IJMMS 2004:25, 1299–1308
PII. S0161171204307325
http://ijmms.hindawi.com

EDGE-DISJOINT HAMILTONIAN CYCLES IN TWO-DIMENSIONAL TORUS

MYUNG M. BAE, BADER F. ALBDAIWI, and BELLA BOSE

Received 20 July 2003

The torus is one of the popular topologies for the interconnecting processors to build high-performance multicomputers. This paper presents methods to generate edge-disjoint Hamiltonian cycles in 2D tori.

2000 Mathematics Subject Classification: 68M10.

1. Introduction. A multicomputer system consists of multiple nodes that communicate by exchanging messages through an interconnection network. At a minimum, each node normally has one or more processing elements, a local memory, and a communication module. A popular topology for the interconnection network is the *torus*. Also called a *wrap-around mesh* or a *toroidal mesh*, this topology includes the k-ary n-cube which is an n-dimensional torus with the restriction that each dimension is of the same size, k, and the hypercube, which is a k-ary n-cube with $k = 2$; a mesh is a subgraph of a torus.

Several parallel machines, both commercial and experimental, have been designed with a toroidal interconnection network. Included among these machines are the following: the iWarp (torus) [5], Cray T3D and T3E (3D torus) [13], the Mosaic (k-ary n-cube) [14], and the Tera parallel computer (torus) [2].

Some topological properties of torus and k-ary n-cubes based on Lee distance are given in [6, 7]. The existence of disjoint Hamiltonian cycles in the cross-product of various graphs has been discussed in [1, 4, 8, 9, 10, 11, 15]; however, a straightforward way of generating such cycles was not known until the results in [3], where some simple ways of generating edge-disjoint Hamiltonian cycles in k-ary n-cubes are presented. In this paper, some simple solutions to this problem are described for 2D torus. For example, Figure 1.1 gives two edge-disjoint cycles in $C_3 \times C_4$.

The rest of the paper is organized as follows. Section 2 gives some preliminaries about the definition of torus. Section 3 discusses the results on edge-disjoint Hamiltonian cycles on the 2D torus. Section 4 is the conclusion of this paper.

2. Preliminaries. This section contains definitions and mathematical background that will be useful in subsequent sections.

2.1. Lee distance, cross-product, and torus. Let $A = a_{n-1}a_{n-2}\cdots a_0$ be an n-dimensional mixed-radix vector over Z_K, where $K = k_{n-1}\times k_{n-2}\times\cdots\times k_0$, that is, all $a_i \in Z_{k_i}$,

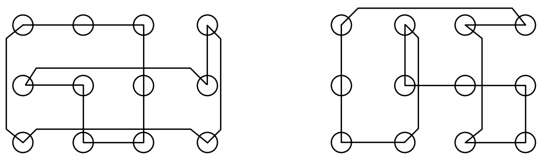


FIGURE 1.1. Two disjoint Hamiltonian cycles in $C_3 \times C_4$.

for $i = 0, 1, \ldots, n-1$. The Lee weight of A in mixed-radix notation is defined as

$$W_L(A) = \sum_{i=0}^{n-1} |a_i|, \tag{2.1}$$

where $|a_i| = \min(a_i, k_i - a_i)$, for $i = 0, 1, \ldots, n-1$.

The Lee distance between the two vectors A and B is denoted by $D_L(A,B)$ and is defined to be $W_L(A-B)$. That is, the Lee distance between the two vectors is the Lee weight of their digitwise difference. In other words, $D_L(A,B) = \sum_{i=0}^{n-1} \min(a_i - b_i, b_i - a_i)$, where $a_i - b_i$ and $b_i - a_i$ are $\bmod k_i$ operations. For example, when $K = 4 \times 6 \times 3$, $W_L(321) = \min(3, 4-3) + \min(2, 6-2) + \min(1, 3-1) = 1 + 2 + 1 = 4$, and $D_L(123, 321) = W_L(123 - 321) = W_L(202) = 3$.

A k-ary n-cube graph (C_k^n) and an n-dimensional torus ($T_{k_1,k_2,\ldots,k_n}$) are $2n$-regular graphs containing k^n and $k_1 k_2 \cdots k_n$ nodes, respectively; it is assumed that $k \geq 3$ and $k_i \geq 3$ for $i = 1, 2, \ldots, n$. Each node in a C_k^n is labeled with a distinct n-digit radix-k vector while each node in a $T_{k_1,k_2,\ldots,k_n}$ is labeled with a distinct n-digit mixed-radix vector. If u and v are two nodes in the graph, then there is an edge between them *if and only if* $D_L(u,v) = 1$. From the definition of Lee distance, it can be seen that every node in a C_k^n or a $T_{k_1,k_2,\ldots,k_n}$ shares an edge with two nodes in every dimension, resulting in a regular graph of degree $2n$.

