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## Research Article

# Separation Axioms in Intuitionistic Fuzzy Topological Spaces

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In this paper we have studied separation axioms  $T_i$ ,  $i = 0, 1, 2$  in an intuitionistic fuzzy topological space introduced by Coker. We also show the existence of functors  $\mathcal{B} : \text{IF-Top} \rightarrow \text{BF-Top}$  and  $\mathcal{D} : \text{BF-Top} \rightarrow \text{IF-Top}$  and observe that  $\mathcal{D}$  is left adjoint to  $\mathcal{B}$ .

## 1. Introduction

Fuzzy sets were introduced by Zadeh [1] in 1965 as follows: a fuzzy set  $A$  in a nonempty set  $X$  is a mapping from  $X$  to the unit interval  $[0, 1]$ , and  $A(x)$  is interpreted as the degree of membership of  $x$  in  $A$ . Atanassov [2] generalized this concept and introduced intuitionistic fuzzy sets which take into account both the degrees of membership and of nonmembership subject to the condition that their sum does not exceed 1. Çoker [3] subsequently initiated a study of intuitionistic fuzzy topological spaces.

In this paper we have searched for appropriate definitions of the separation axioms  $T_i$ ,  $i = 0, 1, 2$  in intuitionistic fuzzy topological spaces.

Hausdorffness in an intuitionistic fuzzy topological space has been introduced earlier by Çoker [3], Bayhan and Çoker [4], and Lupianez [5]. In [4], the authors have given six possible definitions of Hausdorffness including that given in [3], and a comparative study has been done. In this paper we have introduced another definition which generalizes the corresponding definition in a fuzzy topological space given in [6]. Our definition is more general than those given in [3, 5], and it turns out to be equivalent to  $FT_2(vi)$  in [4].

$T_1$ -ness in an intuitionistic fuzzy topological space has been defined earlier in [4] in six possible ways. Out of those, we have chosen  $FT_1(ii)$  as it generalizes the most appropriate definition of  $T_1$ -ness in a fuzzy topological space (cf. definition 5.1, [7]). We have also introduced a suitable definition of  $T_0$ -ness in an intuitionistic topological space.

The appropriateness of the definitions has been established by proving several basic desirable results; for example, they satisfy hereditary, productive, and projective properties. We have also shown that the functor  $\mathcal{B} : \text{IF-Top} \rightarrow \text{BF-Top}$  preserves these separation properties.

## 2. Preliminaries

Throughout  $X$  denotes a nonempty set,  $I$  denotes the unit interval  $[0, 1]$ , and  $I_0$  and  $I_1$  denote the intervals  $(0, 1]$  and  $[0, 1)$ , respectively. A fuzzy set in  $X$  is a function from  $X$  to  $I$ . The collection of all fuzzy sets in  $X$  is denoted by  $I^X$ . For any  $A \in I^X$ ,  $A'$  denotes the fuzzy complement of  $A$ , and the constant fuzzy set in  $X$ , taking value  $\alpha \in I$ , is denoted by  $\underline{\alpha}$ . A crisp subset of  $X$  will be identified with its characteristic function. If  $Y \subseteq X$ , then  $A \in I^Y$  will be identified with the fuzzy set in  $X$  which takes the same value as  $A$  if  $x \in Y$  and zero if  $x \notin Y$ .

*Definition 1* (Atanassov [2]). Let  $X$  be a nonempty set. An intuitionistic fuzzy set (IFS, in short)  $A$  is an ordered pair  $(\mu_A, \nu_A)$  of fuzzy sets in  $X$ . Here  $(\mu_A, \nu_A)(x) = (\mu_A(x), \nu_A(x))$  and  $\mu_A(x)$ ,  $\nu_A(x)$ , respectively, denote the degree of membership and the degree of nonmembership of  $x \in X$  to the set  $A$  and  $0 \leq \mu_A(x) + \nu_A(x) \leq 1$  for each  $x \in X$ .

We identify an ordinary fuzzy set  $A \in I^X$  with the intuitionistic fuzzy set  $(A, A')$ .

*Definition 2* (Atanassov [2]). Let  $X$  be a nonempty set and  $A, B$  be given by  $(\mu_A, \nu_A)$  and  $(\mu_B, \nu_B)$ , respectively,

- (a)  $A \subseteq B$  if  $\mu_A(x) \leq \mu_B(x)$  and  $\nu_A(x) \geq \nu_B(x)$  for all  $x \in X$ ,
- (b)  $A = B$  if  $A \subseteq B$  and  $B \subseteq A$ ,
- (c)  $\bar{A} = (\nu_A, \mu_A)$ ,
- (d)  $A \cap B = (\mu_A \cap \mu_B, \nu_A \cup \nu_B)$ ,
- (e)  $A \cup B = (\mu_A \cup \mu_B, \nu_A \cap \nu_B)$ .

**Definition 3** (Çoker [3]). Let  $\{A_i : i \in J\}$  be an arbitrary family of IFSs in  $X$ . Then

- (a)  $\cap A_i = (\cap \mu_{A_i}, \cup \nu_{A_i})$ ,
- (b)  $\cup A_i = (\cup \mu_{A_i}, \cap \nu_{A_i})$ ,
- (c)  $0_{\sim} = (\underline{0}, \underline{1})$ ,  $1_{\sim} = (\underline{1}, \underline{0})$ .

**Definition 4** (Çoker [3]). Let  $X$  and  $Y$  be two nonempty sets and  $f : X \rightarrow Y$  be a function. If  $A$  and  $B$  be IFSs in  $X$  and  $Y$ , respectively, then

- (a)  $f(A) = (f(\mu_A), (1 - f(1 - \nu_A)))$ ,
- (b)  $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B))$ .

It is easy to verify that  $f^{-1}(\cap A_i) = \cap f^{-1}(A_i)$  and  $f^{-1}(\cup A_i) = \cup f^{-1}(A_i)$ .

**Definition 5** (Wong [8]). A fuzzy point  $x_r$  in  $X$  is a fuzzy set in  $X$  taking value  $r \in (0, 1)$  at  $x$  and zero elsewhere, and  $x$  and  $r$  are, respectively, called the support and value of  $x$ .

A fuzzy point  $x_r$  is said to belong to a fuzzy set  $A$  (notation :  $x_r \in A$ ) if  $r < A(x)$  (cf. [6]).

Two fuzzy points are said to be distinct if their supports are distinct.

**Definition 6.** Let  $X$  be a nonempty set and  $x \in X$  a fixed element in  $X$ . If  $\alpha \in (0, 1)$  and  $\beta \in (0, 1)$  are two fixed real numbers such that  $\alpha + \beta \leq 1$ , then the IFS  $x_{(\alpha, \beta)} = (x_\alpha, 1 - x_{(1-\beta)})$  is called an intuitionistic fuzzy point (IFP, in short) in  $X$ , and  $x$  is called its support. Two IFPs are said to be distinct if their supports are distinct.

Let  $x_{(\alpha, \beta)}$  be an IFP in  $X$  and  $A = (\mu_A, \nu_A)$  be an IFS in  $X$ . Then  $x_{(\alpha, \beta)}$  is said to belong to  $A$  (notation :  $x_{(\alpha, \beta)} \in A$ , in short) if  $\alpha < \mu_A(x)$ ,  $\beta > \nu_A(x)$  (cf. [9]).

We identify a fuzzy point  $x_r$  in  $X$  by the intuitionistic fuzzy point  $x_{(r, (1-r))}$  in  $X$ .

