Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 823264, 9 pages
http://dx.doi.org/10.1155/2015/823264

Research Article

A Unified Reproducing Kernel Method and Error Estimation for Solving Linear Differential Equation with Functional Constraints

Xinjian Zhang and Xiongwei Liu

College of Science, National University of Defense Technology, Changsha, Hunan 410073, China

Correspondence should be addressed to Xiongwei Liu; nudtlxw@163.com

Received 26 May 2015; Accepted 30 July 2015

Academic Editor: Dane Quinn

A unified reproducing kernel method for solving linear differential equations with functional constraint is provided. We use a specified inner product to obtain a class of piecewise polynomial reproducing kernels which have a simple unified description. Arbitrary order linear differential operator is proved to be bounded about the special inner product. Based on space decomposition, we present the expressions of exact solution and approximate solution of linear differential equation by the polynomial reproducing kernel. Error estimation of approximate solution is investigated. Since the approximate solution can be described by polynomials, it is very suitable for numerical calculation.

1. Introduction

The reproducing kernel method of solving differential equation is one of the important topics of reproducing kernel numerical methods. In recent years, reproducing kernel theories are widely applied to solve all kinds of the differential equations, which provide new description forms for exact solutions and approximate solutions of many classical equations. For example, Zhang and Cui [1] and Li [2] obtained the analytical solutions described as series for the steady state convecting equations and boundary value problems, separately. Wang et al. [3] not only got the exact solution described as series for a class of boundary value problems of ordinary differential equations, but also proposed iterative method for the approximate solutions. Gao et al. [4] and Li and Wu [5] investigated the reproducing kernel methods to solve singular second-order initial/boundary value problems and multipoint boundary value problems. Zhang [6] and Wu and Lin [7] specially introduced the reproducing kernel methods of solving linear differential equation based on reproducing kernel theory. Shi et al. [8] and Du and Zhang [9] obtained the analytical solutions and the corresponding approximate solutions of initial value problems of linear ordinary differential equations and error estimation was considered. Li and Wu [10] considered the error estimation of solving boundary value problem of linear equations. Castro et al. [11] introduced a new method to solve general initial value problems by means of reproducing kernel theory.

However, these studies were generally concentrated on constructing the corresponding reproducing kernels for solving specific equations. There was no unified description for these methods. Therefore, how to construct a common method of reproducing kernel for a class of differential equations is worthy of further study about theories and methods. In this paper, for a class of frequently used reproducing kernel space $W_2^s[0,1]$, a class of piecewise polynomial reproducing kernels is presented by a specific inner product. These reproducing kernels have a simple unified expression for all $W_2^s[0,1]$. They are suitable for numerical calculation and independent of the concrete form of the operator L. According to the given inner product, the arbitrary kth-order linear differential operator is proved to be bounded operator of $W_2^n[0,1] \rightarrow W_2^m[0,1]$ (if only $k = n - m$). Based on space decomposition, $\{\psi_i\}$ (defined as (1)) were proved to always be complete in a subspace. Finally, we give the error estimation of approximate solution. These results provide a unified reproducing kernel theory and method for solving arbitrary linear differential equation with arbitrary linear functional constraint.

2. Preliminary

For the convenience of description, we first give a brief description of the problem concerned in this paper. Let H_1, H_2 be reproducing kernel Hilbert spaces composed of functions defined on a certain Ω. The inner products are $\langle\cdot,\cdot\rangle_{H_1}$, $\langle\cdot,\cdot\rangle_{H_2}$ and reproducing kernels are $K_{H_1}(t,\tau)$, $K_{H_2}(t,\tau)$, separately. L denotes a linear differential operator from H_1 to H_2. For the given $u \in H_2$, under certain conditions, we need to find $f \in H_1$ such that $Lf(t) = u(t)$. The main idea of solving this kind of equation by reproducing kernel method is presented as follows:

(i) For supposing $L^* : H_2 \to H_1$ to be conjugate operator of L, prove L to be a linear bounded operator from H_1 to H_2.

(ii) For the specified dense sequence $\{x_i\} \subset \Omega$, which is composed of distinct numbers, let

$$\sigma_i(t) = K_{H_2}(t, x_i), \quad \psi_i(t) = L^*\sigma_i(t), \quad i = 1, 2, \ldots, \tag{1}$$

and then prove $\{\psi_i\}$ to be complete about H_1.

(iii) Based on (ii), orthonormalize $\{\psi_i\}$ to obtain $\{\widetilde{\psi}_i\}$ and suppose $\widetilde{\psi}_i = \sum_{j=1}^{i} \beta_{ij}\psi_j$; then, from (1) and properties of reproducing kernel and conjugate operator, we can get the solution of $Lf(t) = u(t)$ as

$$\begin{aligned} f(t) &= \sum_{i=1}^{\infty} \langle f, \widetilde{\psi}_i\rangle_{H_1} \widetilde{\psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{i} \beta_{ij} \left\langle Lf, K_{H_2}(\cdot, x_i)\right\rangle_{H_2} \widetilde{\psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{i} \beta_{ij} \left[Lf(x_i)\right] \widetilde{\psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{i} \beta_{ij} u(x_i) \widetilde{\psi}_i(t). \end{aligned} \tag{2}$$

From (2), we can obtain the exact expression of the solution and can do numerical calculation by values of the given $u(x_i)$.

According to the above process, we can see that if we want to solve differential equation, (i) and (ii) must be true. To verify these two conditions, the frequently used method is to firstly design an inner product and corresponding reproducing kernel according to the specified differential operator L. Then, the boundedness of L is considered and, for ensuring $\{\psi_i\}$ to be complete, L has to be invertible. For each specific differential equation, we have the same process. Moreover, operator L usually is not invertible. The forms of reproducing kernels obtained by solving high-order differential equation are complicated and have no unified structural properties. These also make it difficult to verify the boundedness of operator L and the completeness of $\{\psi_i\}$.