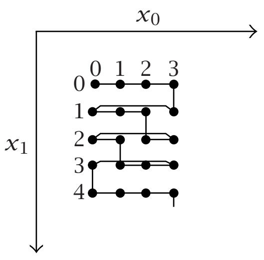
Since the Hamming distance, $D_H(A,B)$, between the two vectors A and B is the number of positions in which A and B differ, $D_L(A,B) = D_H(A,B)$ when $k_i = 2$ or 3, for all i, and $D_L(A,B) \geq D_H(A,B)$ when some $k_i > 3$.

The k-ary n-cube and the torus can also be seen as the cross-product of cycles. The *cross-product* of G_1 and G_2, denoted by $G = G_1 \otimes G_2$, is defined as follows [6, 12]:

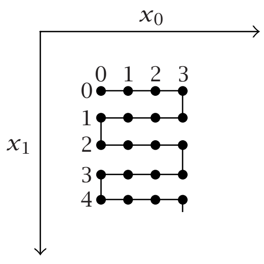
$$\begin{gathered} V = \{(u,v) \mid u \in V_1,\ v \in V_2\}, \\ E = \{((u_1,v_1),(u_2,v_2)) \mid ((u_1,u_2) \in E_1 \text{ and } v_1 = v_2) \text{ or } (u_1 = u_2 \text{ and } (v_1,v_2) \in E_2)\}, \end{gathered} \tag{2.2}$$

where $G = (V,E)$, $G_1 = (V_1,E_1)$, and $G_2 = (V_2,E_2)$. A cycle of length k is denoted by C_k, and each node in C_k is labeled with a radix k number, $0, \ldots, k-1$. There is an edge between vertices u and v *if and only if* $D_L(u,v) = 1$. Thus, a k-ary n-cube (C_k^n) and an n-dimensional torus ($T_{k_1,k_2,\ldots,k_n}$) can be defined as a product of cycles as follows:

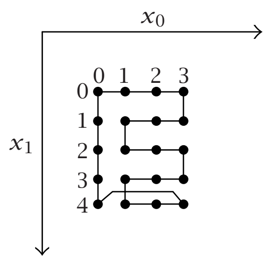
$$\begin{gathered} C_k^n = \underbrace{C_k \otimes C_k \otimes \cdots \otimes C_k}_{n \text{ times}} = \otimes_{i=1}^n C_k, \\ T_{k_1,k_2,\ldots,k_n} = C_{k_1} \otimes C_{k_2} \otimes \cdots \otimes C_{k_n}. \end{gathered} \tag{2.3}$$



(a) Spiral.



(b) Maze.



(c) Maze + feedback.

FIGURE 3.1. Basic mappings.

3. Edge-disjoint Hamiltonian cycles in a 2D torus. When edge-disjoint Hamiltonian cycles are used in a communication algorithm, their effectiveness is improved if more than one cycle exists. As mentioned earlier, the existence of disjoint Hamiltonian cycles in the cross-product of various graphs has been discussed in the literature [1, 4, 8, 9, 10, 11, 15]; however, all these methods do not give a straightforward way of generating such disjoint cycles. This section contains the functions that generate these disjoint cycles for the 2D torus.

Two Gray codes, G_1 and G_2, over Z_k^n are said to be *independent if* two words, a and b, are adjacent in G_1 (or G_2), then they are not adjacent in G_2 (or G_1). If $k \geq 3$, we can have at most n sets of independent Gray codes; for all $k_i = 2$, this number is $\lfloor n/2 \rfloor$. Note that the independent Gray codes form the edge-disjoint cycles.

In order to develop generating functions for 2D torus, we first define several basic mappings: *spiral* and *maze*, and some basic vector operations: *reverse, mod*, and *translate*.

Using these basic building blocks, we first show the generating functions of the edge-disjoint Hamiltonian cycles in 2D torus (T_{k_1,k_0}) whose sides are either both even or both odd, and then extend the result to the general 2D torus. This 2D Hamiltonian decomposition can also be used as a basis for 3D Hamiltonian decomposition.

Let X be an integer, $x_1 = \lfloor X/k \rfloor$, and $x_0 = X \bmod k$.

BASIC OPERATIONS.

(i) $\text{reverse}((x_1,x_0)) = (x_1,x_0)^R = (x_0,x_1)$.

(ii) $(x_1,x_0) \bmod (k_1,k_0) = (x_1 \bmod k_1, x_0 \bmod k_0)$.

(iii) $\text{translate}(+) : (x_1,x_0) + (d_1,d_0) = (x_1+d_1, x_0+d_0)$.

