Research Article

The Dynamics of a Predator-Prey System with State-Dependent Feedback Control

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A Lotka-Volterra-type predator-prey system with state-dependent feedback control is investigated in both theoretical and numerical ways. Using the Poincaré map and the analogue of the Poincaré criterion, the sufficient conditions for the existence and stability of semitrivial periodic solutions and positive periodic solutions are obtained. In addition, we show that there is no positive periodic solution with period greater than and equal to three under some conditions. The qualitative analysis shows that the positive period-one solution bifurcates from the semitrivial solution through a fold bifurcation. Numerical simulations to substantiate our theoretical results are provided. Also, the bifurcation diagrams of solutions are illustrated by using the Poincaré map, and it is shown that the chaotic solutions take place via a cascade of period-doubling bifurcations.

1. Introduction

In the last decades, some impulsive systems have been studied in population dynamics such as impulsive birth [1, 2], impulsive vaccination [3, 4], and chemotherapeutic treatment of disease [5, 6]. In particular, the impulsively controlled prey-predator population systems have been investigated by a number of researchers [7–15]. Thus the field of research of impulsive differential equations seems to be a new growing interesting area in recent years. Many authors in the articles cited above have shown theoretically and numerically that prey-predator systems with impulsive control are more efficient and economical than classical ones to control the prey (pest) population. However, the majority of these studies only consider impulsive control at fixed time intervals to eradiate the prey (pest) population. Such control measure of prey (pest) management is called fixed-time control strategy, modeled by impulsive differential equations. Although this control measure is better than classical one, it has shortcomings, regardless of the growth rules of the prey (pest) and the cost of management. In recent years, in order to overcome such drawbacks, several researchers

have started paying attention to another control measure based on the state feedback control strategy, which is taken only when the amount of the monitored prey (pest) population reaches a threshold value [2, 16–19]. Obviously, the latter control measure is more reasonable and suitable for prey (pest) control.

In order to investigate the dynamic behaviors of a population model with the state feedback control strategy, an autonomous Lotka-Volterra system, which is one of the most basic and important models, is considered. Actually, the principles of Lotka-Volterra models have remained valid until today and many theoretical ecologists adhere to their principles (cf. [8, 20–22]).

Thus, in this paper, we consider the following Lotka-Volterra type prey-predator system with impulsive state feedback control:

$$x'(t) = x(t)\big(a - bx(t) - cy(t)\big), \quad y'(t) = y(t)(-D + ex(t)), \quad x \neq h,$$

$$\Delta x(t) = -px(t), \quad \Delta y(t) = qy(t) + r, \quad x = h,$$

$$(1.1)$$

where all parameters except q and r are positive constants. Here, $x(t)$ and $y(t)$ are functions of the time representing population densities of the prey and the predator, respectively, a is the inherent net birth rate per unit of population per unit time of the prey, b is the self-inhibition coefficient, c is the per capita rate of predation of the predator, D denotes the death rate of the predator, e is the rate of conversion of a consumed prey to a predator, $0 < p < 1$ presents the fraction of the prey which die due to the harvesting or pesticide, and so forth, and $q > -1$ and $r \geq 0$ represent the amount of immigration or stock of the predator. We denote by h the economic threshold and $\Delta x(t) = x(t^+) - x(t)$ and $\Delta y(t) = y(t^+) - y(t)$. When the amount of the prey reaches the threshold h at time t_h, controlling measures are taken and hence the amounts of the prey and predator immediately become $(1 - p)h$ and $(1 + q)y(t_h) + r$, respectively.

The main purpose of this research is to investigate theoretically and numerically the dynamical behaviors of system (1.1).

This paper is organized as follows. In the next section, we present a useful lemma and notations and construct a Poincaré map to discuss the dynamics of the system. In Section 3, the sufficient conditions for the existence of a semi-periodic solution of system (1.1) with $r = 0$ are established via the Poincaré criterion. On the other hand, in Section 4, we find out some conditions for the existence and stability of stable positive period-one solutions of system (1.1). Further, under some conditions, we show that there exists a stable positive periodic solution of period 1 or 2; however, there is no positive periodic solutions with period greater than and equal to three. In order to testify our theoretical results by numerical simulations, in Section 5, we give some numerical examples and the bifurcation diagrams of solutions that show the existence of a chaotic solution of system (1.1). Finally, we have a discussion in Section 6.