3. Polynomial Reproducing Kernel in $W_2^s[0,1]$

For any positive integer s, let

$$\begin{aligned} W_2^s[0,1] = \Big\{ & f(t), t \in [0,1] : f^{(s-1)}(t) \\ & \text{is absolutely continuous on } [0,1] \text{ and } f^{(s)}(t) \\ & \in L^2[0,1] \Big\}, \end{aligned} \tag{3}$$

where $L^2[0,1]$ is a square integrable function space. According to [6], every $f(t) \in W_2^s[0,1]$ can be written as

$$f(t) = \sum_{i=1}^{s} f^{(i-1)}(0) e_i(t) + \int_0^1 g_s(t,\tau) f^{(s)}(\tau)\, d\tau, \tag{4}$$

where

$$e_i(t) = \frac{t^{i-1}}{(i-1)!}, \quad g_s(t,\tau) = \frac{1}{(s-1)!}(t-\tau)_+^{s-1}. \tag{5}$$

For any $u \in L^2[0,1]$, we have

$$\frac{d^i}{dt^i}\int_0^1 g_s(t,\tau) u(\tau)\, d\tau = \int_0^1 \left[\frac{\partial^i}{\partial t^i} g(t,\tau)\right] u(\tau)\, d\tau, \quad 0 \le i \le s-1, \qquad \frac{d^s}{dt^s}\int_0^1 g_s(t,\tau) u(\tau)\, d\tau = u(t). \tag{6}$$

$W_2^s[0,1]$ is a reproducing kernel Hilbert space with respect to inner product

$$\langle f, h\rangle_s = \sum_{i=1}^{s} f^{(i-1)}(0) \cdot h^{(i-1)}(0) + \int_0^1 f^{(s)}(t) \cdot h^{(s)}(t)\, dt \tag{7}$$

and has a reproducing kernel

$$K_s(t,\tau) = \sum_{i=1}^{s} e_i(t) e_i(\tau) + \int_0^1 g_s(t,x) \cdot g_s(\tau,x)\, dx. \tag{8}$$

For the convenience of numerical calculation, we explore more concrete form of reproducing kernel $K_s(t,\tau)$. Suppose that

$$E_s(t,\tau) = \int_0^1 g_s(t,x) \cdot g_s(\tau,x)\, dx. \tag{9}$$

The key of calculating reproducing kernel $K_s(t,\tau)$ is to calculate $E_s(t,\tau)$. Let

$$\widehat{e}_i(t) = \frac{(-1)^{s-i} t^{s-i}}{(s-i)!}, \quad i = 1, 2, \ldots, s; \tag{10}$$

then we can obtain

$$\begin{aligned} g_s(t,x) &= \frac{1}{(s-1)!}(t-x)_+^{s-1} \\ &= \sum_{i=1}^{s} e_i(t) \cdot \widehat{e}_i(x)(t-x)_+^0, \end{aligned} \tag{11}$$

where $e_i(t)$ is as the definition in (5).

It follows that when $t \geq \tau$,

$$\begin{aligned} E_s(t,\tau) &= \int_0^\tau \left[\sum_{i=1}^{s} e_i(t)\widehat{e}_i(x)\right] \cdot \left[\sum_{j=1}^{s} e_j(\tau)\widehat{e}_j(x)\right] dx \\ &= \sum_{i=1}^{s} e_i(t) \int_0^\tau \left[\sum_{j=1}^{s} e_j(\tau)\widehat{e}_j(x)\right] \cdot \widehat{e}_i(x)\, dx \\ &= \frac{1}{(s-1)!}\sum_{i=1}^{s} e_i(t) \int_0^\tau (\tau-x)^{s-1} \cdot \widehat{e}_i(x)\, dx, \end{aligned} \tag{12}$$

$$\begin{aligned} &\int_0^\tau (\tau-x)^{s-1} \cdot \widehat{e}_i(x)\, dx \\ &\quad = \frac{(-1)^{s-i}}{(s-i)!} \int_0^\tau (\tau-x)^{s-1} \cdot x^{s-i} dx \\ &\quad = \frac{(-1)^{s-i}}{(s-i)!} \tau^{2s-i} \cdot B(s+1-i, s), \end{aligned}$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ is a Beta function. As a consequence, we have

$$E_s(t,\tau) = \frac{1}{(s-1)!}\sum_{i=1}^{s} \frac{(-1)^{s-i} B(s+1-i,s)}{(s-i)!(i-1)!} t^{i-1}\tau^{2s-i}, \quad t \geq \tau. \tag{13}$$

When $t < \tau$, we can get $E_s(t,\tau) = E_s(\tau,t)$ by symmetry. By Beta function, we have

$$E_s(t,\tau) = \begin{cases} \displaystyle\sum_{i=1}^{s} \frac{(-1)^{s-i}}{(2s-i)!(i-1)!} t^{i-1}\tau^{2s-i}, & t \geq \tau, \\ \displaystyle\sum_{i=1}^{s} \frac{(-1)^{s-i}}{(2s-i)!(i-1)!} t^{2s-i}\tau^{i-1}, & t < \tau. \end{cases} \tag{14}$$

Then, from (8), we can get

$$K_s(t,\tau) = \sum_{i=1}^{s} \frac{t^{i-1}\tau^{i-1}}{[(i-1)!]^2} + E_s(t,\tau). \tag{15}$$

Obviously, $K_s(t,\tau)$ is a piecewise polynomial of two variables.

Lemma 1. *For any dense sequence $\{x_i\}$ composed of distinct numbers in $[0,1]$, function sequence*

$$\phi_i(t) = K_s(t, x_i), \quad i = 1, 2, \ldots, \tag{16}$$

is a linearly independent complete system in $W_2^s[0,1]$.

Proof. For any positive integer N, if there are constants $c_1, c_2, \ldots, c_N$ such that

$$\begin{aligned} &\phi = c_1\phi_1 + c_2\phi_2 + \cdots + c_N\phi_N = \theta_s \\ &\qquad (\theta_s \text{ represents the zero element of } W_2^s[0,1]), \end{aligned} \tag{17}$$

namely, $\|\phi\|_s = 0$, then, for any $f \in W_2^s[0,1]$, it can follow that

$$0 = \langle f, \phi\rangle_s = c_1 f(x_1) + c_2 f(x_2) + \cdots + c_N f(x_N). \tag{18}$$

Take f in the above equation as $1, x, \ldots, x^{N-1}$ in turn; we can obtain $c_1 = c_2 = \cdots = c_N = 0$. Therefore, $\phi_1, \phi_2, \ldots, \phi_N$ are linearly independent.

For any $f \in W_2^s[0,1]$, if $\langle f, \phi_i\rangle_s = 0$ $(i = 1, 2, \ldots)$, that is, $f(x_i) = \langle f, K_s(\cdot, x_i)\rangle_s = 0$, since $\{x_i\}$ are dense and f is continuous in $[0,1]$, then $f(t) = 0$ and $\phi_1, \phi_2, \ldots$ are complete.