(iv) $\text{translate}(-) : (x_1,x_0) - (d_1,d_0) = (x_1-d_1, x_0-d_0)$.

SPIRAL MAPPING. Figure 3.1(a) is the graphical view of this mapping. The cycle is produced by generating $G_s(X;k)$ for successive values of X starting at $X = 0$:

$$\begin{gathered} G_s(X;k) = G_s((x_1,x_0);k) = (x_1,(x_0-x_1) \bmod k), \\ G_s^{-1}((y_1,y_0);k) = (y_1,(y_1+y_0) \bmod k). \end{gathered} \tag{3.1}$$

Maze mapping. Figure 3.1(b) is the graphical view of this mapping. The cycle is produced by generating $G_m(X;k)$ for successive values of X starting at $X = 0$:

$$\begin{aligned} G_m(X;k) = G_m((x_1,x_0);k) &= \begin{cases} (x_1,x_0), & \text{if } x_1 \text{ is even,} \\ (x_1,k-1-x_0), & \text{if } x_1 \text{ is odd,} \end{cases} \\ G_m^{-1}(Y;k) = G_m^{-1}((y_1,y_0);k) &= \begin{cases} (y_1,y_0), & \text{if } y_1 \text{ is even,} \\ (y_1,k-1-y_0), & \text{if } y_1 \text{ is odd.} \end{cases} \end{aligned} \tag{3.2}$$

Maze with feedback. Figure 3.1(c) is the graphical view of this mapping. The cycle is produced by generating $G_{mf}(X;k)$ for successive values of X starting at $X = 0$:

$$G_{mf}(X;k_1,k_0) = \begin{cases} G_m(X;k_0-1), & \text{if } x < k_1k_0-k_1, \\ -X, & \text{if } k_1k_0-k_1 \le x. \end{cases} \tag{3.3}$$

3.1. Special case 1: $k_1 = k_0 = k$. In a $T_{k,k}$, the Hamiltonian cycles can be described as follows:

$$\begin{aligned} h_0(X,T_{k,k}) &= (x_1,x_0-x_1) \bmod (k,k) = G_s(X,k), \\ h_1(X,T_{k,k}) &= h_0^R(X,T_{k,k}) = (x_0-x_1,x_1) \bmod (k,k), \end{aligned} \tag{3.4}$$

where $x_1 = \lfloor X/k \rfloor$ and $x_0 = X \bmod k$.

3.2. Special case 2: $k_1 = mk_0$ and $\mathrm{GCD}(k_1,k_0-1) = 1$

Theorem 3.1. *In a mixed-radix number system $Z_{k_1\times k_0}$, if $\mathrm{GCD}(k_1,k_0-1) = 1$ and $k_1 = mk_0$, for $m \ge 1$, then the following two functions generate the independent Gray codes:*

$$\begin{aligned} f_0(X;k_1,k_0) &= (x_1 \bmod k_1, (x_0+(k_0-1)x_1) \bmod k_0), \\ f_1(X;k_1,k_0) &= ((x_0+(k_0-1)x_1) \bmod k_1, x_1 \bmod k_0), \end{aligned} \tag{3.5}$$

where $X = (x_1,x_0)$, $x_1 \in Z_{k_1}$, and $x_0 \in Z_{k_0}$.

Proof. The proof has three parts.

(1) If $X' \ne X''$, then it is required to prove that $f_0(X';k_1,k_0) \ne f_0(X'';k_1,k_0)$ and $f_1(X';k_1,k_0) \ne f_1(X'';k_1,k_0)$. Let $X' = (x_1',x_0')$, $X'' = (x_1'',x_0'')$, $x_1',x_1'' \in Z_{k_1}$, and x_0', $x_0'' \in Z_{k_0}$.

(a) Suppose $f_0(X';k_1,k_0) = f_0(X'';k_1,k_0)$. Since $x_1' = x_1'' \bmod k_1$ and $x_1',x_1'' \in Z_{k_1}$, $x_1' = x_1''$. For the second component, $x_0' + (k_0-1)x_1' = x_0'' + (k_0-1)x_1'' \bmod k_0$, and hence $x_0' = x_0''$. Thus $f_0(X';k_1,k_0) \ne f_0(X'';k_1,k_0)$ if $X' \ne X''$.