2. Preliminaries

Many considerable investigators have studied the dynamic behaviors of system (1.1) without the state feedback control. (cf. [23, 24].) It has a saddle $(0,0)$, one locally stable focus $(D/e, (ae-bD)/ce)$ and a saddle $(a/b, 0)$ if the condition $D/e < a/b$ holds. Since the carrying

capacity of the prey population $x(t)$ is b/a, so it is meaningful that the economical threshold h is less than b/a. Thus, throughout this paper, we set up the following two assumptions:

$$\text{(A1)} \ \frac{D}{e} < \frac{a}{b}, \qquad \text{(A2)} \ h \leq \frac{b}{a}. \tag{2.1}$$

From the biological point of view, it is reasonable that system (1.1) is considered to control the prey population in the biological meaning space $\{(x,y) : x \geq 0, y \geq 0\}$.

The smoothness properties of f, which denotes the right hand of (1.1), guarantee the global existence and uniqueness of a solution of system (1.1) (see [25, 26] for the details).

Let $R = (-\infty, \infty)$ and $R_+^2 = \{(x,y) \mid x \geq 0, y \geq 0\}$. Firstly, we denote the distance between the point p and the set S by $d(p,S) = \inf_{p_0 \in S}|p - p_0|$ and define, for any solution $z(t) = (x(t), y(t))$ of system (1.1), the positive orbit of $z(t)$ through the point $z_0 \in R_+^2$ as

$$O^+(z_0, t_0) = \left\{ z \in R_+^2 \mid z = z(t), t \geq t_0, z(t_0) = z_0 \right\}. \tag{2.2}$$

Now, we introduce some definitions (cf. [27]).

Definition 2.1 (orbital stability). $z^*(t)$ is said to be orbitally stable if, given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that, for any other solution $z(t)$ of system (1.1) satisfying $|z^*(t_0) - z(t_0)| < \delta$, then $d(z(t), O^+(z_0, t_0)) < \epsilon$ for $t > t_0$.

Definition 2.2 (asymptotic orbital stability). $z^*(t)$ is said to be asymptotically orbitally stable if it is orbitally stable and for any other solution $z(t)$ of system (1.1), there exists a constant $\eta > 0$ such that, if $|z^*(t_0) - z(t_0)| < \eta$, then $\lim_{t \to \infty} d(z(t), O^+(z_0, t_0)) = 0$.

In order to discuss the orbital asymptotical stability of a positive periodic solution of system (1.1), a useful lemma, which follows from Corollary 2 of Theorem 1 given in Simeonov and Bainov [28], is considered as follows.

Lemma 2.3 (analogue of the Poincaré criterion). *The T-periodic solution $x = \varphi(t), y = \zeta(t)$ of system*

$$\begin{aligned} x'(t) = P(x,y), \quad y'(t) = Q(x,y), \quad &\text{if } \phi(x,y) \neq 0, \\ \Delta x = \alpha(x,y), \quad \Delta y = \beta(x,y), \quad &\text{if } \phi(x,y) = 0, \end{aligned} \tag{2.3}$$

is orbitally asymptotically stable if the multiplier μ_2 satisfies the condition $|\mu_2| < 1$, where

$$\mu_2 = \prod_{k=1}^{q} \Delta_k \exp\left[\int_0^T \frac{\partial P}{\partial x}(\zeta(t), \eta(t)) + \frac{\partial Q}{\partial y}(\zeta(t), \eta(t))dt\right],$$

$$\Delta_k = \frac{P_+\left((\partial\beta/\partial y)\varpi - (\partial\beta/\partial x)\varrho + \varpi\right) + Q_+\left((\partial\alpha/\partial x)\varrho - (\partial\alpha/\partial y)\varpi + \varrho\right)}{P\varpi + Q\varrho}, \tag{2.4}$$

where ϖ denotes $(\partial\phi/\partial x)$ and ϱ denotes $(\partial\phi/\partial y)$ and $P, Q, \partial\alpha/\partial x, \partial\alpha/\partial y, \partial\beta/\partial x, \partial\beta/\partial y, \partial\phi/\partial x$, and $\partial\phi/\partial y$ are calculated at the point $(\varphi(\tau_k), \zeta(\tau_k))$, $P_+ = P(\varphi(\tau_k^+), \zeta(\tau_k^+))$, and

