t) = (A+\Delta A_i)e_i(t) + f(x_i(t)) - f(s(t)) + c\sum_{j=1}^{N} b_{(0)ij}\Gamma_0 e_j(t) + c\sum_{l=1}^{m-1}\sum_{j=1}^{N} b_{(l)ij}\Gamma_l e_j(t-\tau_l) + \eta_i(t), \tag{17}$$

then we get

$$E(\dot{e}_i(t)) = (A+\Delta A_i)E[e_i(t)] + E[f(x_i(t)) - f(s(t))] + c\sum_{j=1}^{N} b_{(0)ij}\Gamma_0 E[e_j(t)] + c\sum_{l=1}^{m-1}\sum_{j=1}^{N} b_{(l)ij}\Gamma_l E[e_j(t-\tau_l)] + E[\eta_i(t)]. \tag{18}$$

Finally, we get the theorem as follows.

Theorem 8. *When there is a noise or perturbation, considering the network* (16)*, if the following condition holds*

$$\frac{\lambda_{\max}(A)+L+c\lambda_{\max}(P_0)+(c/2)\sum_{l=1}^{l=m-1}\lambda_{\max}\left(P_l P_l^T\right)}{\|A\|_F} + \frac{(c/2)(m-1)+\|A\|_F}{\|A\|_F}<\nu<1, \tag{19}$$

then $E[e_i(t)]$ *approaches to zero.*

The proof process of Theorem 8 is similar to the proof process of Theorem 6, so here it is omitted.

4. Adaptive Synchronization

In this section, a control scheme is developed to synchronize a delayed complex multilinks network with similar nodes, which do not satisfy the synchronization criterion (15). And the following adaptive controllers are used:

$$u_i = -d_i e_i(t), \quad 1 \leq i \leq N. \tag{20}$$

And the updating laws are

$$\dot{d}_i = k_i e_i(t)^T e_i(t), \quad 1 \leq i \leq N, \tag{21}$$

where k_i $(1 \leq i \leq N)$ are positive constants. The adaptive controllers (20) are widely used in solving many synchronous problems.

Then the controlled network can be characterized as

$$\dot{x}_i(t) = A_i x_i(t) + f(x_i(t)) + c\sum_{j=1}^{N} b_{(0)ij}\Gamma_0 x_j(t) + c\sum_{l=1}^{m-1}\sum_{j=1}^{N} b_{(l)ij}\Gamma_l x_j(t-\tau_l) + u_i, \quad i = 1, 2, \dots, N. \tag{22}$$

According to system (22), the following error dynamical system can be derived:

$$\dot{e}_i(t) = (A + \Delta A_i) e_i(t) + f(x_i(t)) - f(s(t)) + c\sum_{j=1}^{N} b_{(0)ij}\Gamma_0 e_{j(t)} + c\sum_{l=1}^{m-1}\sum_{j=1}^{N} b_{(l)ij}\Gamma_l e_j(t-\tau_l) - d_i e_i(t). \tag{23}$$

It is clear to see that the synchronization of the controlled complex network (22) is achieved if the zero solution of the error system (23) is globally asymptotically stable, which is ensured by the following theorem.

Theorem 9. *Consider the network* (22) *under the actions of the controllers* (20) *and the updating laws* (21). *If the following condition holds:*

$$\lambda_{\max}(A) + (1-\nu)\|A\|_F + L + c\lambda_{\max}(P_0) + \frac{c}{2}\sum_{l=1}^{l=m-1}\lambda_{\max}(P_l P_l^T) + \frac{c}{2}(m-1) < d^*, \tag{24}$$

where d^* is a sufficiently large positive constant to be determined, then the system (1) *is synchronized.*

Proof. Construct the following Lyapunov function:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i(t)^T e_i(t) + \frac{c}{2}\sum_{l=1}^{m-1}\int_{t-\tau_l}^{t}\sum_{i=1}^{N} e_i(\theta)^T e_i(\theta)\,d\theta + \frac{1}{2}\sum_{i=1}^{N}\frac{(d_i - d^*)^2}{k_i}. \tag{25}$$

Clearly, $V(t)$ is positive. Then the derivative of $V(t)$ is obtained as

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{N} e_i(t)^T \dot{e}_i(t) + \frac{c}{2}\sum_{l=1}^{m-1}\sum_{i=1}^{N}\left[e_i(t)^T e_i(t) - e_i(t-\tau_l)^T e_i(t-\tau_l)\right] + \sum_{i=1}^{N}\left[d_i e_i(t)^T e_i(t) - d^* e_i(t)^T e_i(t)\right] \\ &= \sum_{i=1}^{N}\Big\{ e_i(t)^T A e_i(t) + e_i(t)^T \Delta A_i e_i(t) + e_i(t) \times \left[f(x_i(t)) - f(s(t))\right] + c e_i(t)^T\sum_{j=1}^{N} b_{(0)ij}\Gamma_0 e_j(t) \\ &\quad + c e_i(t)^T\sum_{l=1}^{m-1}\sum_{j=1}^{N} b_{(l)ij}\Gamma_l e_j(t-\tau_l) - d_i e_i(t)^T e_i(t) + \frac{c}{2}\sum_{l=1}^{m-1} e_i(t)^T e_i(t) \\ &\quad - \frac{c}{2}\sum_{l=1}^{m-1} e_i(t-\tau_l)^T e_i(t-\tau_l) + d_i e_i(t)^T e_i(t) - d^* e_i(t)^T e_i(t)\Big\}.\end{aligned} \tag{26}$$