(b) Suppose $f_1(X';k_1,k_0) = f_1(X'';k_1,k_0)$, $x_1' = x_1'' \bmod k_0$, that is, $k_0 \mid (x_1'-x_1'')$, and $x_0' + (k_0-1)x_1' = (x_0'' + (k_0-1)x_1'') \bmod k_1$, that is, $x_0'-x_0''+(k_0-1)(x_1'-x_1'') = 0 \bmod k_1$. Since $|x_0'-x_0''| < k_0$ and $k_0 \mid (x_1'-x_1'')$, $x_0'-x_0'' = 0 \bmod k_0$. Further, $x_1' - x_1'' = 0 \bmod k_1$ because $\mathrm{GCD}(k_0-1,k_1) = 1$. Thus $f_1(X';k_1,k_0) \ne f_1(X'';k_1,k_0)$ if $X' \ne X''$.

This implies that f_0 and f_1 are one-to-one mappings over Z_k^2.

(2) f_0 and f_1 generate cycles H_0 and H_1, respectively. In other words, the mappings of two numbers X and $X+1$ by f_0 or f_1 must generate an edge in H_0 or H_1. There would be the following subcases.

(a) Case $X=(x_1,x_0)$ and $X+1=(x_1,x_0+1)$. Since $f_0(X)=(x_1,x_0+(k_0-1)x_1)$ and $f_1(X+1)=(x_1,x_0+1+(k_0-1)x_1)$, we have $e_0=(f_0(X),f_0(X+1))=((x_1,x_0+(k_0-1)x_1),(x_1,x_0+1+(k_0-1)x_1))$ and $D_L(f_0(X),f_0(X+1))=1$. Thus e_0 is an edge of H_0.

(b) Case $X'=(x_1',k_0-1)$ and $X'+1=(x_1'+1,0)$. Similar to case (a), we have $e_1=((x_1',(k_0-1)(x_1'+1)),(x_1'+1,(k_0-1)(x_1'+1)))$ and $D_L(f_0(X'),f_0(X'+1))=1$. Thus e_1 is an edge of H_0.

(c) Case $X''=(x_1'',x_0'')$ and $X''+1=(x_1'',x_0''+1)$. Similarly, we have $e_2=(f_1(X''),f_1(X''+1))=((x_0''+(k_0-1)x_1'',x_1''),(x_0''+1+(k_0-1)x_1'',x_1''))$ and $D_L(f_1(X''),f_1(X''+1))=1$. Thus e_2 is an edge of H_1.

(d) Case $X'''=(x_1''',k_0-1)$ and $X'''+1=(x_1'''+1,0)$. We have $e_3=(f_1(X'''),f_1(X'''+1))=(((k_0-1)(x_1'''+1),x_1'''),((k_0-1)(x_1'''+1),(x_1'''+1)))$ and $D_L(f_1(X'''),f_1(X'''+1))=1$. Thus e_3 is an edge of H_1.

Since f_0 is one-to-one and $D_L(f_0(X),f_0(X+1))=1$, f_0 generates a Hamiltonian cycle. Similarly, f_1 also generates another Hamiltonian cycle.

(3) The edges which are generated by f_0 and f_1 must be unique so that the cycles, H_0 and H_1, become edge-disjoint. In other words, the edges e_0, e_1, e_2, and e_3 (described in the above case (2)) must be different. The proof is by contradiction. Suppose that e_0 (in H_0) is the same as e_2 (in H_1). For that, one of (3.6a) or (3.6b) must hold:

$$\begin{aligned} x_1 &= x_0''+(k_0-1)x_1'',\\ x_0+(k_0-1)x_1 &= x_1'',\\ x_1 &= x_0''+1+(k_0-1)x_1'',\\ x_0+1+(k_0-1)x_1 &= x_1'', \end{aligned} \tag{3.6a}$$

$$\begin{aligned} x_1 &= x_0''+1+(k_0-1)x_1'',\\ x_0+(k_0-1)x_1 &= x_1'',\\ x_1 &= x_0''+(k_0-1)x_1'',\\ x_0+1+(k_0-1)x_1 &= x_1''. \end{aligned} \tag{3.6b}$$

However, either (3.6a) or (3.6b) cannot be true. Similarly, one of e_0 and e_1 (in H_0) cannot be the same as one of e_2 and e_3 (in H_1). Therefore H_0 and H_1 are edge-disjoint. □

The inverses of f_0 and f_1 are as follows:

$$\begin{aligned} f_0^{-1}(Y;k_1,k_0) &= (y_1 \bmod k_1,(y_0-(k_0-1)y_1) \bmod k_0),\\ f_1^{-1}(Y;k_1,k_0) &= ((y_0-(k_0-1)y_1) \bmod k_1, y_1 \bmod k_0), \end{aligned} \tag{3.7}$$

where $Y=(y_1,y_0)$, $y_1\in Z_{k_1}$, and $y_0\in Z_{k_0}$.