This completes the proof of Lemma 1. □

Remark 2. The above proof is independent of the inner product of $W_2^s[0,1]$ and the concrete form of reproducing kernel. Therefore, Lemma 1 is always true for arbitrary inner product of $W_2^s[0,1]$ and the corresponding reproducing kernel.

4. Reproducing Kernel Method for Solution of Linear Differential Equation

Suppose that kth-order linear differential operator is

$$L = D^k + a_{k-1}(t) D^{k-1} + \cdots + a_1(t) D + a_0(t), \quad t \in [0,1], \tag{19}$$

where $a_j(t) \in C^j[0,1]$ and integers m and n satisfy $0 \leq m < n$ and $k = n - m$. It is obvious that L is a linear surjective map from $W_2^n[0,1]$ to $W_2^m[0,1]$. The null space Ker L is a k-dimensional subspace of $W_2^n[0,1]$.

When $m = 0$, then $k = n$. Denote $W_2^0[0,1]$ by $L^2[0,1]$; that is, $W_2^0[0,1] = L^2[0,1]$. The corresponding inner product is

$$\langle f, h\rangle_0 = \int_0^1 f(t) \cdot h(t)\, dt = \langle f, h\rangle_{L^2}. \tag{20}$$

Theorem 3. *Suppose that $a_i^{(m)}(t)$ $(0 \leq i \leq k-1)$ exist and are bounded on $[0,1]$; then the differential operator L is a bounded linear operator from $W_2^n[0,1]$ to $W_2^m[0,1]$.*

Proof. We only need to prove that L is bounded. For any $f \in W_2^n[0,1]$ and $0 \le i \le n-1$, there is

$$\begin{aligned}\left[f^{(i)}(t)\right]^2 &\le 2\left[f^{(i)}(0)\right]^2 + 2\left[\int_0^t f^{(i+1)}(\tau)\,d\tau\right]^2 \\ &\le 2\left[f^{(i)}(0)\right]^2 + 2\int_0^1 \left|f^{(i+1)}(t)\right|^2 dt.\end{aligned} \tag{21}$$

It follows that

$$\int_0^1 \left|f^{(i)}(t)\right|^2 dt \le 2\left[f^{(i)}(0)\right]^2 + 2\int_0^1 \left|f^{(i+1)}(t)\right|^2 dt. \tag{22}$$

According to the above inequality, for $i \le n$, we have

$$\begin{aligned}\int_0^1 \left|f^{(i)}(t)\right|^2 dt &\le 2\left[f^{(i)}(0)\right]^2 + 2^2\left[f^{(i+1)}(0)\right]^2 + \cdots \\ &\quad + 2^{n-i}\left[f^{(n-1)}(0)\right]^2 + 2^{n-i}\int_0^1 \left|f^{(n)}(t)\right|^2 dt \\ &\le 2^{n-i}\left[\left[f^{(i)}(0)\right]^2 + \left[f^{(i+1)}(0)\right]^2 + \cdots \right. \\ &\quad \left. + \left[f^{(n-1)}(0)\right]^2 + \int_0^1 \left|f^{(n)}(t)\right|^2 dt\right] \\ &\le 2^{n-i}\left[\sum_{i=1}^{n}\left[f^{(i-1)}(0)\right]^2 + \int_0^1 \left[f^{(n)}(t)\right]^2 dt\right] \\ &= 2^{n-i}\|f\|_n^2.\end{aligned} \tag{23}$$

When $m \ge 1$, the norm of Lf in $W_2^m[0,1]$ is

$$\|Lf\|_m^2 = \sum_{i=1}^{m}\left[(Lf)^{(i-1)}(0)\right]^2 + \int_0^1 \left[(Lf)^{(m)}(t)\right]^2 dt. \tag{24}$$

Since

$$\begin{aligned}\left[(Lf)^{(i)}\right]^2 &= \left[\left(f^{(k)} + a_{k-1}f^{(k-1)} + a_{k-2}f^{(k-2)} + \cdots \right.\right. \\ &\quad \left.\left. + a_1 f^{(1)} + a_0 f\right)^{(i)}\right]^2 = \left[f^{(k+i)}\right. \\ &\quad + \sum_{j=0}^{i} C_i^j \left[a_{k-1}^{(j)} f^{(k-1+i-j)} + a_{k-2}^{(j)} f^{(k-2+i-j)} + \cdots \right. \\ &\quad \left.\left. + a_0^{(j)} f^{(i-j)}\right]\right]^2 \le 2\left[f^{(k+i)}\right]^2 \\ &\quad + 2\left[\sum_{j=0}^{i} C_i^j \left[a_{k-1}^{(j)} f^{(k-1+i-j)} + a_{k-2}^{(j)} f^{(k-2+i-j)} + \cdots \right.\right. \\ &\quad \left.\left. + a_0^{(j)} f^{(i-j)}\right]\right]^2,\end{aligned} \tag{25}$$

it is easily known that, for arbitrary n numbers $c_1, c_2, \ldots, c_n$, we have

$$\left(c_1 + c_2 + \cdots + c_n\right)^2 \le 2^{n-1}\left(c_1^2 + c_2^2 + \cdots + c_n^2\right) \tag{26}$$

and $C_i^j \le m!$ for $0 \le i \le m$. Because $a_i^{(m)}(t)$ $(0 \le i \le k-1)$ are bounded, there exists a constant $A > 0$ (we might as well put $A > 1$) such that

$$\left|a_i^{(j)}(t)\right| \le A \quad \left(0 \le i \le k-1,\ 0 \le j \le m\right). \tag{27}$$

Then, we can get

$$\begin{aligned}&\left[\sum_{j=0}^{i} C_i^j \left[a_{k-1}^{(j)} f^{(k-1+i-j)} + a_{k-2}^{(j)} f^{(k-2+i-j)} + \cdots \right.\right. \\ &\quad \left.\left. + a_0^{(j)} f^{(i-j)}\right]\right]^2 \le (m!)^2\, 2^i \sum_{j=0}^{i}\left[\sum_{l=1}^{k} a_{k-l}^{(j)} f^{(k-l+i-j)}\right]^2 \\ &\quad \le (m!)^2\, 2^m 2^{k-1}\sum_{j=0}^{i}\sum_{l=1}^{k}\left[a_{k-l}^{(j)} f^{(k-l+i-j)}\right]^2 \le (m!)^2 \\ &\quad \cdot 2^{n-1} A \sum_{j=0}^{i}\sum_{l=1}^{k}\left[f^{(k+i-l-j)}\right]^2 \le (m!)^2\, 2^{n-1} A \cdot (i+1) \\ &\quad \cdot \left[\left[f^{(k+i-1)}\right]^2 + \left[f^{(k+i-2)}\right]^2 + \cdots + \left[f^{(1)}\right]^2 + f^2\right].\end{aligned} \tag{28}$$