Let $e(t) = (e_1(t), e_2(t), \dots, e_N(t))$, then we get

$$\begin{aligned}\dot{V}(t) \le{}& \left[e(t)^T A e(t) + e(t)^T \Delta A_i e(t) + L e(t)^T e(t)\right] + c e(t)^T (B_0 \otimes \Gamma_0) e(t) \\ &+ c e(t)^T (B_1 \otimes \Gamma_1) e(t-\tau_1) + c e(t)^T \times (B_2 \otimes \Gamma_2) e(t-\tau_2) \\ &+ \cdots + c e(t)^T (B_{m-1} \otimes \Gamma_{m-1}) e(t-\tau_{m-1}) \\ &+ \frac{c}{2}(m-1) e(t)^T e(t) - \frac{c}{2} e(t-\tau_1)^T e(t-\tau_1) - \frac{c}{2} e(t-\tau_2)^T e(t-\tau_2) \\ &- \cdots - \frac{c}{2} e(t-\tau_{m-1})^T e(t-\tau_{m-1}) - d^* e(t)^T e(t).\end{aligned} \tag{27}$$

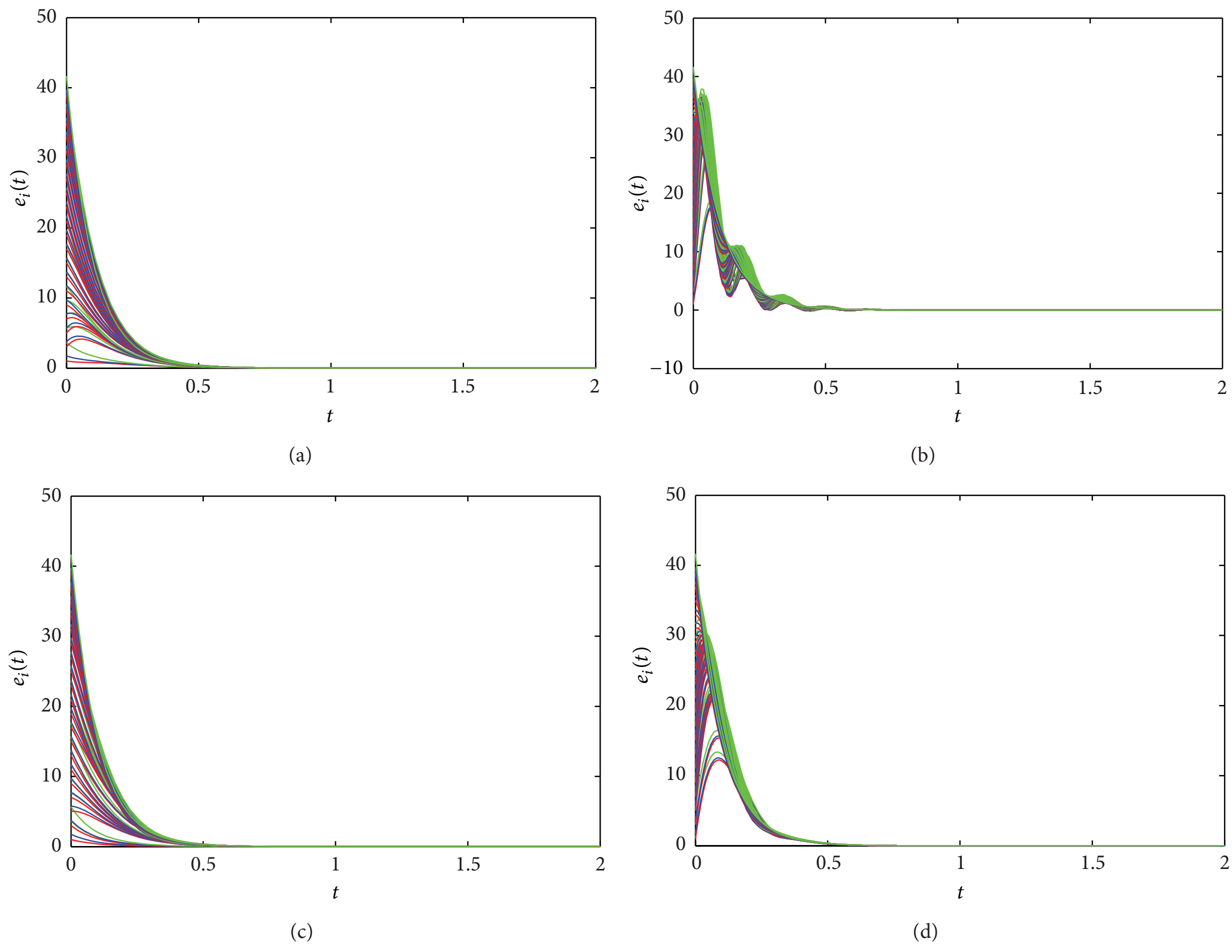

Figure 1: Separate synchronous variables e_{i1}, e_{i2}, e_{i3} ($1 \leqslant i \leqslant 30$) of different network models where e_{i1} are shown by the red line, e_{i2} are shown by the blue line, and e_{i3} are shown by the green line. $c = 2$; $\tau = 0.01$. (a) B_0 and B_1 are small-world network models, and the rewiring probability among nodes is 0.3 and 0.6. (b) B_0 and B_1 are scale-free network models, and their minimum degrees are 2 and 3. (c) B_0 and B_1 are random network models, and their connection probability among nodes is 0.1 and 0.3. (d) B_0 is a small-world network model, and the rewiring probability among nodes is 0.3. B_1 is a scale-free network model, and the minimum degree is 2. Those similar nodes can achieve synchronization without controllers (color online).

Let

$$\begin{aligned} P_0 &= B_0 \otimes \Gamma_0, \\ P_1 &= B_1 \otimes \Gamma_1, \\ P_2 &= B_1 \otimes \Gamma_2, \\ &\vdots \\ P_{m-1} &= B_{m-1} \otimes \Gamma_{m-1}, \end{aligned} \tag{28}$$

where $\otimes$ represents the Kronecker product. Then by Lemma 4, we have

$$\begin{aligned} \dot{V}(t) \le{}& \lambda_{\max}(A)\, e(t)^T e(t) \\ &+ \lambda_{\max}(\Delta A_i)\, e(t)^T e(t) + L e(t)^T e(t) \\ &+ c\lambda_{\max}(P_0)\, e(t)^T e(t) \\ &+ \frac{c}{2}\lambda_{\max}\left(P_1 P_1^T\right) e(t)^T e(t) \\ &+ \cdots + \frac{c}{2}\lambda_{\max}\left(P_{m-1} P_{m-1}^T\right) e(t)^T e(t) \\ &+ \frac{c}{2}(m-1)\, e(t)^T e(t) - d^* e(t)^T e(t) \\ \le{}& \Bigg[\lambda_{\max}(A) + \left\|\Delta A_i\right\|_F + L \\ &+ c\lambda_{\max}(P_0) + \frac{c}{2}\sum_{l=1}^{m-1}\lambda_{\max}\left(P_l P_l^T\right) \\ &+ \frac{c}{2}(m-1) - d^*\Bigg] e(t)^T e(t) \\ \le{}& \Bigg[\lambda_{\max}(A) + (1-\nu)\|A\|_F + L \\ &+ c\lambda_{\max}(P_0) + \frac{c}{2}\sum_{l=1}^{m-1}\lambda_{\max}\left(P_l P_l^T\right) \\ &+ \frac{c}{2}(m-1) - d^*\Bigg] e(t)^T e(t). \end{aligned} \tag{29}$$