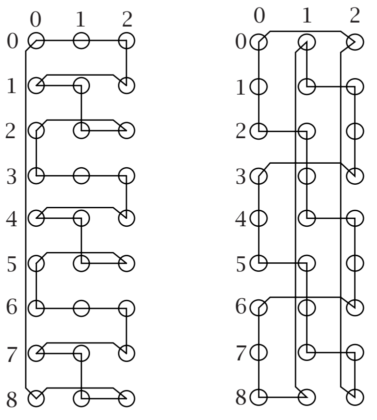


FIGURE 3.2. Edge-disjoint Hamiltonian cycles in $T_{9,3}$ produced by f_0 and f_1 of Theorem 3.1.

COROLLARY 3.2. *There are two independent Gray codes in $T_{k^r,k}$ for $k \geq 3$ and $r \geq 1$ and are generated by the functions h_0 and h_1, where*

$$\begin{aligned} h_0(x_1,x_0) &= (a_1,a_0) = (x_1,(x_0-x_1) \bmod k), \\ h_1(x_1,x_0) &= (b_1,b_0) = ((x_0+(k-1)x_1) \bmod k^r, x_1 \bmod k). \end{aligned} \tag{3.8}$$

The inverse functions are given by

$$\begin{aligned} h_0^{-1}(a_1,a_0) &= (x_1,x_0) = (a_1,(a_1+a_0) \bmod k), \\ h_1^{-1}(b_1,b_0) &= (x_1,x_0) = ((b_1-x_0)(k-1)^{-1} \bmod k^r, (b_1-b_0(k-1)) \bmod k) \\ &= ((b_1-x_0)(k-1)^{-1} \bmod k^r, (b_1+b_0) \bmod k) \\ &= ((b_1-((b_1+b_0) \bmod k))(k-1)^{-1} \bmod k^r, (b_1+b_0) \bmod k), \end{aligned} \tag{3.9}$$

where $(k-1)^{-1}$ is the multiplicative inverse of $(k-1)$ under $\bmod k^r$ (note that for $k \geq 3$, $k-1$ and k^r are relatively prime and so the inverse exists).

EXAMPLE 3.3. Figure 3.2 shows the two edge-disjoint Hamiltonian cycles in $T_{9\times 3}$ produced by f_0 and f_1 of Theorem 3.1.

3.3. $k_1 = k+2r$ and $k_0 = k+2s$. Without loss of generality, we assume $k_1 = k+2r$ and $k_0 = k+2s$ for some $k \geq 3$, $r \geq 0$, and $s \geq 0$.

DEFINITION 3.4. If $k_1 = k+2r$ and $k_0 = k+2s$ for some k, $r \geq 0$, and $s \geq 0$, define a function $h_0(X;T_{k_1,k_0})$ as follows:

$$\begin{aligned} h_0'(X;T_{k_1,k_0}) &= \begin{cases} G_s(X+2s;k_0)-(0,2s), & \text{if } 0 \leq X < p_\alpha, \\ G_m^R(X-p_\alpha;k_1-k+2)+(k-1,k), & \text{if } p_\alpha \leq X < p_\beta, \\ G_m(X-p_\beta;k)+(k,0), & \text{if } p_\beta \leq X, \end{cases} \\ h_0(X;T_{k_1,k_0}) &= h_0'(X;T_{k_1,k_0}) \bmod (k_1,k_0), \end{aligned} \tag{3.10}$$

where $p_\alpha = (k-2)k_0+2k-1$ and $p_\beta = k_1k_0 - k(k_1-k)$.

Note that $h_0(p_\alpha;T_{k_1,k_0}) = (k-1, k \bmod k_0)$ and $h_0(p_\beta;T_{k_1,k_0}) = (k \bmod k_1, 0)$.

Theorem 3.5. *The function $h_0(X;T_{k_1,k_0})$ generates a Hamiltonian cycle, $H_0(T_{k_1,k_0})$, in a 2D torus (T_{k_1,k_0}).*

Proof. The proof has two parts.

(1) If $X \neq X'$, then $h_0(X;T_{k_1,k_0}) \neq h_0(X';T_{k_1,k_0})$.

Assume $Y = (y_1,y_0) = h_0(X;T_{k_1,k_0})$. By Definition 3.4, the range of Y can be found from the range of X. If these ranges are disjoint, then the claim will be true:

$$\begin{aligned}
R_1 &= \{h_0(X) \mid 0 \le X < p_\alpha\} \\
&= \{(y_1,y_0) \mid y_1 = 0,\ 0 \le y_0 < k\} \cup \{(y_1,y_0) \mid 0 < y_1 < k-1,\ 0 \le y_0 < k_0\} \\
&\quad \cup \{(y_1,y_0) \mid y_1 = k-1,\ 1 \le y_0 < k\}, \\
R_2 &= \{h_0(X) \mid p_\alpha \le X < p_\beta\} \\
&= \{(y_1,y_0) \mid y_1 = 0,\ k \le y_0 < k_0\} \cup \{(y_1,y_0) \mid k-1 \le y_1 < k_1,\ k \le y_0 < k_0\}, \\
R_3 &= \{h_0(X) \mid p_\beta \le X\} = \{(y_1,y_0) \mid k \le y_1 < k_1,\ 0 \le y_0 < k\}.
\end{aligned} \tag{3.11}$$

Since R_1, R_2, and R_3 are mutually exclusive, the claim is true.