Let $C_1 = (m!)^2 2^n (m+1) A$; we have

$$\begin{aligned}&\left[(Lf)^{(i)}\right]^2 \\ &\quad \le C_1\left[\left[f^{(k+i)}\right]^2 + \left[f^{(k+i-1)}\right]^2 + \cdots + \left[f^{(1)}\right]^2 + f^2\right] \\ &\quad \le C_1\left[\left[f^{(n)}\right]^2 + \left[f^{(n-1)}\right]^2 + \cdots + \left[f^{(1)}\right]^2 + f^2\right], \\ &\qquad 0 \le i \le m.\end{aligned} \tag{29}$$

Note that C_1 is independent of i, so, from (29) and (23), we can obtain

$$\begin{aligned}&\sum_{i=1}^{m}\left[(Lf)^{(i-1)}(0)\right]^2 \\ &\quad \le mC_1\left[\left[f^{(n)}(0)\right]^2 + \left[f^{(n-1)}(0)\right]^2 + \cdots + f^2(0)\right], \\ &\int_0^1 \left[(Lf)^{(m)}(t)\right]^2 dt \\ &\quad \le C_1\int_0^1 \left[\left[f^{(n)}(t)\right]^2 + \left[f^{(n-1)}(t)\right]^2 + \cdots + f^2(t)\right] dt \\ &\quad \le C_1\left[1 + 2 + \cdots + 2^n\right]\|f\|_n^2 \le 2^{n+1}C_1\|f\|_n^2.\end{aligned} \tag{30}$$

Substituting the above two formulas into (24) and letting $C = [2^{n+1} + m]C_1$, we can get

$$\|Lf\|_m^2 \le mC_1\sum_{i=0}^{n-1}\left[f^{(i)}(0)\right]^2 + 2^{n+1}C_1\|f\|_n^2 \le C\|f\|_n^2. \tag{31}$$

It shows that L is a bounded operator from $W_2^n[0,1]$ to $W_2^m[0,1]$.

When $m=0$, then $k=n$, and the norm of Lf in $W_2^0[0,1]$ is $\|Lf\|_0^2 = \int_0^1 [Lf(t)]^2 dt$. According to the conditions, we can know that $a_i(t)$ is bounded on $[0,1]$. Therefore, there exists $A_0 > 0$ (we might as well put $A_0 > 1$) such that $|a_i(t)| \le A_0$ $(0 \le i \le n-1)$.

From (29) and (23), we can obtain

$$\begin{aligned}\int_0^1 [Lf(t)]^2 dt &\le 2^n A_0 \int_0^1 \left[\left[f^{(n)}(t)\right]^2 + \left[f^{(n-1)}(t)\right]^2 \right.\\ &\quad \left. + \cdots + \left[f^{(1)}(t)\right]^2 + [f(t)]^2\right] dt \le 2^n A_0 [1+2 \\ &\quad + \cdots + 2^n] \|f\|_n^2 \le 2^{n+k+1} A_0 \|f\|_n^2.\end{aligned} \quad (32)$$

This shows that L is a bounded operator from $W_2^n[0,1]$ to $W_2^0[0,1]$.

This completes the proof of Theorem 3. □

The null space $\operatorname{Ker} L$ of L is k-dimensional subspace of $W_2^n[0,1]$. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are continuous linear functional and linearly independent on $\operatorname{Ker} L$ and suppose that the values of $r_1, r_2, \ldots, r_k$ are given. Now we consider the following definite solutions problem.

For a given $u \in W_2^m[0,1]$, find $f \in W_2^n[0,1]$ such that it satisfies

$$\begin{aligned} Lf(t) &= u(t), \\ \lambda_i f &= r_i \quad (i = 1, 2, \ldots, k). \end{aligned} \quad (33)$$

Denoted by $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, Λ is a vector-valued linear functional on $W_2^n[0,1]$; namely, $\Lambda f = (\lambda_1 f, \ldots, \lambda_k f)$.

Lemma 4. *$W_2^n[0,1]$ has direct sum decomposition:*

$$\begin{aligned} W_2^n[0,1] &= \operatorname{Ker} L + \operatorname{Ker} \Lambda, \\ \operatorname{Ker} L \cap \operatorname{Ker} \Lambda &= \{0\}. \end{aligned} \quad (34)$$

Proof. Because $\lambda_1, \lambda_2, \ldots, \lambda_k$ are linearly independent on $\operatorname{Ker} L$, there is a basis $\varphi_1, \varphi_2, \ldots, \varphi_k$ of $\operatorname{Ker} L$ which is dual to $\lambda_1, \lambda_2, \ldots, \lambda_k$; namely,

$$\begin{aligned} &L\varphi_i(t) = 0, \\ &\lambda_i \varphi_j = \delta_{ij}, \\ &\qquad 1 \le i, \ j \le k. \end{aligned} \quad (35)$$

Since $\operatorname{Ker} L$ is a closed subspace of $W_2^n[0,1]$, every $f \in W_2^n[0,1]$ has a unique decomposition form:

$$\begin{aligned} f(t) &= f_1(t) + f_2(t), \\ f_1 &\in \operatorname{Ker} L, \\ f_2 &\in (\operatorname{Ker} L)^{\perp}. \end{aligned} \quad (36)$$

From (35), we can obtain $f_1(t) = \sum_{i=1}^k (\lambda_i f_1) \varphi_i(t)$; then

$$f_2(t) = f(t) - \sum_{i=1}^k (\lambda_i f_1) \varphi_i(t). \quad (37)$$

Let

$$\begin{aligned} f_L(t) &= \sum_{i=1}^k (\lambda_i f) \varphi_i(t), \\ f_\Lambda(t) &= f_2(t) - \sum_{i=1}^k (\lambda_i f_2) \varphi_i(t); \end{aligned} \quad (38)$$

we have

$$f(t) = \sum_{i=1}^k (\lambda_i f) \varphi_i(t) + f_\Lambda(t) = f_L(t) + f_\Lambda(t). \quad (39)$$

From (35) again, we can get $f_L \in \operatorname{Ker} L$ and $f_\Lambda \in \operatorname{Ker} \Lambda$.