**Proposition 7.** An intuitionistic fuzzy set  $A$  in  $X$  is the union of all intuitionistic fuzzy points belonging to  $A$ .

The proof is on similar lines as in [10, Theorem 2.4] and hence is omitted.

Replacing fuzzy sets by intuitionistic fuzzy sets in Chang's definition of a fuzzy topological space, we get the following.

**Definition 8** (Çoker [3]). An intuitionistic fuzzy topology (IFT, in short) on a nonempty set  $X$  is a family  $\tau$  of IFSs in  $X$  satisfying the following axioms:

- (1)  $0_{\sim}, 1_{\sim} \in \tau$ ,
- (2)  $G_1 \cap G_2 \in \tau$ , for all  $G_i \in \tau$ ,  $i = 1, 2$ ,
- (3)  $\cup G_i \in \tau$  for any arbitrary family  $\{G_i \in \tau : i \in J\}$ .

The pair  $(X, \tau)$  is called an intuitionistic fuzzy topological space (IFTS, in short), members of  $\tau$  are called intuitionistic fuzzy open sets (IFOS, in short) in  $X$ , and their complements are called intuitionistic fuzzy closed sets (IFCS, in short).

**Definition 9.** Let  $(X, \tau)$  be an IFTS. A subfamily  $\mathcal{B} \subseteq \tau$  is called a base for  $\tau$  if every  $U \in \tau$  can be written as a union of members of  $\mathcal{B}$ .

**Proposition 10.** Let  $(X, \tau)$  be an IFTS, and then a subfamily  $\mathcal{B} \subseteq \tau$  is a base for  $\tau$  if and only if for all  $U \in \tau$  and intuitionistic fuzzy point  $x_{(\alpha, \beta)} \in U$ ,  $\exists B \in \mathcal{B}$  such that  $x_{(\alpha, \beta)} \in B \subseteq U$ .

The proof is easy omitted.

**Definition 11.** Let  $(X, \tau)$  be an IFTS. Then a subfamily  $\mathcal{J} \subseteq \tau$  is called a subbase for  $\tau$  if the family of finite intersections of members of  $\mathcal{J}$  forms a base for  $\tau$ .

Given any collection  $\mathcal{J}$  of IFSs in  $X$ , containing  $0_{\sim}$  and  $1_{\sim}$ , the set  $\tau$  consisting of arbitrary unions of finite intersections of members of  $\mathcal{J}$  forms an IFT on  $X$ . This is the smallest IFT on  $X$  containing  $\mathcal{J}$  and is called the IFT generated by  $\mathcal{J}$ .

**Definition 12** (S. J. Lee and E. P. Lee [10]). An IFS  $N$  in an IFTS  $(X, \tau)$  is called an intuitionistic fuzzy neighborhood (IFN, in short) of an IFP  $x_{(\alpha, \beta)}$  if  $\exists U \in \tau$  such that  $x_{(\alpha, \beta)} \in U \subseteq N$ .

**Proposition 13.** Let  $(X, \tau)$  be an IFTS. Then an IFS  $A$  in  $X$  is an IFOS if and only if  $A$  is an IFN of each of IFP  $x_{(\alpha, \beta)} \in A$ .

The proof is on similar lines as in ([10], Theorem 2.6) and hence is omitted.

**Definition 14** (S. J. Lee and E. P. Lee [10]). A map  $f : (X, \tau) \rightarrow (Y, \delta)$  between IFTSs is called intuitionistic fuzzy continuous if  $f^{-1}(U) \in \tau$ , for all  $U \in \delta$ .

**Definition 15** (Abu Safia et al. [12]). Let  $X$  be a nonempty set and  $\tau_1, \tau_2$  be two fuzzy topologies on  $X$ . Then  $(X, \tau_1, \tau_2)$  is called a bifuzzy topological space (BFTS, in short).

A map  $f : (X, \tau_1, \tau_2) \rightarrow (Y, \delta_1, \delta_2)$  between two BFTSs is said to be FP continuous if  $f^{-1}(U_i) \in \tau_i$ , for all  $U_i \in \delta_i$ ,  $i = 1, 2$ .

**Definition 16** (Bayhan and Çoker [4]). Let  $A = (\mu_A, \nu_A)$  and  $B = (\mu_B, \nu_B)$  be IFSs in  $X$  and  $Y$ , respectively, and then  $A \times B$  is the IFS in  $X \times Y$  defined as follows

$$A \times B = (\mu_A \times \mu_B, \nu_A * \nu_B), \quad (1)$$

where  $(\mu_A \times \mu_B)(x, y) = \min(\mu_A(x), \mu_B(y))$ , for all  $(x, y) \in X \times Y$  and  $(\nu_A * \nu_B)(x, y) = \max(\nu_A(x), \nu_B(y))$ , for all  $(x, y) \in X \times Y$ .

This definition can be extended to an arbitrary family of IFSs as follows.

If  $\{A_i = (\mu_{A_i}, \nu_{A_i}), i \in J\}$  is a family of IFSs in  $X_i$ , then their product is defined as the IFS in  $\Pi X_i$  given by

$$\Pi A_i = (\Pi \mu_{A_i}, \Pi^* \nu_{A_i}), \quad (2)$$

where  $\Pi \mu_{A_i}(x) = \inf \mu_{A_i}(x_i)$ , for all  $x = \Pi x_i \in X$  and  $\Pi^* \nu_{A_i}(x) = \sup \nu_{A_i}(x_i)$ , for all  $x = \Pi x_i \in X$ .

**Definition 17** (Bayhan and Çoker [4]). Let  $(x_i, \tau_i), i = 1, 2$  be two IFTSs, and then the product IFT  $\tau_1 \times \tau_2$  on  $X_1 \times X_2$  is defined as the IFT generated by  $\{p_i^{-1}(U_i) : U_i \in \tau_i, i = 1, 2\}$  where  $p_i : X_1 \times X_2 \rightarrow X_i, i = 1, 2$  are the projection maps, and the IFTS  $(X_1 \times X_2, \tau_1 \times \tau_2)$  is called the product IFTS.

This definition can be extended to an arbitrary family of IFTSs as follows.

Let  $\{(X_i, \tau_i) : i \in J\}$  be a family of IFTSs. Then the product intuitionistic fuzzy topology  $\tau$  on  $X = \Pi X_i$  is the one having  $\{p_j^{-1}(U_j) : U_j \in \tau_j, j \in J\}$  as a subbase where  $p_j : X \rightarrow X_j$  is the  $j$ th projection map.  $(X, \tau)$  is called the product IFTS of the family  $\{(X_i, \tau_i) : i \in J\}$ .

**Definition 18.** A fuzzy topological space  $(X, \tau)$  is called

- (a)  $T_0$  if for all  $x, y \in X, x \neq y, \exists U \in \tau$  such that either  $U(x) = 1, U(y) = 0$ , or  $U(y) = 1, U(x) = 0$ ,
- (b)  $T_1$  if for all  $x, y \in X, x \neq y, \exists U, V \in \tau$  such that  $U(x) = 1, U(y) = 0, V(y) = 1$ , and  $V(x) = 1$ ,
- (c)  $T_2$  (Hausdorff) if for all pair of distinct fuzzy points  $x_r, y_s$  in  $X, \exists U, V \in \tau$  such that  $x_r \in U, y_s \in V$ , and  $U \cap V = 0$ ,
- (d)  $q-T_2$  ( $q$ -Hausdorff) if for any pair of distinct fuzzy points  $x_r$ , and  $y_s, \exists U, V \in \tau$  such that  $x_r \in U, y_s \in \tau$  and  $U \subseteq V'$ .