(2) $D_L(h_0(X;T_{k_1,k_0}), h_0(X';T_{k_1,k_0})) = 1$ if $X = X'+1$. In each subrange, the proof is trivial. The only case that needs to be considered is the situation where there are transitions from one subrange to another subrange. $h_0(p_\alpha - 1;T_{k_1,k_0}) = (k-1,k-1)$, $h_0(p_\alpha;T_{k_1,k_0}) = (k-1,k)$, $h_0(p_\beta - 1;T_{k_1,k_0}) = (k-1,0)$, and $h_0(p_\beta;T_{k_1,k_0}) = (k,0)$. □

Corollary 3.6 (inverse of $h_0(X;T_{k_1,k_0})$). *$h_0^{-1}((y_1,y_0);T_{k_1,k_0})$ is described as follows:*

$$h_0^{-1}(Y;T_{k_1,k_0}) = \begin{cases} G_s^{-1}(Y+(0,2s);k_0) - 2s, & \text{if } Y \in R_1, \\ G_m^{-1}((Y-(k-1,k))^R;k_1-k+2) + p_\alpha, & \text{if } Y \in R_2, \\ G_m^{-1}(Y-(k,0);k) + p_\beta, & \text{if } Y \in R_3. \end{cases} \tag{3.12}$$

Theorem 3.7. *If $H_0(T_{k_1,k_0})$ and $H_1(T_{k_1,k_0})$ are the edge-disjoint Hamiltonian cycles, and $H_0(T_{k_1,k_0})$ is generated by the function $h_0(X;T_{k_1,k_0})$, then the generator function $h_1(X;T_{k_1,k_0})$ for $H_1(T_{k_1,k_0})$ is defined as*

$$h_1(X;T_{k_1,k_0}) = h_0^R(X;T_{k_0,k_1}). \tag{3.13}$$

Proof. We prove the theorem by induction on k_1 and k_0.

Base step. Consider $T_{k,k}$. We get $h_0(X;T_{k,k}) = G_s(X;k)$ and $h_1^R(X;T_{k,k}) = G_s^R(X;k)$. Thus $H_0(T_{k,k})$ and $H_1(T_{k,k})$ are edge-disjoint.

Inductive step. Assume that a T_{k_1,k_0}, where $k_1 = k+2r$ and $k_0 = k+2s$, has two edge-disjoint Hamiltonian cycles generated by $h_0(X;T_{k_1,k_0})$ and $h_1(X;T_{k_1,k_0})$.

Case 1. $k_1' = k_1 + 2$: Figure 3.3 illustrates the process of adding two rows to a $T_{3,7}$.

Case 2. $k_0' = k_0 + 2$: similar to the previous case. □

Example 3.8. Figure 3.3 shows two edge-disjoint Hamiltonian cycles in $T_{5,7}$ produced by Theorem 3.7.

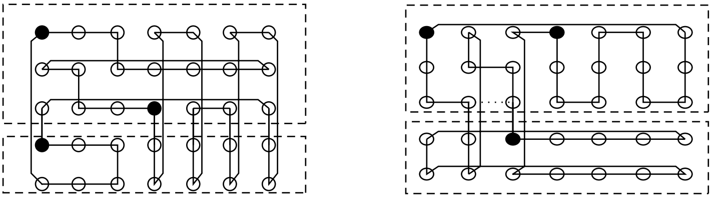


FIGURE 3.3. H_0 and H_1 in $T_{5,7}$ ($k = 3$).

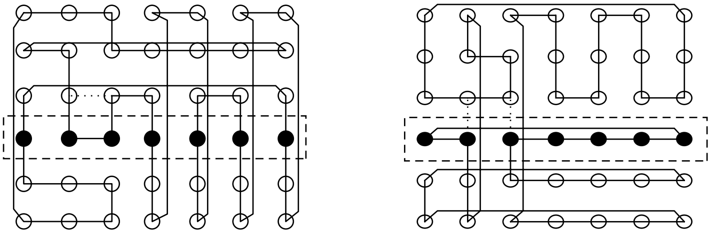


FIGURE 3.4. H_0 and H_1 in $T_{6,7}$ ($k = 3$).