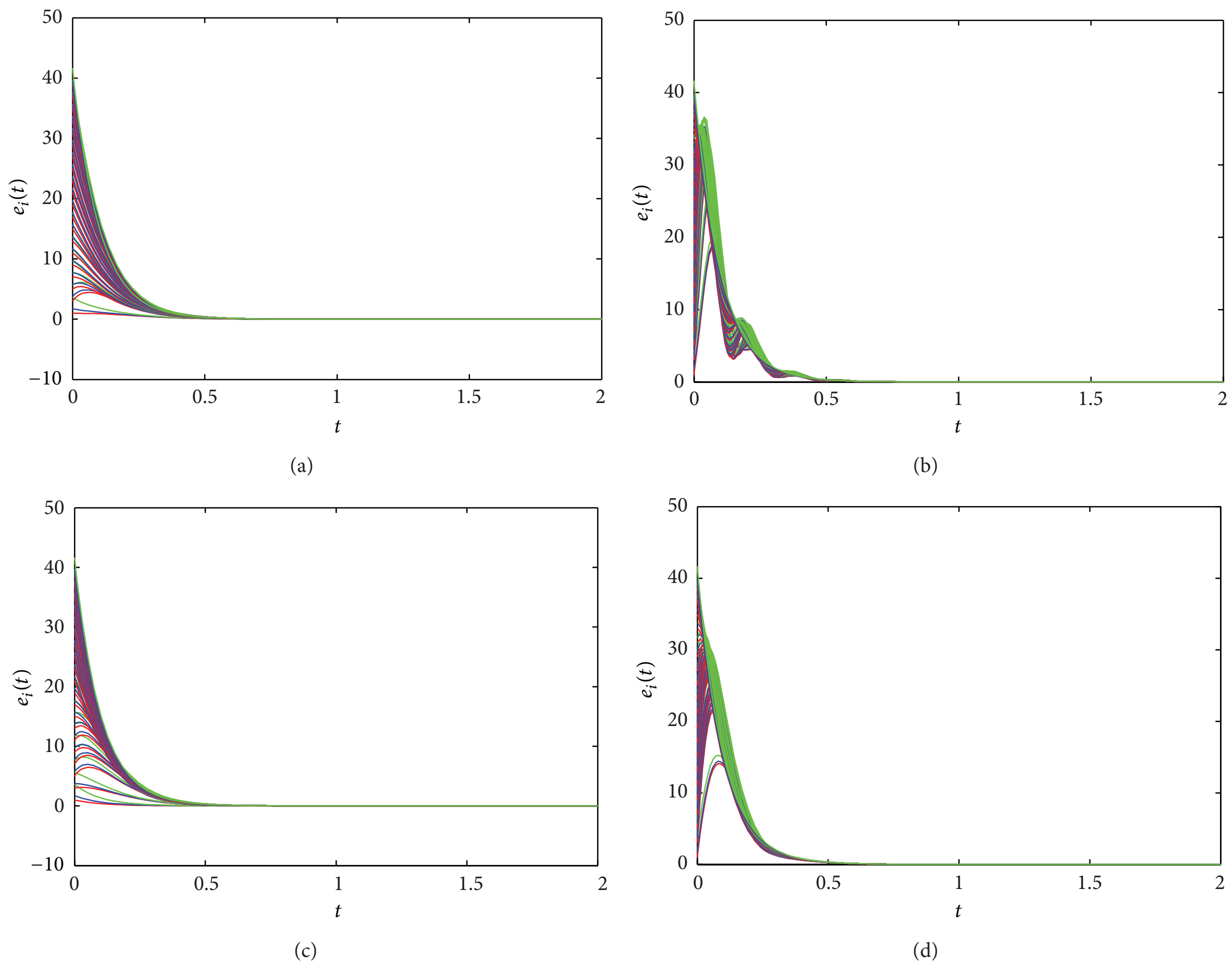

Figure 2: The similar nodes can achieve synchronization without controllers when the Brownian motion satisfies $E\omega(t) = 0$, $D\omega(t) = 1$. And separate synchronous variables e_{i1}, e_{i2}, e_{i3} ($1 \leqslant i \leqslant 30$) of different network models where e_{i1} are shown by the red line, e_{i2} are shown by the blue line, and e_{i3} are shown by the green line. $c = 2$; $\tau = 0.01$. The network models of (a)–(d) are the same with Figures 1(a)–1(d) (color online).

Therefore, if we have

$$\lambda_{\max}(A) + (1-\nu)\|A\|_F + L + c\lambda_{\max}(P_0) + \frac{c}{2}\sum_{l=1}^{m-1}\lambda_{\max}\left(P_l P_l^T\right) + \frac{c}{2}(m-1) < d^*, \tag{30}$$

then $\dot{V}(t) \leq 0$. Here we complete the proof. □

Remark 10. If there is not a nonlinear function in system (1), then the network (1) is transferred into

$$\dot{x}_i(t) = A_i x_i(t) + c\sum_{j=1}^{N} b_{(0)ij}\Gamma_0 x_{j(t)} + c\sum_{l=1}^{m-1}\sum_{j=1}^{N} b_{(l)ij}\Gamma_l x_j(t-\tau_l), \quad i = 1, 2, \ldots, N. \tag{31}$$

Likewise, we can design the controllers as in (20) and (21). If the following condition holds:

$$\lambda_{\max}(A) + (1-\nu)\|A\|_F + c\lambda_{\max}(P_0) + \frac{c}{2}\sum_{l=1}^{m-1}\lambda_{\max}\left(P_l P_l^T\right) + \frac{c}{2}(m-1) < d^*, \tag{32}$$

then the system (31) is synchronized, where d^* is a sufficiently large positive constant to be determined.

Remark 11. The single-link network is a special case of multilinks networks [19]. When there is not a delay, the network (1) is transferred into the following single-link network:

$$\dot{x}_i(t) = A_i x_i(t) + f(x_i(t)) + c\sum_{j=1}^{N} b_{(0)ij}\Gamma_0 x_j(t), \quad i = 1, 2, \ldots, N, \tag{33}$$