Here definitions (d), (c), (b), and (a) are from [5–7, 13], respectively.

**Definition 19.** Let  $(X, \tau_1, \tau_2)$  be a BFTS. Then it is called

- (a)  $T_0$  if for all  $x, y \in X, x \neq y, \exists U \in \tau_1 \cup \tau_2$  such that  $U(x) = 1, U(y) = 0$  or  $U(x) = 0, U(y) = 1$ ,
- (b)  $T_1$  if for all  $x, y \in X, x \neq y, \exists U \in \tau_1$  and  $V \in \tau_2$  such that  $U(x) = 1, U(y) = 0$  and  $V(x) = 0, V(y) = 1$ ,
- (c)  $T_2$  if for all pair of distinct fuzzy points  $x_r, y_s$  in  $X, \exists U \in \tau_1, V \in \tau_2$  such that  $x_r \in U, y_s \in V$  and  $U \cap V = 0$ .

Here definitions (a) and (b) are from [14], and (c) is from [15].

For the categorical concepts used here, we refer the reader to [16].

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**Definition 20.** An IFTS  $(X, \tau)$  is called

- (a)  $T_0$  if for all  $x, y \in X, x \neq y, \exists U = (\mu_U, \nu_U), V = (\mu_V, \nu_V) \in \tau$  such that  $(\mu_U, \nu_U)(x) = (1, 0), (\mu_U, \nu_U)(y) = (0, 1)$  or  $(\mu_V, \nu_V)(x) = (0, 1), (\mu_V, \nu_V)(y) = (1, 0)$ ,
- (b) (Bayhan and Çoker [4]).  $T_1$  if for all  $x, y \in X, x \neq y, \exists U = (\mu_U, \nu_U), V = (\mu_V, \nu_V) \in \tau$  such that  $(\mu_U, \nu_U)(x) = (1, 0), (\mu_U, \nu_U)(y) = (0, 1), (\mu_V, \nu_V)(x) = (0, 1)$  and  $(\mu_V, \nu_V)(y) = (1, 0)$ ,
- (c)  $T_2$  (Hausdorff) if for all pair of distinct intuitionistic fuzzy points  $x_{(\alpha, \beta)}, y_{(\gamma, \delta)}$  in  $X, \exists U, V \in \tau$  such that  $x_{(\alpha, \beta)} \in U, y_{(\gamma, \delta)} \in V$  and  $U \cap V = 0_{\sim}$ ,
- (d)  $q-T_2$  if for every pair of distinct intuitionistic fuzzy points  $x_{(\alpha, \beta)}, y_{(\gamma, \delta)}$  in  $X, \exists U$  and  $V \in \tau$  such that  $x_{(\alpha, \beta)} \in U, y_{(\gamma, \delta)} \in V$  and  $U \subseteq V'$ .

**Example 21.** Let  $X = \{a, b\}$  and let  $\tau = \{0_{\sim}, A, B, 1_{\sim}\}$ , where  $A = \langle x, (a/1, b/0), (a/0, b/1) \rangle$  and  $B = \langle x, (a/1, b/0), (a/0, b/1) \rangle$ , then  $(X, \tau)$  is an IFTS, and it is  $T_0, T_1, T_2$  (Hausdorff) and  $q-T_2$ .

We have  $T_2 \Rightarrow T_1 \Rightarrow T_0$  and  $T_2 \Rightarrow q-T_2$ , but none of the implication are reversible.

Now we associate a BFTS with an IFTS and vice versa on parallel lines as in Bayhan and Çoker [11].

Let  $(X, \tau)$  be an IFTS and  $\tau_1 = \{\mu_A \mid \exists \nu_A \in I^X \text{ such that } (\mu_A, \nu_A) \in \tau\}, \tau_2 = \{\underline{1} - \nu_A \mid \exists \mu_A \in I^X \text{ such that } (\mu_A, \nu_A) \in \tau\}$ . It is easy to see that  $(X, \tau_1)$  and  $(X, \tau_2)$  are fuzzy topological spaces in Chang's sense.

$(X, \tau_1, \tau_2)$  is called the bifuzzy topological space associated with the IFTS  $(X, \tau)$ .

**Proposition 22.** Let  $(X, \tau_1, \tau_2)$  be a BFTS and  $\tau_{\tau_1, \tau_2} = \{(U, V') \mid U \in \tau_1, V \in \tau_2 \text{ and } U \subseteq V'\}$ . Then  $(X, \tau_{\tau_1, \tau_2})$  is an IFTS and  $(\tau_{\tau_1, \tau_2})_1 = \tau_1, (\tau_{\tau_1, \tau_2})_2 = \tau_2$ .

*Proof.* Clearly members of  $\tau_{\tau_1, \tau_2}$  are intuitionistic fuzzy sets, and  $0_{\sim}$  and  $1_{\sim}$  belong to it. Now let  $(U_i, V'_i) \in \tau_{\tau_1, \tau_2}, i = 1, 2$  then  $(U_1, V'_1) \cap (U_2, V'_2) = (U_1 \cap U_2, V'_1 \cup V'_2) = (U_1 \cap U_2, (V_1 \cap V_2)') \in \tau_{\tau_1, \tau_2}$ . Further let  $\{(U_i, V'_i) : i \in J, \text{ where } J \text{ is arbitrary}\} \subseteq \tau_{\tau_1, \tau_2}$ . Then  $\bigcup \{(U_i, V'_i) : i \in J, \text{ where } J \text{ is arbitrary}\} \subseteq \tau_{\tau_1, \tau_2}$ . Thus  $\bigcup \{(U_i, V'_i) : i \in J, \text{ where } J \text{ is arbitrary}\} \subseteq \tau_{\tau_1, \tau_2}$ . Thus  $(X, \tau_{\tau_1, \tau_2})$  is an IFTS.

Now let  $U \in \tau_1$  then  $(U, \phi) \in \tau_{\tau_1, \tau_2}$ . Therefore  $\tau_1 \subseteq (\tau_{\tau_1, \tau_2})_1$ . Conversely let  $U \in (\tau_{\tau_1, \tau_2})_1$  then  $\exists V \in I^X$  such that  $(U, V) \in \tau_{\tau_1, \tau_2} \Rightarrow U \in \tau_1$ , so  $(\tau_{\tau_1, \tau_2})_1 \subseteq \tau_1$ . Thus  $(\tau_{\tau_1, \tau_2})_1 = \tau_1$ . Similarly we can show that  $(\tau_{\tau_1, \tau_2})_2 = \tau_2$ .  $\square$

The IFTS  $(X, \tau_{\tau_1, \tau_2})$  is called the IFTS associated with the BFTS  $(X, \tau_1, \tau_2)$ .

**Proposition 23.** Let  $(X, \tau)$  and  $(Y, \delta)$  be two IFTSs and  $f : (X, \tau) \rightarrow (Y, \delta)$  be IF-continuous. Then  $f : (X, \tau_1, \tau_2) \rightarrow (Y, \delta_1, \delta_2)$  is FP-continuous (Here  $(X, \tau_1, \tau_2)$  and  $(Y, \delta_1, \delta_2)$  are BFTSs associated with  $(X, \tau)$  and  $(Y, \delta)$ , resp.).