If $f \in \operatorname{Ker} L \cap \operatorname{Ker} \Lambda$, there is $f(t) = \sum_{i=1}^k (\lambda_i f) \varphi_i(t)$ and $\lambda_i f = 0$ $(1 \le i \le k)$. It follows that $f(t) = 0$. Therefore, $\operatorname{Ker} L \cap \operatorname{Ker} \Lambda = \{0\}$.

This completes the proof of Lemma 4. □

According to the proof of Lemma 4, we have

$$\begin{aligned} &Lf_\Lambda(t) = Lf(t), \\ &\lambda_i f_L = \lambda_i f, \\ &\qquad i = 1, 2, \ldots, k. \end{aligned} \quad (40)$$

Since $L : \operatorname{Ker} \Lambda \to W_2^m[0,1]$ is a bounded linear operator, there exists a conjugate operator $L^* : W_2^m[0,1] \to \operatorname{Ker} \Lambda$. For any dense sequence $\{x_i\}$ composed of distinct numbers in $[0,1]$, when $m \ge 1$, suppose that

$$\begin{aligned} &\sigma_i(t) = K_m(t, x_i), \\ &\psi_i(t) = L^* \sigma_i(t), \\ &\qquad i = 1, 2, \ldots; \end{aligned} \quad (41)$$

then we have the following lemma.

Lemma 5. *As in Lemma 4, suppose that $f(t) = f_L(t) + f_\Lambda(t)$; then*

(i) *for any $f \in W_2^n[0,1]$, $\langle f, \psi_i \rangle_n = 0$ $(i = 1, 2, \ldots) \Leftrightarrow f = f_L$;*

(ii) *for any $f_\Lambda \in \operatorname{Ker} \Lambda$, $\langle f_\Lambda, \psi_i \rangle_n = 0$ $(i = 1, 2, \ldots) \Leftrightarrow f_\Lambda = 0$.*

Proof. Suppose that $f \in W_2^n[0,1]$. If $\langle f, \psi_i \rangle_n = 0$ $(i = 1, 2, \ldots)$, then, from (41), we have

$$\begin{aligned} \langle Lf_\Lambda, K_m(\cdot, x_i) \rangle_m &= \langle Lf, K_m(\cdot, x_i) \rangle_m = \langle f, \psi_i \rangle_n \\ &= 0 \quad (i = 1, 2, \ldots). \end{aligned} \quad (42)$$

Since $Lf_\Lambda \in W_2^m[0,1]$ and from Lemma 1, $\{K_m(\cdot, x_i)\}$ are complete in $W_2^m[0,1]$, so $Lf_\Lambda = 0$. As f_Λ satisfies $\lambda_i f_\Lambda = 0$ $(1 \le i \le k)$, therefore, $f_\Lambda = 0$; that is to say, $f = f_L$. By contrast, if $f = f_L$, namely, $Lf = 0$, then $\langle f, \psi_i \rangle_n = \langle Lf, K_m(\cdot, x_i) \rangle_m = 0$ $(i = 1, 2, \ldots)$. It follows that (i) is true. Meanwhile, according to the proof, we can also know that (ii) is true.

Thus, we finish the proof of Lemma 5. □

Theorem 6. *Suppose that $H_n = \text{Span}\{\psi_i\} + \text{Ker}\,\Lambda$; then $\{\psi_i\}$ are linearly independent and complete in H_n.*

Proof. Clearly, $\{\psi_i\} \subset H_n$. We first prove that $\{\psi_i\}$ are linearly independent. Suppose that there are a positive integer N and constants $c_1, c_2, \ldots, c_N$ such that $\psi = c_1\psi_1 + c_2\psi_2 + \cdots + c_N\psi_N = \theta_n$; namely,

$$L^*\left(c_1\sigma_1 + c_2\sigma_2 + \cdots + c_N\sigma_N\right) = \theta_n,$$
$$c_1\sigma_1 + c_2\sigma_2 + \cdots + c_N\sigma_N \in \text{Ker}\,L^*. \tag{43}$$

Since $\text{Ker}\,L^* = (\text{Rang}L)^{\perp}$ and L is surjective, $(\text{Rang}L)^{\perp} = (W_2^m[0,1])^{\perp} = \theta_m$, we have

$$c_1\sigma_1 + c_2\sigma_2 + \cdots + c_N\sigma_N = \theta_m. \tag{44}$$

From Lemma 1, $\sigma_1, \sigma_2, \ldots, \sigma_N$ are linearly independent; then $c_1 = c_2 = \cdots = c_N = 0$. Therefore, $\{\psi_i\}$ are linearly independent.

Next, we prove that $\{\psi_i\}$ is a complete system of H_n.

Suppose that $f \in H_n$ satisfies $\langle f, \psi_i\rangle_n = 0$ $(i = 1, 2, \ldots)$. When $f \in \text{Span}\{\psi_i\}$, it is obvious that $f = 0$. From Lemma 5, it is easily obtained that when $f = f_\Lambda$, $f = 0$ and when $f = h + f_\Lambda$ $(h \in \text{Span}\{\psi_i\})$, $f = f_L$; namely, $f = h = h_L$. Then, $f \in \text{Span}\{\psi_i\}$ and $\langle f, \psi_i\rangle_n = 0$ $(i = 1, 2, \ldots)$. As a result, $f = 0$. Therefore, $\{\psi_i\}$ is a complete system of H_n.

This completes the proof of Theorem 6. □

Theorem 7. *Denote $\{\psi_i\}$ orthonormalized in $W_2^n[0,1]$ by $\{\widetilde{\psi}_i\}$ and suppose that*

$$\widetilde{\psi}_i(t) = \sum_{j=1}^{i}\beta_{ij}\psi_j(t) \quad (i = 1, 2, \ldots); \tag{45}$$

then, the solution $\overline{f}(t)$ of (33) can be expressed as

$$\overline{f}(t) = \sum_{i=1}^{k} r_i\varphi_i(t) + \sum_{i=1}^{\infty}\sum_{j=1}^{i}\beta_{ij}u\left(x_j\right)\widetilde{\psi}_i(t). \tag{46}$$

Proof. From Lemma 4 and (39), $\overline{f}(t)$ can be expressed as

$$\overline{f}(t) = \sum_{i=1}^{k} r_i\varphi_i(t) + \overline{f}_\Lambda(t). \tag{47}$$