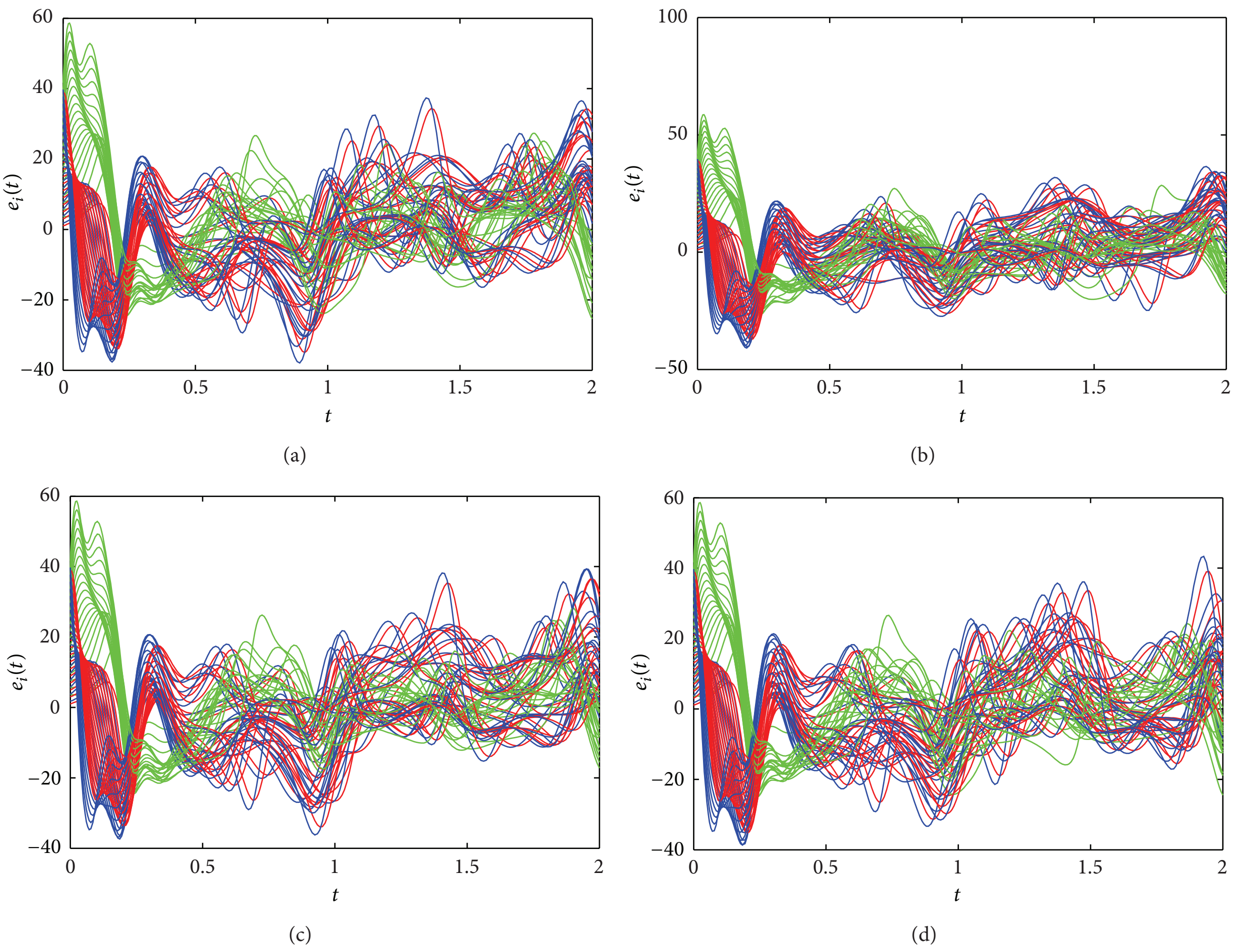

FIGURE 3: Separate synchronous variables e_{i1}, e_{i2}, e_{i3} $(1 \leqslant i \leqslant 30)$ of different network models where e_{i1} are shown by the red line, e_{i2} are shown by the blue line, and e_{i3} are shown by the green line. $c = 2$; $\tau = 0.01$. The dynamics of nodes satisfy the Lü system. The nodes of the network cannot achieve synchronization without controllers. The network models of (a)–(d) are the same with Figures 1(a)–1(d) (color online).

and the controllers are designed as in (20)-(21). If the following condition holds:

$$\lambda_{\max}(A) + (1-\nu)\|A\|_F + L + c\lambda_{\max}(P_0) < d^*, \quad (34)$$

then the system (33) is synchronized, where d^* is a sufficiently large positive constant to be determined.

5. Numerical Simulation

In this section, we use some examples to explain the influence of the proposed criteria, and we consider a network consisting of 30 similar nodes. The multilinks network with 2 properties can be described as follows:

$$\dot{x}_i(t) = A_i x_i(t) + f(x_i(t)) + c\sum_{j=1}^{30} b_{(0)ij}\Gamma_0 x_j(t) + c\sum_{j=1}^{30} b_{(1)ij}\Gamma_1 x_j(t-\tau), \quad (35)$$

where $1 \leqslant i \leqslant 30$, $B_0 = (b_{(0)ij})_{30\times 30}$, and $B_1 = (b_{(1)ij})_{30\times 30}$ are symmetrically diffusive coupling matrixes with $b_{(0)ij}$, $b_{(1)ij} = 0$ or 1. $\Gamma_0 = \Gamma_1 = \mathrm{diag}(1,1,1)$, $c = 2$, $\tau = 0.01$, $f(x_i(t)) = (0.6\sin(x_{i1}), 0.6\sin(x_{i2}), 0.6\sin(x_{i3}))^T$. According to Assumption 3, we can know that $L = 0.6$:

$$A_i = \begin{pmatrix} -10 + 0.1 * \mathrm{rand} & 0.1 * \mathrm{rand} & 0.1 * \mathrm{rand} \\ 0.1 * \mathrm{rand} & -10 + 0.1 * \mathrm{rand} & 0.1 * \mathrm{rand} \\ 0.1 * \mathrm{rand} & 0.1 * \mathrm{rand} & -10 + 0.1 * \mathrm{rand} \end{pmatrix}, \quad (36)$$