*Proof.* Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be IF-continuous. To show that  $f : (X, \tau_1, \tau_2) \rightarrow (Y, \delta_1, \delta_2)$  is FP-continuous, take  $U \in \delta_1$  then  $\exists V \in I^Y$  such that  $(U, V) \in \delta$ . Hence since  $f$  is IF-continuous,  $f^{-1}(U, V) \in \tau$ , that is,  $(f^{-1}(U), f^{-1}(V)) \in \tau \Rightarrow f^{-1}(U) \in \tau_1$ . Further take  $V_1 \in \delta_2$  then  $\exists U_1 \in I^Y$  such that  $(U_1, V_1) \in \delta \Rightarrow f^{-1}(U_1, V_1) \in \tau \Rightarrow (f^{-1}(U_1), f^{-1}(V_1)) \in \tau \Rightarrow (f^{-1}(U_1), (f^{-1}(V_1))') \in \tau \Rightarrow f^{-1}(V_1) \in \tau_2$ . Thus  $f$  is FP-continuous.  $\square$

**Proposition 24.** Let  $(X, \tau_1, \tau_2)$  and  $(Y, \delta_1, \delta_2)$  be two BFTSs and  $(X, \tau_{\tau_1, \tau_2})$  and  $(Y, \delta_{\delta_1, \delta_2})$  be the associated IFTSs respectively. Then  $f : (X, \tau_1, \tau_2) \rightarrow (Y, \delta_1, \delta_2)$  is FP-continuous if and only if  $f : (X, \tau_{\tau_1, \tau_2}) \rightarrow (Y, \delta_{\delta_1, \delta_2})$  is IF-continuous.

*Proof.* Let  $f : (X, \tau_1, \tau_2) \rightarrow (Y, \delta_1, \delta_2)$  be FP-continuous. To show that  $f : (X, \tau_{\tau_1, \tau_2}) \rightarrow (Y, \delta_{\delta_1, \delta_2})$  is IF continuous, take any  $U = (\mu_U, \nu_U) \in \delta_{\delta_1, \delta_2}$ , where  $\mu_U \in \delta_1, \nu_U \in \delta_2$  and  $\mu_U \subseteq \nu_U$ . Now using FP-continuity of  $f, f^{-1}(\mu_U) \in \tau_1$  and  $f^{-1}(\nu_U) \in \tau_2$ . So  $(f^{-1}(\mu_U), (f^{-1}(\nu_U))') \in \tau_{\tau_1, \tau_2} \Rightarrow (f^{-1}(\mu_U), f^{-1}(\nu_U)) \in \tau_{\tau_1, \tau_2} \Rightarrow f^{-1}(\mu_U, \nu_U) \in \tau_{\tau_1, \tau_2}$ , that is,  $f^{-1}(U) \in \tau_{\tau_1, \tau_2}$ , showing that  $f$  is IF-continuous.

The converse follows from the previous Proposition 23 in view of the fact that  $(\tau_{\tau_1, \tau_2})_1 = \tau_1, (\tau_{\tau_1, \tau_2})_2 = \tau_2$  and  $(\delta_{\delta_1, \delta_2})_1 = \delta_1$  and  $(\delta_{\delta_1, \delta_2})_2 = \delta_2$ .  $\square$

The category of all BFTS together with FP-continuous functions will be denoted by BF-Top and the category of all IFTS together with IF-continuous function will be denoted by IF-Top.

Now we define  $\mathcal{B} : \text{IF-Top} \rightarrow \text{BF-Top}$  as follows:

$\mathcal{B}(X, \tau) = (X, \tau_1, \tau_2), \mathcal{B}(f) = f$ , for all morphism  $f$  and  $\mathcal{D} : \text{BF-Top} \rightarrow \text{IF-Top}$  as follows:

$\mathcal{D}(X, \tau_1, \tau_2) = (X, \tau_{\tau_1, \tau_2}), \mathcal{D}(f) = f$ , for all morphism  $f$ .

It can be checked easily that  $\mathcal{B}$  and  $\mathcal{D}$  are covariant functors, and in view of Proposition 22 we have the following remark.

*Remark 25.*  $\mathcal{B} \circ \mathcal{D} = Id_{\text{BF-Top}}$ , the identity functor.

**Theorem 26.** The functor  $\mathcal{D} : \text{BF-Top} \rightarrow \text{IF-Top}$  is left adjoint to the functor  $\mathcal{B} : \text{IF-Top} \rightarrow \text{BF-Top}$ .

The proof is on parallel lines as in ([11], Theorem 3.10) and hence is omitted.

**Proposition 27.** The following statements are equivalent in an IFTS  $(X, \tau)$ :

- (1)  $(X, \tau)$  is  $T_1$ ,
- (2)  $(\{x\}, \{x\}')$  is intuitionistic fuzzy closed in  $(X, \tau)$ , for all  $x \in X$ .

*Proof.* (1)  $\Rightarrow$  (2) We show that  $(\{x\}', \{x\})$  is intuitionistic fuzzy open in  $X$ . Choose any IFP  $\gamma_{(\alpha, \beta)}$  in  $(\{x\}', \{x\})$  then,  $\gamma \neq x$ . Hence  $\exists$  IFOSSs  $U, V \in \tau$  such that  $U(x) = (\mu_U(x), \nu_U(x)) = (1, 0), U(\gamma) = (\mu_U(\gamma), \nu_U(\gamma)) = (0, 1)$  and  $V(x) = (\mu_V(x), \nu_V(x)) = (0, 1), V(\gamma) = (\mu_V(\gamma), \nu_V(\gamma)) = (1, 0)$ . Now  $\gamma_{(\alpha, \beta)} \in V \subseteq (\{x\}', \{x\})$ . Therefore in view of Proposition 13,  $(\{x\}', \{x\})$  is an IFOSS.

(2)  $\Rightarrow$  (1) Choose  $x, \gamma \in X$  such that  $x \neq \gamma$ . Then  $(\{x\}, \{x\}')$  and  $(\{\gamma\}, \{\gamma\}')$  are IFOSSs and hence  $U = (\{x\}', \{x\}), V = (\{\gamma\}', \{\gamma\})$  are IFOSSs such that  $U(x) = (0, 1), U(\gamma) = (1, 0), V(x) = (0, 1)$  and  $V(\gamma) = (1, 0)$  showing that  $(X, \tau)$  is  $T_1$ .  $\square$

**Proposition 28.** If an IFTS  $(X, \tau)$  is  $T_1$ , then its associated BFTS is  $T_1$ .

*Proof.* Let  $(X, \tau)$  be  $T_1$ . Then for all  $x \neq \gamma, \exists U = (\mu_U, \nu_U)$  and  $V = (\mu_V, \nu_V) \in \tau$  such that  $(\mu_U, \nu_U)(x) = (1, 0), (\mu_U, \nu_U)(\gamma) = (0, 1), (\mu_V, \nu_V)(x) = (0, 1)$ , and  $(\mu_V, \nu_V)(\gamma) = (1, 0)$ . Now  $(\mu_U, \nu_U)(x) = (1, 0) \Rightarrow \mu_U(x) = 1, \nu_U(x) = 0$ .