Since $\overline{f}_\Lambda \in H_n$, from Theorem 6, we suppose that $\overline{f}_\Lambda(t) = \sum_{i=1}^{\infty}\langle\overline{f}_\Lambda, \widetilde{\psi}_i\rangle_n\widetilde{\psi}_i(t)$; according to the properties of conjugate operator and reproducing kernel, it follows that

$$\begin{aligned}
\overline{f}_\Lambda(t) &= \sum_{i=1}^{\infty}\left\langle\overline{f}_\Lambda, \widetilde{\psi}_i\right\rangle_n\widetilde{\psi}_i(t)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\beta_{ij}\left\langle\overline{f}_\Lambda, \psi_j\right\rangle_n\widetilde{\psi}_i(t)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\beta_{ij}\left\langle\overline{f}_\Lambda, L^*K_m\left(\cdot, x_j\right)\right\rangle_n\widetilde{\psi}_i(t)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\beta_{ij}\left\langle L\overline{f}, K_m\left(\cdot, x_j\right)\right\rangle_m\widetilde{\psi}_i(t)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\beta_{ij}\left\langle u, K_m\left(\cdot, x_j\right)\right\rangle_m\widetilde{\psi}_i(t)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\beta_{ij}u\left(x_j\right)\widetilde{\psi}_i(t).
\end{aligned} \tag{48}$$

So we obtain (46).

Thus, the proof of Theorem 7 is completed. □

From (48), we can obtain the approximation of $\overline{f}_\Lambda(t)$ as

$$\left(\overline{f}_\Lambda\right)_N(t) = \sum_{i=1}^{N}\sum_{j=1}^{i}\beta_{ij}u\left(x_j\right)\widetilde{\psi}_i(t). \tag{49}$$

Remark 8. Solving (33) by using (48) and (49) depends on the orthonormalization of $\{\psi_i\}$. This can sometimes be troublesome. Therefore, we hope that $(\overline{f}_\Lambda)_N(t)$ can directly be written as a linear combination of $\{\psi_i\}$.

Note that when $j > N$, $\langle\widetilde{\psi}_j, \psi_i\rangle_n = 0$ $(1 \le i \le N)$, so when $j \le N$, there is

$$\begin{aligned}
\left\langle\left(\overline{f}_\Lambda\right)_N, \psi_j\right\rangle_n &= \left\langle\overline{f}_\Lambda, \psi_j\right\rangle_n = \left\langle L\overline{f}_\Lambda, K_m\left(\cdot, x_j\right)\right\rangle_m\\
&= u\left(x_j\right).
\end{aligned} \tag{50}$$

From (49) and Abel transformation of the partial sum of series, we have

$$\begin{aligned}
\left(\overline{f}_\Lambda\right)_N(t) &= \sum_{i=1}^{N}\left\langle\overline{f}_\Lambda, \widetilde{\psi}_i\right\rangle_n\widetilde{\psi}_i(t)\\
&= \sum_{i=1}^{N}\left[\sum_{j=i}^{N}\left\langle\overline{f}_\Lambda, \widetilde{\psi}_j\right\rangle_n\beta_{ji}\right]\psi_i(t)\\
&\triangleq \sum_{i=1}^{N}a_i\psi_i(t).
\end{aligned} \tag{51}$$

Let two sides of the above equation do inner product with ψ_j $(1 \le j \le N)$; by (49), we can obtain the system of equations

$$\sum_{i=1}^{N}\left\langle\psi_i, \psi_j\right\rangle_n a_i = u\left(x_j\right), \quad j = 1, 2, \ldots, N, \tag{52}$$

where the coefficient matrix $A = (\langle\psi_i, \psi_j\rangle_n)_{N\times N}$ is symmetric and positive definite. Then, there exists only a set of solutions $a_1, a_2, \ldots, a_N$. By this solution, we can obtain the approximate solution $(\overline{f}_\Lambda)_N(t)$.

In order to obtain matrix A, we must calculate $\langle \psi_i, \psi_j \rangle_n$ $(1 \le i, j \le N)$. For this purpose, we first deduce the concrete form of ψ_i. According to the properties of conjugate operator and reproducing kernel, we have

$$\begin{aligned}\psi_i(t) &= \langle \psi_i, K_n(\cdot, t)\rangle_n = \langle L^* K_m(\cdot, x_i), K_n(\cdot, t)\rangle_n \\ &= \langle K_m(\cdot, x_i), LK_n(\cdot, t)\rangle_m = L_{(\tau)} K_n(\tau, t)\big|_{\tau = x_i}.\end{aligned} \tag{53}$$

Using the reproducing kernel expression (14) and (15), we obtain the concrete form of ψ_i as

$$\psi_i(t) = \sum_{j=1}^{n} \frac{q_{i,j-1} t^{j-1}}{[(j-1)!]^2} + \begin{cases} \displaystyle\sum_{j=1}^{n} \frac{(-1)^{n-j} q_{i,j-1}}{(2n-j)!\,(j-1)!} t^{2n-j}, & t \le x_i, \\ \displaystyle\sum_{j=1}^{n} \frac{(-1)^{n-j} q_{i,2n-j}}{(2n-j)!\,(j-1)!} t^{j-1}, & t > x_i, \end{cases} \tag{54}$$

where

$$q_{ij} = \left(Lt^j\right)\big|_{\tau = x_i}, \quad j = 0, 1, \ldots, 2n-1. \tag{55}$$

In order to calculate $\langle \psi_i, \psi_j \rangle_n$ $(1 \le i, j \le N)$, from (54), we have

$$\begin{aligned}&\psi_i^{(l-1)}(0) = \frac{q_{i,l-1}}{(l-1)!}, \quad l = 1, 2, \ldots, n, \\ &\psi_i^{(n)}(t) = \begin{cases} \displaystyle\sum_{j=1}^{n} \frac{(-1)^{j-1} q_{i,n-j}}{(n-j)!\,(j-1)!} t^{j-1}, & t \le x_i, \\ 0, & t > x_i, \end{cases} \\ &\sum_{l=1}^{n} \psi_i^{(l-1)}(0) \cdot \psi_j^{(l-1)}(0) = \sum_{l=1}^{n} \frac{q_{i,l-1}}{(l-1)!} \cdot \frac{q_{j,l-1}}{(l-1)!}.\end{aligned} \tag{56}$$