where the function of rand can produce a random number between 0 and 1. According to the definition of similar nodes, we know A_i, $1 \leqslant i \leqslant 30$ are matrices of the similar nodes. And

$$A = \begin{pmatrix} -10 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -10 \end{pmatrix} \quad (37)$$

is the matrix of the basic node. So

$$\Delta A_i = \begin{pmatrix} 0.1 * \mathrm{rand} & 0.1 * \mathrm{rand} & 0.1 * \mathrm{rand} \\ 0.1 * \mathrm{rand} & 0.1 * \mathrm{rand} & 0.1 * \mathrm{rand} \\ 0.1 * \mathrm{rand} & 0.1 * \mathrm{rand} & 0.1 * \mathrm{rand} \end{pmatrix}. \quad (38)$$

According to the precise calculation, $\lambda_{\max}(A) = -10$, $\lambda_{\max}(P_0) = 0$, $\lambda_{\max}(P_1^T P_1) = 18.5139$. Based on

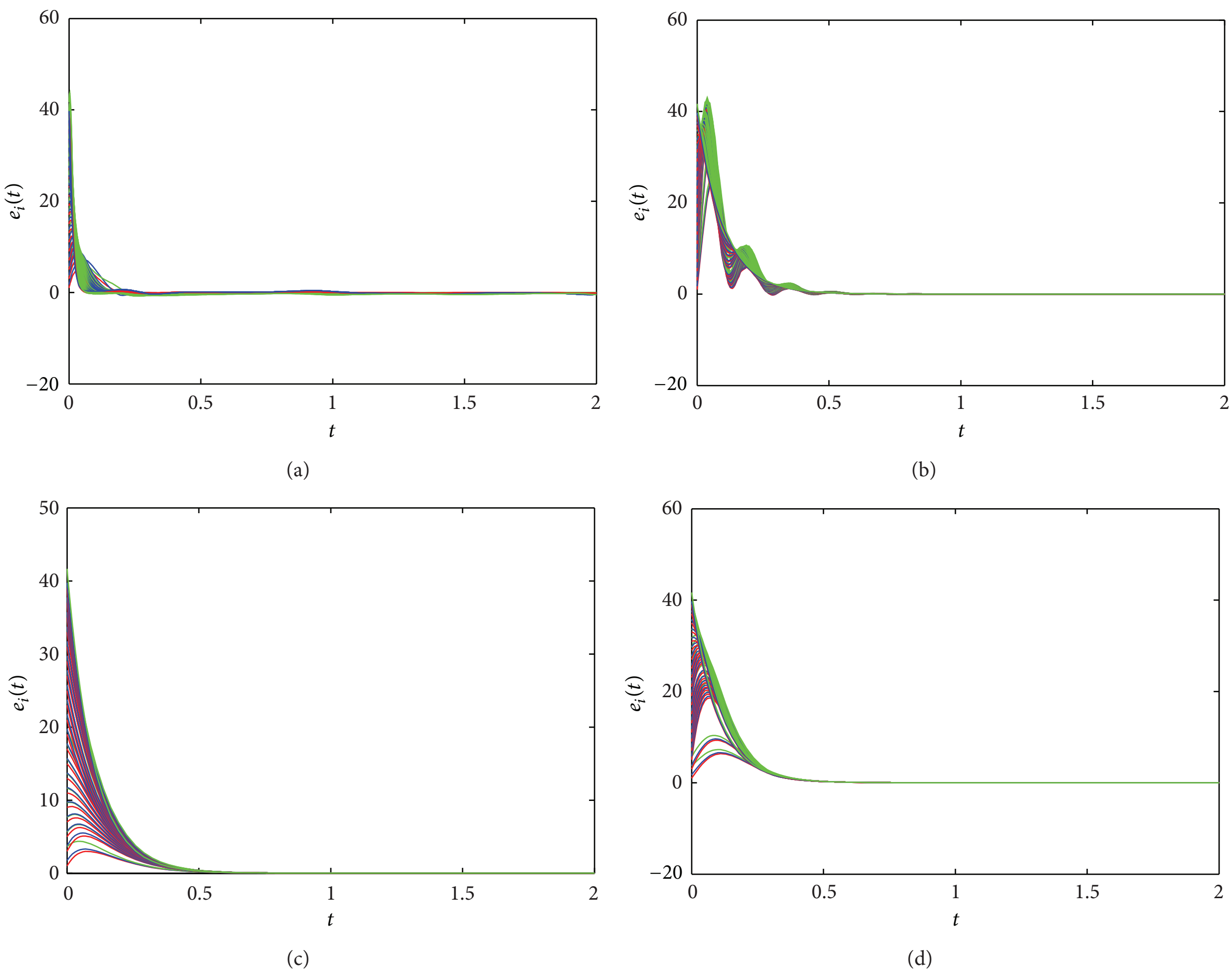

Figure 4: Separate synchronous variables e_{i1}, e_{i2}, e_{i3} $(1 \leqslant i \leqslant 30)$ of different network models where e_{i1} are shown by the red line, e_{i2} are shown by the blue line, and e_{i3} are shown by the green line. $c = 2$; $\tau = 0.01$. The dynamics of nodes satisfy the Lü system. The nodes are controlled by the adaptive controllers (20) and (21). The network models of (a)–(d) are the same with Figures 1(a)–1(d) (color online).