Similarly,  $\mu_U(\gamma) = 0, \nu_U(\gamma) = 1, \mu_V(x) = 0, \nu_V(x) = 1$ , and  $\mu_V(\gamma) = 1, \nu_V(\gamma) = 0$ . Now  $\mu_U \in \tau_1, (1 - \nu_U) \in \tau_2$  and further  $\mu_U(x) = 1, \mu_U(\gamma) = 0, (1 - \nu_U)(x) = 0$ , and  $(1 - \nu_U)(\gamma) = 1$  showing that  $(X, \tau_1, \tau_2)$  is a  $T_1$  space.  $\square$

Similarly it can be shown that if an IFTS  $(X, \tau)$  is  $T_0$ , then its associated BFTS is  $T_0$ .

**Theorem 29.** Let  $\{(X_i, \tau_i) : i \in J\}$  be a family of  $T_1$  IFTSs and  $(X, \tau)$  be their product IFTS. Then  $(X, \tau)$  is  $T_1$  if and only if  $(X_i, \tau_i)$  is  $T_1$ , for all  $i \in J$ .

*Proof.* Let  $(X_i, \tau_i)$  be  $T_1$  for all  $i \in J$ . To show that  $(X, \tau)$  is  $T_1$ , choose  $x, \gamma \in X, x \neq \gamma$ . Let  $x = \prod x_i, \gamma = \prod \gamma_i$  then  $\exists j \in J$  such that  $x_j \neq \gamma_j$ . Now since  $(X_j, \tau_j)$  is  $T_1, \exists U_j = (\mu_{U_j}, \nu_{U_j})$  and  $V_j = (\mu_{V_j}, \nu_{V_j}) \in \tau_j$  such that  $\mu_{U_j}(x_j) = 1, \nu_{U_j}(x_j) = 0, \mu_{U_j}(\gamma_j) = 0, \nu_{U_j}(\gamma_j) = 1$ , and  $\mu_{V_j}(\gamma_j) = 1, \nu_{V_j}(\gamma_j) = 0, \mu_{V_j}(x_j) = 0, \nu_{V_j}(x_j) = 1$ . Now consider the basic IFOSSs  $\prod U_k$  and  $\prod V_k$  where  $U_k = V_k = (1, 0)$  for  $k \in J, k \neq j$  and  $U_k = U_j$  and  $V_k = V_j$  when  $k = j$ . Then  $\prod U_k(x) = (\inf \mu_{U_k}(x_k), \sup \nu_{U_k}(x_k)) = (1, 0), \prod U_k(\gamma) = (\inf \mu_{U_k}(\gamma_k), \sup \nu_{U_k}(\gamma_k)) = (0, 1)$ . Similarly we can show that  $\prod V_k(\gamma) = (1, 0)$  and  $\prod V_k(x) = (0, 1)$ . Therefore  $(X, \tau)$  is  $T_1$ .

Conversely let  $(X, \tau)$  be  $T_1$ . To show that  $(X_j, \tau_j)$  is  $T_1$ , choose  $x_j, \gamma_j \in X_j$  such that  $x_j \neq \gamma_j$ . Now consider  $x = \prod x_i, \gamma = \prod \gamma_i$  where  $x_i = \gamma_i, i \neq j$  and the  $j$ th coordinate of  $x, \gamma$  are  $x_j$  and  $\gamma_j$ , respectively. Then  $x \neq \gamma$ , therefore since  $(X, \tau)$  is  $T_1, \exists U = (\mu_U, \nu_U)$  and  $V = (\mu_V, \nu_V) \in \tau$  such that  $U(x) = (1, 0), U(\gamma) = (0, 1), V(x) = (0, 1)$ , and  $V(\gamma) = (1, 0)$ . Now consider the intuitionistic fuzzy points  $x_{(\alpha, 1-\alpha)} \in U$  and  $\gamma_{(\gamma, 1-\gamma)} \in V$ . Then  $\exists$  basic IFOSSs  $\prod U_i^\alpha$  and  $\prod V_i^\gamma$  in  $X$  such that  $x_{(\alpha, 1-\alpha)} \in \prod U_i^\alpha \subseteq U$  and  $\gamma_{(\gamma, 1-\gamma)} \in \prod V_i^\gamma \subseteq V$ :

$$\begin{aligned} x_{(\alpha, 1-\alpha)} \in \prod U_i^\alpha \subseteq U &\implies \alpha < \inf_i \mu_{U_i^\alpha}(x_i), \\ 1 - \alpha > \sup_i \nu_{U_i^\alpha}(x_i) &\implies \alpha < \mu_{U_i^\alpha}(x_i), \\ (1 - \alpha) > \nu_{U_i^\alpha}(x_i), &\quad \forall i \in J. \end{aligned} \quad (3)$$

Similarly we can show that

$$\begin{aligned} \gamma_{(\gamma, 1-\gamma)} \in \prod V_i^\gamma \subseteq V &\implies \gamma < \mu_{V_i^\gamma}(\gamma_i), (1 - \gamma) > \nu_{V_i^\gamma}(\gamma_i), \\ &\quad \forall i \in J. \end{aligned} \quad (4)$$

Now  $V_j = \cup\{U_j^\gamma : \gamma \in (0, 1)\}$  is such that  $\mu_{V_j}(y_j) = 1$ ,  $\nu_{V_j}(y_j) = 0$ . Further, since  $x_i = y_i$  for  $i \neq j$ , from (3) and (4), it follows that

$$\alpha < \mu_{U_i^\alpha}(y_i), \quad (1 - \alpha) > \nu_{U_i^\alpha}(y_i), \quad \forall i \in J, \quad (5)$$

$$i \neq j, \alpha \in (0, 1),$$

$$\gamma < \mu_{V_i^\gamma}(x_i), \quad (1 - \gamma) > \nu_{V_i^\gamma}(x_i), \quad \forall i \in J, \quad (6)$$

$$i \neq j, \gamma \in (0, 1).$$

Therefore

$$U(y) = (0, 1) \implies \Pi U_i^\alpha(y) = (0, 1)$$

$$\implies \inf_i \mu_{U_i^\alpha}(y_i) = 0,$$

$$\sup_i \nu_{U_i^\alpha}(y_i) = 1, \quad \forall \alpha \in (0, 1)$$

$$\implies \mu_{U_j^\alpha}(y_j) = 0, \quad \nu_{U_j^\alpha}(y_j) = 1, \quad \forall \alpha \in (0, 1) \quad (7)$$

(in view of (5))

$$\implies \sup_\alpha \mu_{U_j^\alpha}(y_j) = 0, \quad \inf_\alpha \nu_{U_j^\alpha}(y_j) = 1.$$

Thus  $\mu_{U_j}(y_j) = 0$  and  $\nu_{U_j}(y_j) = 1$ . That is,  $U_j(y_j) = (0, 1)$ . Similarly it can be shown that  $V_j(x_j) = (0, 1)$ . Hence  $(X_j, \tau_j)$  is  $T_1$ .  $\square$

The following theorem can be proved in a similar way.

**Theorem 30.** Let  $\{(X_i, \tau_i) : i \in J\}$  be a family of  $T_0$  IFTSs and  $(X, \tau)$  be their product IFTS. Then  $(X, \tau)$  is  $T_0$  if and only if  $(X_i, \tau_i)$  is  $T_0$ , for all  $i \in J$ .

**Proposition 31.** An IFTS  $(X, \tau)$  is Hausdorff, and then its associated BFTS  $(X, \tau_1, \tau_2)$  is Hausdorff.