$Q_+ = Q(\varphi(\tau_k^+), \zeta(\tau_k^+))$. *Also $\phi(x, y)$ is a sufficiently smooth function on a neighborhood of the points $(\varphi(\tau_k), \zeta(\tau_k))$ such that $\mathrm{grad}\phi(x, y) \neq 0$ and τ_k is the moment of the kth jump, where $k = 1, 2, \ldots, q$.*

From now on, we construct two Poincaré maps to discuss the dynamics of system (1.1). For this, we introduce two cross-sections $\sum_1 = \{(x, y) : x = (1 - p)h, y \geq 0\}$ and $\sum_2 = \{(x, y) : x = h, y \geq 0\}$. In order to establish the Poincaré map of $\sum_2$ via an approximate formula, suppose that system (1.1) has a positive period-1 solution $z(t) = (\varphi(t), \zeta(t))$ with period T and the initial condition $z_0 = A^+((1 - p)h, y_0) \in \sum_1$, where $y(0) \equiv y_0 > 0$. Then the periodic trajectory intersects the Poincaré section $\sum_2$ at the point $A(h, y_1)$ and then jumps to the point A^+ due to the impulsive effects with $\Delta x(t) = -px(t)$ and $\Delta y(t) = qy(t) + r$. Thus

$$\varphi(0) = (1 - p)h, \qquad \zeta(0) = y_0, \qquad \varphi(T) = h, \qquad \zeta(T) = y_1 = \frac{y_0}{1 + q} - r. \qquad (2.5)$$

Now, we consider another solution $\overline{z}(t) = (\overline{\varphi}(t), \overline{\zeta}(t))$ with the initial condition $\overline{z}_0 = A_0((1 - p)h, y_0 + \delta_{y_0})$. Suppose that this trajectory which starts form A_0 first intersects $\sum_2$ at the point $A_1(h, \overline{y}_1)$ when $t = T + \delta_t$ and then jumps to the point $A_1^+((1 - p)h, \overline{y}_2)$ on $\sum_1$. Then we have

$$\overline{\varphi}(0) = (1 - p)h, \qquad \overline{\zeta}(0) = y_0 + \delta_{y_0}, \qquad \overline{\varphi}(T + \delta_t) = h, \qquad \overline{\zeta}(T + \delta_t) = \overline{y}_1. \qquad (2.6)$$

Set $u(t) = \overline{\varphi}(t) - \varphi(t)$ and $v(t) = \overline{\zeta}(t) - \zeta(t)$, then $u_0 = u(0) = \overline{\varphi}(0) - \varphi(0) = 0$ and $v_0 = v(0) = \overline{\zeta}(0) - \zeta(0)$. Let $v_1 = \overline{y}_2 - y_0$ and $v_0^* = \overline{y}_1 - y_1$. It is well known that, for $0 < t < T$, the variables $u(t)$ and $v(t)$ are described by the relation

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + o\left(u_0^2 + v_0^2\right) = \Phi(t) \begin{pmatrix} 0 \\ v_0 \end{pmatrix} + o\begin{pmatrix} 0 \\ v_0^2 \end{pmatrix}, \qquad (2.7)$$

where the fundamental solution matrix $\Phi(t)$ satisfies the matrix equation

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} a - 2b\varphi(t) - c\zeta(t) & -c\varphi(t) \\ e\zeta(t) & -D + e\varphi(t) \end{pmatrix} \Phi(t) \qquad (2.8)$$

with $\Phi(0) = I$(the identity matrix). Set $g_1(t) = \varphi(t)(a - b\varphi(t) - c\zeta(t))$ and $g_2(t) = \zeta(t)(-D + e\varphi(t))$. We can express the perturbed trajectory in a first-order Taylor expansion

$$\overline{\varphi}(T + \delta_t) \approx \varphi(T) + u(T) + g_1(T)\delta_t,$$
$$\overline{\zeta}(T + \delta_t) \approx \zeta(T) + v(T) + g_2(T)\delta_t. \qquad (2.9)$$