the stability analysis, we get $(\lambda_{\max}(A) + L + c\lambda_{\max}(P_0) + (c/2)\sum_{l=1}^{l=m-1}\lambda_{\max}(P_l P_l^T) + (c/2)(m-1) + \|A\|_F)(\|A\|_F)^{-1} = (-100 + 0.6 + 18.5139 + 1 + \sqrt{30000})/\sqrt{30000} = 0.5388$. According to (38), because $0 < \text{rand} < 1$, the biggest changes are

$$\Delta A_i = \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{pmatrix}. \tag{39}$$

Then we compute the $\nu = 0.9983$, $0.5388 < \nu < 1$, so the similar nodes can achieve synchronization which satisfies Theorem 6. From Figures 1(a)–1(d), we know that the similar nodes in the network achieved synchronization under different network models.

Furthermore, in order to verify Theorem 9, we choose the model (16) as the second example, where the Brownian motion satisfies $E\omega(t) = 0$, $D\omega(t) = 1$, and the parameters are the same with the first example. Figures 2(a)–2(d) plot the synchronous errors converge to 0 in finite time under different network models with noise, which reflects that similar nodes have a certain robustness. In our future work, we will consider the model (35) with Gaussian noise [25] or $1/f$ noise [26], and the stochastic bounded model like [27] in the complex network with similar nodes will be studied.

Next, another example as the third one describes the controlled network using Lü systems and considers the network consisting of 30 nodes. The node dynamical system is $\dot{x}_i = (-36x_{i1} + 36x_{i2};\ 20x_{i2} - x_{i1}x_{i3}; -3x_{i3} + x_{i1}x_{i2})$, for $i = 1, 2, \ldots, 30$. And ΔA_i are the same with the first example. Since Lü attractor is bounded, we suppose that all nodes are running in the given bounded region. There exists the constants $M_1 = 25$, $M_2 = 30$, $M_3 = 45$ satisfying $\|x_{ij}\|_2, \|s_j\|_2 \leqslant M_j$ for $1 \leqslant i \leqslant 30$ and $1 \leqslant j \leqslant 3$ [28]. Thus we have

$$\|f(x_i) - f(s)\| \leq \sqrt{2M_1^2 + M_2^2 + M_3^2}\,\|e_i\|_2 \approx 64.6142\|e_i\|_2, \tag{40}$$