*Proof.* Let  $(X, \tau)$  be Hausdorff. To show that  $(X, \tau_1, \tau_2)$  is Hausdorff, choose any two distinct fuzzy points  $x_\alpha, y_{1-\delta}$  in  $X$ . Now choose  $\beta, \gamma \in (0, 1)$ , and then  $x_{(\alpha, \beta)}$  and  $y_{(\gamma, \delta)}$  are distinct intuitionistic fuzzy points in  $X$ . Since  $(X, \tau)$  is Hausdorff,  $\exists U, V \in \tau$  such that  $x_{(\alpha, \beta)} \in U, y_{(\gamma, \delta)} \in V$  and  $U \cap V = 0_\sim$ . Let  $U = (\mu_U, \nu_U), V = (\mu_V, \nu_V)$ , then

$$x_{(\alpha, \beta)} \in U \implies \alpha < \mu_U(x), \quad \beta > \nu_U(x)$$

$$y_{(\gamma, \delta)} \in V \implies \gamma < \mu_V(y), \quad (8)$$

$$\delta > \nu_V(y) \iff 1 - \delta < 1 - \nu_V(y),$$

and also we have  $\mu_U + \nu_U \leq \underline{1}$  and  $\mu_V + \nu_V \leq \underline{1}$ .

Now

$$U \cap V = 0_\sim \iff \mu_U \cap \mu_V = \underline{0}, \quad \nu_U \cup \nu_V = \underline{1}. \quad (9)$$

From (8) we have  $x_\alpha \in \mu_U$  and  $y_{1-\delta} \in \underline{1} - \nu_V$ . Now we show that  $\mu_U \cap (\underline{1} - \nu_V) = \underline{0}$  as follows.

We have,  $\nu_U(x) < 1 \implies \nu_V(x) = 1$  (in view of (9)),  $\implies 1 - \nu_V(x) = 0 \implies (\mu_U \cap (\underline{1} - \nu_V))(x) = 0$ .

Further,  $\nu_V(y) < 1 \implies \nu_U(y) = 1$  (in view of (9)),  $\implies \mu_U(y) = 0$  since  $\mu_U(y) + \nu_U(y) \leq 1 \implies (\mu_U \cap (\underline{1} - \nu_V))(y) = 0$ .

Now take  $z \in X$  such that  $z \neq x, y$ . If  $\mu_U(z) = 0$ , then obviously  $(\mu_U \cap (\underline{1} - \nu_V))(z) = 0$ , and if  $\mu_U(z) \neq 0$ , then  $\nu_U(z) < 1$  (since  $\mu_U + \nu_U \leq \underline{1} \implies \nu_V(z) = 1 \implies 1 - \nu_V(z) = 0 \implies (\mu_U \cap (\underline{1} - \nu_V))(z) = 0$ ).

Thus we have  $\mu_U \in \tau_1, (\underline{1} - \nu_V) \in \tau_2$  such that  $x_\alpha \in \mu_U, y_{1-\delta} \in (\underline{1} - \nu_V)$  and  $\mu_U \cap (\underline{1} - \nu_V) = \underline{0}$ , and hence  $(X, \tau_1, \tau_2)$  is Hausdorff.  $\square$

**Definition 32.** Let  $(X, \tau)$  be an IFTS and  $Y \subseteq X$ . Then  $(Y, \tau | Y)$  is called a subspace of  $(X, \tau)$  where  $\tau | Y = \{U | Y = (\mu_U | Y, \nu_U | Y) : U \in \tau\}$ .

**Proposition 33.** If an IFTS  $(X, \tau)$  is  $T_i, i = 0, 1, 2$ , then its subspace,  $(Y, \tau | Y)$  is also  $T_i, i = 0, 1, 2$ .

The proofs are easy and hence are omitted.

**Proposition 34.** The product IFTS  $(X, \tau)$  of  $\{(X_j, \tau_j) : j \in J\}$  is initial with respect to the family of projections  $\{p_j : X \rightarrow X_j, j \in J\}$ , that is, for any IFTS  $(Y, \eta)$ , a map  $g : (Y, \eta) \rightarrow (X, \tau)$  is IF continuous if and only if the map  $p_j \circ g : (Y, \eta) \rightarrow (X_j, \tau_j)$  is IF continuous for all  $j \in J$ .

*Proof.* Since projection maps are IF continuous and composition of IF-continuous maps are IF-continuous, so  $p_j \circ g$  is IF continuous for all  $j \in J$ .

Conversely, if  $p_j \circ g$  is IF-continuous for all  $j \in J$ , then  $(p_j \circ g)^{-1}(U_j) = g^{-1}(p_j^{-1}(U_j))$  is IFO in  $(Y, \eta)$  for all  $U_j \in \tau_j, j \in J$  showing that inverse image of every subbasic IFOS in  $X$  is IFO in  $Y$  which implies that  $g$  is IF continuous.  $\square$

**Proposition 35.** Let  $\{(X_j, \tau_j) : j \in J\}$  be a family of IFTSs,  $(X_j, (\tau_j)_1, (\tau_j)_2)$  be the BFTS associated with  $(X_j, \tau_j)$ , and  $(X, \tau_1, \tau_2)$  be the BFTS associated with the product IFTS  $(X, \tau)$ . Then  $\tau_1 = \Pi(\tau_j)_1$  and  $\tau_2 = \Pi(\tau_j)_2$ .

*Proof.* The product space  $(X, \tau)$  is generated by  $\{p_j^{-1}(U_j) : U_j \in \tau_j, j \in J\}$  where  $p_j$ 's are projection maps. Let  $U_j = (\mu_{U_j}, \nu_{U_j})$ , and then  $p_j^{-1}(U_j) = (p_j^{-1}(\mu_{U_j}), p_j^{-1}(\nu_{U_j}))$ .

Now members of  $\tau$  are of the form:

$$\bigcup_{l \in L(\text{arbitrary})} \bigcap_{k \in K(\text{finite})} p_k^{-1}(U_k)$$

$$= \bigcup_l \bigcap_k p_k^{-1}(\mu_{U_k}, \nu_{U_k})$$

$$= \bigcup_l \bigcap_k (p_k^{-1}(\mu_{U_k}), p_k^{-1}(\nu_{U_k})) \quad (10)$$

$$= \left( \bigcup_l \bigcap_k p_k^{-1}(\mu_{U_k}), \right.$$

$$\left. \bigcap_l \bigcup_k p_k^{-1}(\nu_{U_k}) \right).$$

So, members of  $\tau_1$  are of the form  $\bigcup_l \bigcap_k p_k^{-1}(\mu_{U_k})$ , hence  $\tau_1 = \Pi(\tau_j)_1$ , and members of  $\tau_2$  are of the form  $\bigcap_l \bigcup_k p_k^{-1}(\nu_{U_k}) = \bigcup_l \bigcap_k p_k^{-1}(\underline{1} - \nu_{U_k})$ , hence  $\tau_2 = \Pi(\tau_j)_2$ .  $\square$

**Theorem 36.** An IFTS  $(X, \tau)$  is Hausdorff if and only if  $(\Delta_X, \Delta'_X)$  is IFCS in  $(X \times X, \tau \times \tau)$ .

*Proof.* We show that  $(\Delta'_X, \Delta_X)$  is IFOS in  $X$ . Choose any  $(x, y)_{(\alpha, \beta)} \in (\Delta'_X, \Delta_X)$  then  $x \neq y$  and  $\alpha < \Delta'_X(x, y)$ ,  $\beta > \Delta_X(x, y)$ . Now  $x_{(\alpha, \beta)}$  and  $y_{(\alpha, \beta)}$  are distinct intuitionistic fuzzy points in  $X$ . Since  $(X, \tau)$  is Hausdorff,  $\exists$  IFOSs  $U = (\mu_U, \nu_U)$  and  $V = (\mu_V, \nu_V)$  in  $\tau$  such that  $x_{(\alpha, \beta)} \in U$ ,  $y_{(\alpha, \beta)} \in V$  and  $U \cap V = 0_{\sim}$ , that is,  $(\mu_U \cap \mu_V, \nu_U \cup \nu_V) = (\underline{0}, \underline{1})$ .