When $x_i \le x_j$, it can be obtained that

$$\begin{aligned}&\int_0^1 \psi_i^{(n)}(t) \cdot \psi_j^{(n)}(t)\, dt \\ &\quad = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \frac{(-1)^{\alpha+\beta} q_{i,n-\alpha} q_{j,n-\beta}}{(n-\alpha)!\,(\alpha-1)!\,(n-\beta)!\,(\beta-1)!} \cdot \int_0^{x_i} t^{\alpha+\beta-2} dt \\ &\quad = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \frac{(-1)^{\alpha+\beta} q_{i,n-\alpha} q_{j,n-\beta}}{(n-\alpha)!\,(\alpha-1)!\,(n-\beta)!\,(\beta-1)!} \cdot \frac{x_i^{\alpha+\beta-1}}{\alpha+\beta-1}.\end{aligned} \tag{57}$$

When $x_i > x_j$, by symmetry, there is

$$\begin{aligned}&\int_0^1 \psi_i^{(n)}(t) \cdot \psi_j^{(n)}(t)\, dt \\ &\quad = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \frac{(-1)^{\alpha+\beta} q_{i,n-\alpha} q_{j,n-\beta}}{(n-\alpha)!\,(\alpha-1)!\,(n-\beta)!\,(\beta-1)!} \cdot \frac{x_j^{\alpha+\beta-1}}{\alpha+\beta-1}.\end{aligned} \tag{58}$$

Thus, substituting (56)–(58) into (7), we can obtain $\langle \psi_i, \psi_j \rangle_n$ $(1 \le i, j \le N)$.

5. Convergence and Error Estimation

In this section, we will consider the convergence and error estimation of the approximate solution given by (49).

Lemma 9. *$\overline{f}_N(t)$ is convergent uniformly to $\overline{f}(t)$ on $[0, 1]$.*

Proof. Since $\{\overline{\psi}_i\}$ is a complete orthonormal system of $W_2^n[0, 1]$, it is obvious that

$$\left\|\overline{f} - \overline{f}_N\right\|_n^2 = \left\|\overline{f}_\Lambda - \left(\overline{f}_\Lambda\right)_N\right\|_n^2 = \sum_{i=N+1}^{\infty} \left|\sum_{j=1}^{i} \beta_{ij} u\left(x_j\right)\right|^2 \longrightarrow 0, \quad N \longrightarrow \infty. \tag{59}$$

From (14) and (15), it can be obtained that

$$K_n(t, t) = \sum_{i=1}^{n} \frac{t^{2i-2}}{[(i-1)!]^2} + \sum_{i=1}^{n} \frac{(-1)^{n-i}}{(2n-i)!\,(i-1)!} t^{2n-1}. \tag{60}$$

Since $K_n(t, t)$ is continuous on $[0, 1]$, there exists a constant M such that $0 \le K_n(t, t) \le M$. Moreover,

$$\begin{aligned}\left|\overline{f}(t) - \overline{f}_N(t)\right|^2 &= \left|\left\langle \overline{f} - \overline{f}_N, K_n(\cdot, t)\right\rangle_n\right|^2 \\ &\le \left\|\overline{f} - \overline{f}_N\right\|_n^2 \left\|K_n(\cdot, t)\right\|_n^2 \\ &= \left\|f - f_N\right\|_n^2 K_n(t, t) \\ &\le M \left\|f - f_N\right\|_n^2.\end{aligned} \tag{61}$$

Hence, $\overline{f}_N(t)$ is convergent uniformly to $\overline{f}(t)$ on $[0, 1]$.

This completes the proof of Lemma 9. □

Theorem 10. *The exact solution and approximate solution of (33) satisfy*

$$\left|\overline{f}(t) - \overline{f}_N(t)\right|^2 \le \left\|\overline{f}_\Lambda\right\|_n^2 \cdot \left[K_n(t, t) - \sum_{i=1}^{N} \left|\overline{\psi}_i(t)\right|^2\right]. \tag{62}$$

Proof. From (48) and (49), we have

$$\left(\overline{f}_\Lambda\right)_N(t) = \sum_{i=1}^{N} \left\langle \overline{f}_\Lambda, \widetilde{\psi}_i \right\rangle_n \widetilde{\psi}_i(t) = \left\langle \sum_{i=1}^{N} \widetilde{\psi}_i(t)\, \widetilde{\psi}_i, \overline{f}_\Lambda \right\rangle_n . \quad (63)$$

It follows that

$$\begin{aligned} \left|\overline{f}(t) - \overline{f}_N(t)\right|^2 &= \left|\overline{f}_\Lambda(t) - \left(\overline{f}_\Lambda\right)_N(t)\right|^2 \\ &= \left|\left\langle K_n(\cdot,t) - \sum_{i=1}^{N} \widetilde{\psi}_i(t)\, \widetilde{\psi}_i, \overline{f}_\Lambda \right\rangle_n\right|^2 \\ &\le \left\|\overline{f}_\Lambda\right\|_n^2 \left\|K_n(\cdot,t) - \sum_{i=1}^{N} \widetilde{\psi}_i(t)\, \widetilde{\psi}_i\right\|_n^2 . \end{aligned} \quad (64)$$

Since $\sum_{i=1}^{N} \widetilde{\psi}_i(t)\widetilde{\psi}_i = \sum_{i=1}^{N} \langle K_n(\cdot,t), \widetilde{\psi}_i\rangle \widetilde{\psi}_i$, then

$$K_n(\cdot,t) - \sum_{i=1}^{N} \widetilde{\psi}_i(t)\, \widetilde{\psi}_i \perp \sum_{i=1}^{N} \widetilde{\psi}_i(t)\, \widetilde{\psi}_i . \quad (65)$$

Therefore,

$$\begin{aligned} &\left\|K_n(\cdot,t) - \sum_{i=1}^{N} \widetilde{\psi}_i(t)\, \widetilde{\psi}_i\right\|_n^2 \\ &= \left\langle K_n(\cdot,t), K_n(\cdot,t)\right\rangle_n - \sum_{i=1}^{N} \widetilde{\psi}_i(t) \left\langle K_n(\cdot,t), \widetilde{\psi}_i\right\rangle_n \\ &= K_n(t,t) - \sum_{i=1}^{N} \left|\widetilde{\psi}_i(t)\right|^2 . \end{aligned} \quad (66)$$

Thus, we prove that (62) is true.

This completes the proof of Theorem 10. □

Remark 11. When we use Theorem 10 to estimate $|\overline{f}(t) - \overline{f}_N(t)|$, the value of $\|\overline{f}_\Lambda\|_n^2$ has to be evaluated. It is usually difficult in practical problems. We will provide a new method of estimation in the following.