3.4. $k_1 = 4+2r$ and $k_0 = 3+2s$. Without loss of generality, we assume $k_1 = 4+2r$ and $k_0 = 3+2s$. We treat T_{k_1,k_0} as a torus obtained after inserting a row to T_{k_1-1,k_0}. For example, $T_{6,7}$ is the torus obtained from $T_{3,3}$ after inserting rows and columns as in Figure 3.4.

Corollary 3.9. *$H_0(T_{k_1,k_0})$ from $H_0(T_{k_1-1,k_0})$: by inserting a row in between the third and the fourth row of T_{k_1-1,k_0}, the function $h_0(X;T_{k_1,k_0})$ which generates a Hamiltonian cycle, $H_0(T_{k_1,k_0})$, can be described as follows:*

$$h_0'(X;T_{k_1,k_0}) = \begin{cases} G_s(X+2s;k_0)-(0,2s), & \text{if } 0 \le X < k_0+3, \\ G_m^R(X-k_0-3;2)+(2,1), & \text{if } k_0+3 \le X < k_0+7, \\ G_m^R(X-k_0-7;k_1-1)+(2,3), & \text{if } k_0+7 \le X < p_\beta, \\ G_m(X-p_\beta;3)+(4,0), & \text{if } p_\beta \le X, \end{cases} \tag{3.14}$$
$$h_0(X;T_{k_1,k_0}) = h_0'(X;T_{k_1,k_0}) \bmod (k_1,k_0),$$

where $p_\beta = k_1k_0 - 3(k_1-4)$.

Note that $h_0(p_\gamma;T_{k_1,k_0}) = (1,0)$. Further note that $p_\alpha = p_\beta$ if $k_0 = 3$, and $0 = p_\beta$ if $k_1 = 4$, and we get

$$\begin{aligned} h_0(0,T_{k_1,k_0}) &= (0,0), \\ h_0(p_\alpha,T_{k_1,k_0}) &= (4 \bmod k_1, 3 \bmod k_0), \\ h_0(p_\beta,T_{k_1,k_0}) &= (4 \bmod k_1, 0). \end{aligned} \tag{3.15}$$

Example 3.10. Figure 3.4 shows two edge-disjoint Hamiltonian cycles in $T_{6,7}$ produced from $T_{5,7}$ by the above corollaries.

Corollary 3.11. *$H_1(T_{k_1,k_0})$ from $H_1(T_{k_1-1,k_0})$: the second function $h_1(X;T_{k_1,k_0})$, where k_1 is even and k_0 is odd, is as follows:*

$$h_1'(X;T_{k_1,k_0}) = \begin{cases} G_s^R(X;3), & \text{if } 0 \le X < 4, \\ G_s^R(p_\alpha - 1 - X; k_1) + (3,1), & \text{if } 4 \le X < p_\alpha, \\ G_m(X - p_\alpha + k_0 - 1; k_0 - 1) + (2,2), & \text{if } p_\alpha \le X < p_\beta, \\ G_m^R(X - p_\beta; 3) + (0,3), & \text{if } p_\beta \le X, \end{cases}$$
$$h_1(X;T_{k_1,k_0}) = h_1'(X;T_{k_1,k_0}) \bmod (k_1,k_0), \tag{3.16}$$

where $p_\alpha = k_1 + 5$, $p_\beta = k_1k_0 - 3(k_0 - 3)$.

Note that $p_\alpha = p_\beta$ if $k_0 = 4$, and $0 = p_\beta$ if $k_1 = 3$, and we get

$$\begin{aligned} h_0(0,T_{k_1,k_0}) &= (0,0), \\ h_0(p_\alpha,T_{k_1,k_0}) &= (3 \bmod k_1, 4 \bmod k_0), \\ h_0(p_\beta,T_{k_1,k_0}) &= (3 \bmod k_1, 0). \end{aligned} \tag{3.17}$$

Corollary 3.12. *$h_0(X,T_{k_1,k_0})$ and $h_1(X,T_{k_1,k_0})$, by Corollaries 3.9 and 3.11, generate two edge-disjoint Hamiltonian cycles in T_{k_1,k_0}, where k_1 is even and k_0 is odd.*

Similar to Corollary 3.6, we can obtain the inverses of $h_0(X,T_{k_1,k_0})$ and $h_1(X,T_{k_1,k_0})$.