Now consider  $U \times V = (\mu_U \times \mu_V, \nu_U * \nu_V)$ , and then  $(x, y)_{(\alpha, \beta)} \in U \times V \subseteq (\Delta'_X, \Delta_X)$  as shown below:  $(x, y)_{(\alpha, \beta)} \in (\mu_U \times \mu_V, \nu_U * \nu_V)$  as  $\alpha < (\mu_U \times \mu_V)(x, y) = \inf(\mu_U(x), \mu_V(y))$  (since  $\alpha < \mu_U(x)$ ,  $\alpha < \mu_V(y)$  both) and  $\beta > (\nu_U * \nu_V)(x, y) = \sup(\nu_U(x), \nu_V(y))$  (since  $\beta > \nu_U(x)$ ,  $\beta > \nu_V(y)$  both).

Further,  $U \times V \subseteq (\Delta'_X, \Delta_X)$  since  $U \times V = (\mu_U \times \mu_V, \nu_U * \nu_V)$  and  $\mu_U \times \mu_V \subseteq \Delta'_X$  as  $(\mu_U \times \mu_V)(x, x) = \inf(\mu_U(x), \mu_V(x)) = 0$ , for all  $x \in X$  and  $\nu_U * \nu_V \supseteq \Delta_X$  as  $(\nu_U * \nu_V)(x, x) = \sup(\nu_U(x), \nu_V(x)) = 1$ , for all  $x \in X$ .

Conversely, let  $(\Delta_X, \Delta'_X)$  be IFCS in  $(X, \tau)$ . Then  $(\Delta'_X, \Delta_X)$  is IFOS in  $X$ . To show that  $(X, \tau)$  is Hausdorff choose any two distinct intuitionistic fuzzy points  $x_{(\alpha, \beta)}$ ,  $y_{(\gamma, \delta)}$  in  $X$  then  $x \neq y$ . Let  $\max(\alpha, \gamma) = \alpha$  and  $\min(\beta, \delta) = \beta$ . Consider the intuitionistic fuzzy point  $(x, y)_{(\alpha, \beta)}$  in  $X \times X$ . Then  $(x, y)_{(\alpha, \beta)} \in (\Delta'_X, \Delta_X)$  and hence  $\exists$  a basic IFOS  $U \times V$  in  $(X \times X, \tau \times \tau)$  such that  $(x, y)_{(\alpha, \beta)} \in U \times V \subseteq (\Delta'_X, \Delta_X)$ .

Now,  $(x, y)_{(\alpha, \beta)} \in U \times V \Rightarrow \alpha < (\mu_U \times \mu_V)(x, y) = \inf(\mu_U(x), \mu_V(y)) \leq \mu_U(x)$ , and  $\beta > (\nu_U * \nu_V)(x, y) = \sup(\nu_U(x), \nu_V(y)) \geq \nu_U(x)$  implies that  $x_{(\alpha, \beta)} \in U$ , similarly  $y_{(\alpha, \beta)} \in V$  which implies that  $y_{(\gamma, \delta)} \in V$ . Now we show that  $U \cap V = 0_{\sim}$ . Since  $U \times V \subseteq (\Delta'_X, \Delta_X)$ ,  $(\mu_U \times \mu_V)(x, x) \leq \Delta'_X(x, x) = 0 \Rightarrow \inf(\mu_U(x), \mu_V(x)) = 0$ , for all  $x \in X$  and  $(\nu_U * \nu_V)(x, x) \geq \Delta_X(x, x) = 1 \Rightarrow \sup(\nu_U(x), \nu_V(x)) = 1$ , for all  $x \in X$ . Thus  $U \cap V = (\mu_U \cap \mu_V, \nu_U \cup \nu_V) = (\underline{0}, \underline{1}) = 0_{\sim}$ .  $\square$

**Definition 37.** A BFTS  $(X, \tau_1, \tau_2)$  is called  $q$ - $T_2$  if for any two distinct fuzzy points  $x_r$  and  $y_s$  there exists  $U \in \tau_1$ ,  $V \in \tau_2$  such that  $x_r \in U$ ,  $y_s \in V$  and  $U \subseteq V'$ .

**Proposition 38.** An IFTS  $(X, \tau)$  is  $q$ - $T_2$ , and then its associated BFTS  $(X, \tau_1, \tau_2)$  is  $q$ - $T_2$ .

*Proof.* Let  $x_\alpha$  and  $y_{1-\delta}$  be any two distinct fuzzy points in  $X$ . Since  $(X, \tau)$  is  $q$ - $T_2$  and  $x_{(\alpha, \beta)}$ ,  $y_{(\gamma, \delta)}$  are distinct intuitionistic fuzzy points in  $X$ , there exist  $G_1 = (\mu_{G_1}, \nu_{G_1})$  and  $G_2 = (\mu_{G_2}, \nu_{G_2})$  such that  $x_{(\alpha, \beta)} \in G_1$ ,  $y_{(\gamma, \delta)} \in G_2$  and  $\mu_{G_1} \subseteq \mu'_{G_2}$ ,  $\nu_{G_1} \supseteq \nu_{G_2}$ .  $\Rightarrow \alpha < \mu_{G_1}(x)$  and  $\delta > \nu_{G_2}(y) \Rightarrow x_\alpha \in \mu_{G_1}$  and  $y_{1-\delta} \in (1 - \nu_{G_2})$ . Thus for fuzzy points  $x_\alpha$  and  $y_{1-\delta}$  in  $X$ ,  $\exists \mu_{G_1} \in \tau_1$  and  $(1 - \nu_{G_2}) \in \tau_2$ . Further, since  $\mu_{G_1} + \nu_{G_1} \subseteq \underline{1}$ , we have  $\mu_{G_1} \subseteq (1 - \nu_{G_1}) \subseteq \nu_{G_2}$  (since  $\nu_{G_1} \supseteq \nu'_{G_2}$ ). Hence  $(X, \tau_1, \tau_2)$  is  $q$ - $T_2$ .  $\square$

**Theorem 39.** Let  $\{(X_i, \tau_i) : i \in J\}$  be a family of IFTs and  $(X, \tau)$  be their product IFTS. Then  $(X, \tau)$  is Hausdorff if and only if each coordinate space  $(X_i, \tau_i)$  is Hausdorff.