As in Lemma 4, suppose that $\varphi_1, \varphi_2, \ldots, \varphi_k$ is a basis of $\operatorname{Ker} L$ and is dual to $\lambda_1, \lambda_2, \ldots, \lambda_k$. According to [6, 7], every $f \in W_2^n[0,1]$ has the decomposition form as

$$f(t) = \sum_{i=1}^{k} \left(\lambda_i f\right) \varphi_i(t) + \int_0^1 G_L(t,\tau) \cdot Lf(\tau)\, d\tau, \quad (67)$$

where $G_L(t,\tau)$ is Green function of differential operator L and satisfies

$$\begin{aligned} &\lambda_i \int_0^1 G_L(t,\tau) \cdot u(\tau)\, d\tau = 0, \quad i = 1, 2, \ldots, k, \\ &L \int_0^1 G_L(t,\tau) \cdot u(\tau)\, d\tau = u(t), \quad \forall u \in L^2[0,1] . \end{aligned} \quad (68)$$

Since $G_L(t,\tau)$ is continuous, there exists a constant C such that $|G_L(t,\tau)| \le C$.

Theorem 12. *Suppose that* $u_i = \sum_{j=1}^{i} \beta_{ij}[u(x_j)]$*; then the exact solution* $\overline{f}(t)$ *and approximate solution* $\overline{f}_N(t)$ *of (33) satisfy*

$$\left|\overline{f}(t) - \overline{f}_N(t)\right| \le C \int_0^1 \left|\sum_{i=N+1}^{\infty} \sum_{l=1}^{i} u_i \beta_{il} \left[L\psi_l(t)\right]\right| dt. \quad (69)$$

The right-hand side of (69) converges to zero as $N \to \infty$.

Proof. From (39), (48), and (67), it follows that, for any $f_\Lambda \in H_n$, there is

$$\begin{aligned} f_\Lambda(t) &= \int_0^1 G_L(t,\tau) \cdot Lf(\tau)\, d\tau \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{i} \beta_{ij} \left[Lf\left(x_j\right)\right] \widetilde{\psi}_i(t) . \end{aligned} \quad (70)$$

Then, we have

$$\begin{aligned} &\left|\overline{f}_\Lambda(t) - \left(\overline{f}_\Lambda\right)_N(t)\right| \\ &\le C \int_0^1 \left|L\left[\overline{f}_\Lambda(\tau) - \left(\overline{f}_\Lambda\right)_N(\tau)\right]\right| d\tau. \end{aligned} \quad (71)$$

From Lemmas 1 and 9, we know that the right-hand side of (71) converges to zero as $N \to \infty$. From (49), we can obtain

$$\left|L\left[\overline{f}_\Lambda(\tau) - \left(\overline{f}_\Lambda\right)_N(\tau)\right]\right| = \left|\sum_{i=N+1}^{\infty} u_i \cdot \left[L\widetilde{\psi}_i(t)\right]\right| = \left|\sum_{i=N+1}^{\infty} \sum_{l=1}^{i} u_i \beta_{il} \left[L\psi_l(t)\right]\right| . \quad (72)$$

Substituting this equation into (71) and $|\overline{f}(t) - \overline{f}_N(t)| = |\overline{f}_\Lambda(t) - (\overline{f}_\Lambda)_N(t)|$, we can get (69) and assert that the right-hand side of (69) converges to zero as $N \to \infty$.

This completes the proof of Theorem 12. □

Remark 13. From (54), $L\psi_l(t)$ can be easily obtained as the piecewise polynomial.

6. Conclusion

When we use the reproducing kernel method to solve differential equation, we always hope that the reproducing kernel is polynomial reproducing kernel. In general, to take the reproducing kernel of $W_2^s[0,1]$ as polynomial reproducing kernel, the inner product taken must be as (7). But if we take this fixed inner product, it is difficult to prove the boundedness of operator and the completeness of $\{\psi_i(t)\}$. How do these two aspects integrate effectively with each other? For the linear differential operator equation with functional constraints, this paper investigates this topic and some meaningful results are presented. Moreover, the proofs of some results do not depend on whether operator L is differential operator. Therefore, these processes can be extended for general linear bounded operator. In this paper, some conclusions are based on $m \ge 1$; they need to be further explored for case $m = 0$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors thank the unknown reviewers for their careful reading and helpful comments. This work is supported by the Research Project of National University of Defense Technology (JC12-02-01).

References

[1] C. P. Zhang and M. G. Cui, "The solution of steady-state convecting equations in the spaces that prossess restoring nucleus," *Applied Mathematics and Mechanics*, vol. 4, pp. 885–891, 1994.

[2] J. Li, "A new method based on reproducing kernel space for boundary value problems," *Chinese Journal of Engineering Mathematics*, vol. 27, no. 5, pp. 918–924, 2010.

[3] Y. L. Wang, L. J. Su, and L. Chao, "Solving a class of ordinary differential eqution using reproducing kernel," *Journal of Natural Science of Heilongjiang University*, vol. 26, no. 2, pp. 150–153, 2009.

[4] E. Gao, S. Song, and X. Zhang, "Solving singular second-orderinitial/boundary value problems in reproducing kernel Hilbert space," *Boundary Value Problems*, vol. 2012, article 3, 2012.

[5] X. Y. Li and B. Y. Wu, "Reproducing kernel method for singular multi-point boundary value problems," *Mathematical Sciences*, vol. 6, article 16, 2012.

[6] X. J. Zhang, *Theory and Application of Reproducing Kernel*, Science Press, Beijing, China, 2010.

[7] B. Y. Wu and Y. Z. Lin, *Applied Reproducing Kernel Space*, Science Press, Beijing, China, 2012.

[8] Y. H. Shi, C. P. Zhang, and M. G. Cui, "The analytic method for initial value problem of system of first order ordinary differential equations," *Journal of Natural Science of Heilongjiang University*, vol. 17, no. 2, pp. 10–12, 2000.

[9] X. P. Du and X. J. Zhang, "Numerical solution for the linear differential equations with initial conditions in reproducing kernel space," *Mathematics in Practice and Theory*, vol. 41, no. 8, pp. 172–176, 2011.

[10] X. Y. Li and B. Y. Wu, "Error estimation for the reproducing kernel method to solve linear boundary value problems," *Journal of Computational and Applied Mathematics*, vol. 243, pp. 10–15, 2013.

[11] L. P. Castro, M. M. Rodrigues, and S. Saitoh, "A fundamental theorem on initial value problems by using the theory of reproducing kernels," *Complex Analysis and Operator Theory*, vol. 9, no. 1, pp. 87–98, 2015.