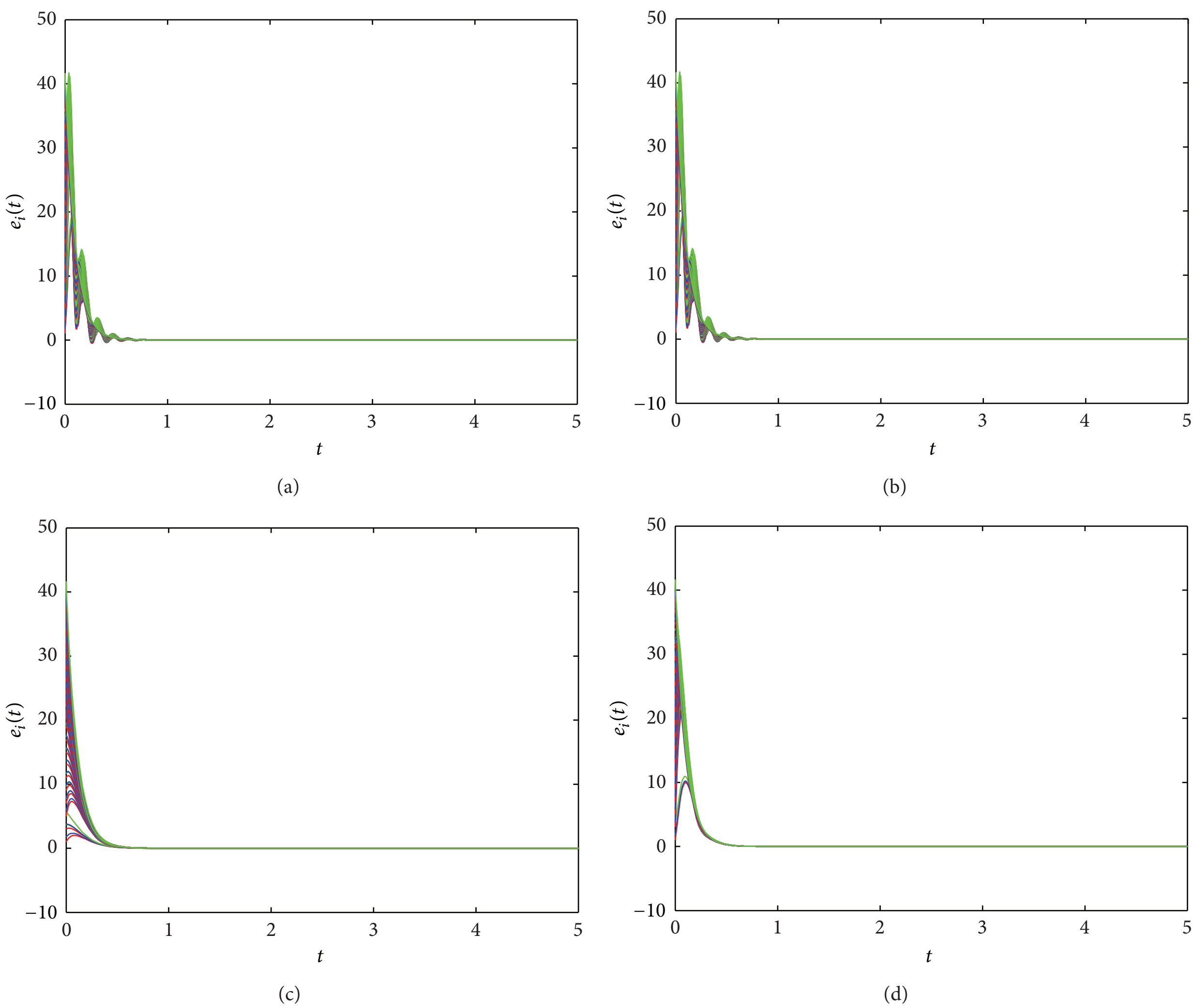

FIGURE 5: Separate synchronous variables e_{i1}, e_{i2}, e_{i3} $(1 \leqslant i \leqslant 30)$ of different network models where $c = 2$, $\tau = 0.01$, and $\tau_1 = 0.01$. (a) B_0, B_1, and B_2 are small-world network models, the rewiring probability among nodes is 0.3, 0.6, and 0.7. (b) B_0, B_1, and B_2 are scale-free network models, and their minimum degrees are 1, 2, and 3. (c) B_0, B_1, and B_2 are random network models, and their connection probability among nodes is 0.1, 0.3, and 0.5. (d) B_0 is a small-world network model, and the rewiring probability among nodes is 0.3. B_1 is a scale-free network model, and the minimum degree is 2. B_2 is a random network model, and the connection probability among nodes is 0.1. The dynamics of nodes satisfy the Lü system. The multilinks network can achieve synchronization under the adaptive controllers (color online).

then we can know that $L = 64.6142$. And other parameters are the same with the first example. We have $\lambda_{\max}(A) = 20$, and

$$\begin{aligned}&\left(\lambda_{\max}(A) + L + c\lambda_{\max}(P_0) + \frac{c}{2}\sum_{l=1}^{l=m-1}\lambda_{\max}\left(P_l P_l^T\right)\frac{c}{2}(m-1) + \|A\|_F\right)\left(\|A\|_F\right)^{-1}\\ &\quad = \frac{20 + 64.6142 + 18.5139 + 1 + \sqrt{3001}}{\sqrt{3001}} = 2.9008.\end{aligned} \tag{41}$$

It does not satisfy Theorem 6. So the nodes cannot achieve synchronization without controllers. Simulation results are given in Figures 3(a)–3(d) which show the evolution process of 30 state variables in three dimensions. And it verified that the similar nodes cannot achieve synchronization without controllers.

According to the adaptive synchronization criteria, we add the adaptive controllers (20) and (21) to these similar nodes of the network. $k_1 = k_2 = k_3 = 1$. The curves of error vectors e_{i1}, e_{i2}, e_{i3} $(i = 1, 2, 3)$ are shown in Figures 4(a)–4(d).