*Proof.* Let  $x_{(\alpha, \beta)}$  and  $y_{(\gamma, \delta)}$  be two distinct intuitionistic fuzzy points in  $X$ . Let  $x = \Pi x_i$ ,  $y = \Pi y_i$  then  $x \neq y$  and hence  $\exists k \in J$  such that  $x_k \neq y_k$ . Consider the distinct intuitionistic fuzzy points  $(x_k)_{(\alpha, \beta)}$  and  $(y_k)_{(\gamma, \delta)}$  in  $(X_k, \tau_k)$ . Since  $(X_k, \tau_k)$  is Hausdorff,  $\exists$  disjoint  $U_k, V_k$  in  $\tau_k$  such that  $(x_k)_{(\alpha, \beta)} \in U_k$ ,  $(y_k)_{(\gamma, \delta)} \in V_k$  and  $U_k \cap V_k = 0_{\sim}$ . Let  $U_k = (\mu_{U_k}, \nu_{U_k})$  and  $V_k = (\mu_{V_k}, \nu_{V_k})$ , and then  $\alpha < \mu_{U_k}(x_k)$ ,  $\beta > \nu_{U_k}(x_k)$  and  $\gamma < \mu_{V_k}(y_k)$ ,  $\delta > \nu_{V_k}(y_k)$ . Equivalently, we have,  $\alpha < p_k^{-1}(\mu_{U_k})(x)$ ,  $\beta > p_k^{-1}(\nu_{U_k})(x)$ , that is,  $x_{(\alpha, \beta)} \in p_k^{-1}(U_k)$  and  $\gamma < p_k^{-1}(\mu_{V_k})(y)$ ,  $\delta > p_k^{-1}(\nu_{V_k})(y)$ , that is,  $y_{(\gamma, \delta)} \in p_k^{-1}(V_k)$  and  $p_k^{-1}(U_k) \cap p_k^{-1}(V_k) = 0_{\sim}$ , since  $U_k \cap V_k = 0_{\sim}$ .  $\square$

Conversely, let  $(X, \tau)$  be Hausdorff. To show that  $(X_j, \tau_j)$  is Hausdorff, and consider two distinct intuitionistic fuzzy points  $(x_j)_{(\alpha, \beta)}$  and  $(y_j)_{(\gamma, \delta)}$  in  $X_j$ , then  $x_j \neq y_j$ . Now consider  $x = \Pi x_i$  and  $y = \Pi y_i$  in  $X$  where  $x_i = y_i$  for  $i \neq j$  and the  $j$ th coordinate of  $x, y$  are  $x_j$  and  $y_j$ , respectively. Since  $(X, \tau)$  is Hausdorff,  $\exists U, V \in \tau$  such that  $x_{(\alpha, \beta)} \in U$ ,  $y_{(\gamma, \delta)} \in V$  and  $U \cap V = 0_{\sim}$ . Now consider the basic IFOSs  $\Pi U_i$  and  $\Pi V_i$  in  $X$  such that  $x_{(\alpha, \beta)} \in \Pi U_i \subseteq U$  and  $y_{(\gamma, \delta)} \in \Pi V_i \subseteq V$ . Here the IFS  $U_i = (\underline{1}, \underline{0})$  except for finitely many  $i$ 's say  $i_1, i_2, \dots, i_k$  and  $V_i = (\underline{1}, \underline{0})$  except for finitely many  $i$ 's say  $i_1, i_2, \dots, i_{k_1}$ .

Now,  $x_{(\alpha, \beta)} \in \Pi U_i \subseteq U \Rightarrow \alpha < \inf\{\mu_{U_{i_1}}(x_{i_1}), \mu_{U_{i_2}}(x_{i_2}), \dots, \mu_{U_{i_k}}(x_{i_k})\}$ ,

$$\begin{aligned} \beta &> \sup\{\nu_{U_{i_1}}(x_{i_1}), \nu_{U_{i_2}}(x_{i_2}), \dots, \nu_{U_{i_k}}(x_{i_k})\} \\ \Rightarrow \alpha &< \mu_{U_{i_m}}(x_{i_m}), \quad \beta > \nu_{U_{i_m}}(x_{i_m}), \quad \forall m = 1, 2, \dots, k. \end{aligned} \quad (11)$$

Similarly,

$$\begin{aligned} y_{(\gamma, \delta)} \in \Pi V_i \subseteq V \Rightarrow \gamma &< \mu_{V_{i_n}}(y_{i_n}), \delta > \nu_{V_{i_n}}(y_{i_n}), \\ \forall n &= 1, 2, \dots, k_1. \end{aligned} \quad (12)$$

We claim that  $j \in \{i_1, i_2, \dots, i_k\}$  and also  $j \in \{i_1, i_2, \dots, i_{k_1}\}$ . If it is not so, then  $x_{i_m} = y_{i_m}$  and  $x_{i_n} = y_{i_n}$ , for all  $m = 1, 2, \dots, k$  and for all  $n = 1, 2, \dots, k_1$  and hence from (11) and (12) we have

$$\begin{aligned} \alpha &< \mu_{U_{i_m}}(x_{i_m}), \quad \beta > \nu_{U_{i_m}}(x_{i_m}), \quad \forall m = 1, 2, \dots, k, \\ \gamma &< \mu_{V_{i_n}}(x_{i_n}), \quad \delta > \nu_{V_{i_n}}(x_{i_n}), \quad \forall n = 1, 2, \dots, k_1 \end{aligned} \quad (13)$$

implying that  $\Pi \mu_{U_i}(x) > 0$ ,  $\Pi \mu_{V_i}(x) > 0$ , therefore  $(\Pi \mu_{U_i} \cap \Pi \mu_{V_i}) \neq \underline{0} \Rightarrow \mu_U \cap \mu_V \neq \underline{0}$  which is a contradiction to the fact that  $U \cap V = 0_{\sim}$ .

So,  $\alpha < \mu_{U_j}(x_j)$ ,  $\beta > \nu_{U_j}(x_j)$ ,  $\gamma < \mu_{V_j}(y_j)$  and  $\delta > \nu_{V_j}(y_j)$  in view of (11) and (12) showing that  $(x_j)_{(\alpha, \beta)} \in U_j$ ,  $(y_j)_{(\gamma, \delta)} \in V_j$ .

Now it remains to show that  $U_j \cap V_j = 0_{\sim}$ , that is,  $\mu_{U_j} \cap \mu_{V_j} = \underline{0}$  and  $\nu_{U_j} \cup \nu_{V_j} = \underline{1}$ . Let  $\exists z_{j_1}$  and  $z_{j_2}$  in  $X_j$  such that  $(\mu_{U_j} \cap \mu_{V_j})(z_{j_1}) > 0$  and  $(\nu_{U_j} \cup \nu_{V_j})(z_{j_2}) < 1$ . Consider  $z_1 = \Pi z_i$  in  $X$  where  $z_i = x_i (= y_i)$  for  $i \neq j$  and  $z_j = z_{j_1}$ .

Now

$$(\mu_{U_j} \cap \mu_{V_j})(z_{j_1}) > 0 \Rightarrow \mu_{U_j}(z_{j_1}) > 0, \quad \mu_{V_j}(z_{j_1}) > 0. \quad (14)$$

In view of (11), (12), and (14) we have  $(\prod \mu_{U_i} \cap \prod \mu_{V_i})(z_1) > 0$  implying that  $U \cap V \neq \emptyset$  which is a contradiction. Hence  $\mu_{U_j} \cap \mu_{V_j} = \underline{0}$ . Now let  $z_2 = \prod z_i$  where  $z_i = x_i (= y_i)$  for  $i \neq j$  and  $z_j = z_{j_2}$ . Since  $(\nu_{U_j} \cup \nu_{V_j})(z_{j_2}) < 1$ ,

We have

$$\nu_{U_j}(z_{j_2}) < 1, \quad \nu_{V_j}(z_{j_2}) < 1. \quad (15)$$

Therefore from (11), (12), and (15) we have  $(\prod \nu_{U_j} \cup \prod \nu_{V_j})(z_2) < 1$  implying that  $U \cap V \neq \emptyset$ , again a contradiction. Hence  $(X_j, \tau_j)$  is Hausdorff.

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