4. Conclusion. In this paper, we present methods to generate the edge-disjoint Hamiltonian cycles in 2D torus. These methods can be used to generate edge-disjoint Hamiltonian cycles in higher-dimensional torus networks. For example, consider a 4D torus $T = (C_{k1} \otimes C_{k2} \otimes C_{k3} \otimes C_{k4})$. This can be decomposed as $T = (H_1 \oplus H_2) \otimes (H_3 \oplus H_4)$, where H_0 and H_1 are disjoint cycles obtained from $(C_{k1} \otimes C_{k2})$; so also H_3 and H_4 from $(C_{k3} \otimes C_{k4})$. Then, T can be written as

$$T = (H_1 \otimes H_3) \oplus (H_2 \otimes H_4) = H_1' \oplus H_2' \oplus H_3' \oplus H_4'. \tag{4.1}$$

All these four cycles (H_i's) are disjoint and of length ($k_1 \times k_2 \times k_3 \times k_4$). Some simple functions to generate these cycles need further research.

Acknowledgment. The third author's work is supported in part by the National Science Foundation under Grant MIP-9705738.

References

[1] B. Alspach, J.-C. Bermond, and D. Sotteau, *Decomposition into cycles. I. Hamilton decompositions*, Cycles and Rays (Montreal, PQ, 1987) (G. Hahn, G. Sabidussi, and R. E. Woodrow, eds.), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 301, Kluwer Academic Publishers, Dordrecht, 1990, pp. 9–18.

[2] R. Alverson, D. Callahan, D. Cummings, B. Koblenz, A. Porterfield, and B. Smith, *The Tera Computer System*, Proceedings of the 1990 international Conference on Supercomputing (Amsterdam, 1990), ACM press, pp. 1–6 .

[3] M. M. Bae and B. Bose, *Gray codes for torus and edge disjoint Hamiltonian cycles*, Proceedings of IEEE 14th International Parallel and Distributed Processing Symposium (Cancun, 2000), IEEE, Cancun, Mexico, 2000, pp. 365–370.

[4] J.-C. Bermond, O. Favaron, and M. Mahéo, *Hamiltonian decomposition of Cayley graphs of degree* 4, J. Combin. Theory Ser. B **46** (1989), no. 2, 142–153.

[5] S. Borkar, R. Cohn, G. Cox, S. Gleason, T. Gross, H. T. Kung, M. Lam, B. Moore, C. Peterson, J. Pieper, L. Rankin, P. S. Tseng, J. Sutton, J. Urbanski, and J. Webb, *iWarp: an integrated solution to high-speed parallel computing*, Proc. IEEE Supercomputing '88 (Fla, 1988), IEEE Computer Society/ACM, Orlando, FL, 1988, pp. 330–339.

[6] B. Bose, B. Broeg, Y. Kwon, and Y. Ashir, *Lee distance and topological properties of k-ary n-cubes*, IEEE Trans. Comput. **44** (1995), no. 8, 1021–1030.

[7] B. Broeg, B. Bose, and V. Lo, *Lee distance, Gray codes, and the torus*, Telecommunication Systems **10** (1998), no. 1, 21–32, Special Issue on High Performance Computing and Interconnection Networks.

[8] M. Y. Chan and S.-J. Lee, *On the existence of Hamiltonian circuits in faulty hypercubes*, SIAM J. Discrete Math. **4** (1991), no. 4, 511–527.

[9] P. Cull, *Tours of graphs, digraphs, and sequential machines*, IEEE Trans. Comput. **29** (1980), no. 1, 50–54.

[10] M. F. Foregger, *Hamiltonian decompositions of products of cycles*, Discrete Math. **24** (1978), no. 3, 251–260.

[11] Y. X. Huang, *On Hamiltonian decompositions of Cayley graphs on cyclic groups*, Graph Theory and Its Applications: East and West (Jinan, 1986), Ann. New York Academy of Science, vol. 576, New York Acad. Sci., New York, 1989, pp. 250–258.

[12] F. T. Leighton, *Introduction to Parallel Algorithms and Architectures. Arrays, Trees, Hypercubes*, Morgan Kaufmann, California, 1992.

[13] W. Oed, *Massively parallel processor system CRAY T3D*, Tech. report, Cray Research GmbH, München, 1993.

[14] C. L. Seitz, *Submicron systems architecture project semi-annual technical report*, Tech. Report Caltec-CS-TR-88-18, California Institute of Technology, California, 1988.

[15] R. Stong, *Hamilton decompositions of Cartesian products of graphs*, Discrete Math. **90** (1991), no. 2, 169–190.

Myung M. Bae: Unix Development Lab., MS/P963, IBM Corp., Poughkeepsie, NY 12601, USA
E-mail address: myungbae@us.ibm.com

Bader F. Albdaiwi: Department of Mathematics and Computer Science, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait
E-mail address: bdaiwi@mcc.sci.kuniv.edu.kw

Bella Bose: Computer Science Department, Oregon State University, Corvallis, OR 97331-3902, USA
E-mail address: bose@cs.orst.edu