To be more persuadable, with the same calculation method, Figures 5(a)–5(d) plot the synchronous errors of networks with links owning 3 properties. Figures 5(a)–5(d) have different network models, and $\tau_1 = 0.01$. This demonstrates that our theorem is not only applicable to multilinks network owning two links properties but also to real networks with multiple links. From Figures 1(a)–1(d) to Figures 5(a)–5(d), we attain that our theorems are feasible in different network models under different conditions. This result is more helpful to real networks not just to model networks. From the above simulation results, we can see that these similar nodes can achieve synchronization under the impacts of the adaptive controllers. In the future, we will consider the possible application of this paper to packet delay issue in computer communications.

6. Conclusion

In this paper, we present the definition of similar nodes and analyze their minimal similarity in the network for the first time. We find the threshold of the minimal similarity of the similar nodes if it is bigger than the threshold, then the similar nodes can achieve synchronization without controllers. Otherwise, we have to add some controllers in order to get synchronization. So some new adaptive synchronization criteria are proposed to realize the synchronization of multilinks networks with similar nodes. Finally, numerical simulations are provided to show the effectiveness and the correctness of the proposed criteria. The model and the principles designed in this paper are very useful to analyze and control the dynamical multilinks networks with similar nodes, such as heart cells networks and neural networks.

Acknowledgments

This paper is supported by the Foundation for the Author of National Excellent Doctoral Dissertation of PR China (Grant no. 200951), the National Natural Science Foundation of China (Grant nos. 61100204, 61070209, and 61121061), and the Asia Foresight Program under NSFC Grant (Grant no. 61161140320).

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