It follows from $\overline{\varphi}(T + \delta_t) = \varphi(T) = h$ that

$$\delta_t = -\frac{u(T)}{g_1(T)} \quad \text{and hence } v_0^* = \overline{y}_1 - y_1 = v(T) - \frac{g_2(T)u(T)}{g_1(T)}. \qquad (2.10)$$

Since $\bar{y}_2 = (1+q)\bar{y}_1 + r$ and $\bar{y}_2 - y_0 = (1+q)(\bar{y}_1 - y_1)$, we obtain $v_1 = (1+q)v_0^*$. So, we can construct a Poincaré map F of $\sum_1$ as follows:

$$v_1 = F_q(v_0) = (1+q)\left[v(T) - \frac{g_2(T)u(T)}{g_1(T)}\right], \tag{2.11}$$

where $u(T)$ and $v(T)$ are calculated according to (2.7).

Now we construct another type of Poincaré maps. Suppose that the point $B_k(h, y_k)$ is on the section $\sum_2$. Then $B_k^+((1-p)h, (1+q)y_k + r)$ is on $\sum_1$ due to the impulsive effects, and the trajectory with the initial point B_k^+ intersects $\sum_2$ at the point $B_{k+1}(h, y_{k+1})$, where y_{k+1} is determined by y_k and the parameters q and r. Thus we can define a Poincaré map F as follows:

$$y_{k+1} = F(q, r, y_k). \tag{2.12}$$

The function F is continuous on q, r, and y_k because of the dependence of the solutions on the initial conditions.

Definition 2.4. A trajectory $O^+(z_0, t_0)$ of system (1.1) is said to be order k-periodic if there exists a positive integer $k \geq 1$ such that k is the smallest integer for $y_0 = y_k$.

Definition 2.5. A solution $z(t) = (x(t), y(t))$ of system (1.1) is called a semitrivial solution if its one component is zero and another is nonzero.

Note that, for each fixed point of the map F in (2.12), there is an associated periodic solution of system (1.1), and vice versa.

3. The Existence and Stability of a Periodic Solution When $r = 0$

In this section, we consider system (1.1) with $r = 0$ as follows:

$$x'(t) = x(t)(a - bx(t) - cy(t)), \quad y'(t) = y(t)(-D + ex(t)), \quad x \neq h, \tag{3.1}$$
$$\Delta x(t) = -px(t), \quad \Delta y(t) = qy(t), \quad x = h.$$

First, let $y(t) = 0$ to calculate a semitrivial periodic solution of system (3.1). Then system (3.1) can be changed into the following impulsive differential equation:

$$x'(t) = x(t)(a - bx(t)), \quad x(t) \neq h, \tag{3.2}$$
$$\Delta x(t) = -px(t), \quad x(t) = h.$$

Under the initial value $x(0) = (1-p)h \equiv x_0$, the solution of the equation $x'(t) = x(t)(a - bx(t))$ can be obtained as $x(t) = a\exp(at)/(\beta + b\exp(at))$, where $\beta = (a - bh(1-p))/(1-p)h$. Assume that $x(T) = h$ and $x(T^+) = x_0$ in order to get a periodic solution of (3.2). Then we have the

period $T = (1/a)\ln((a - bh(1 - p))/(a - bh)(1 - p))$ of a semitrivial periodic solution of (3.1). Thus system (1.1) with $r = 0$ has a semitrivial periodic solution with the period T as follows:

$$\varphi(t) = \frac{a\exp(a(t - (k - 1)T))}{\beta + b\exp(a(t - (k - 1)T))},$$

$$\zeta(t) = 0,$$

(3.3)

where $(k - 1)T < t < kT$.

Using the Poincaré map F defined in (2.12), we will have a criterion for the stability of this semitrivial periodic solution $(\varphi(t), \zeta(t))$.

Theorem 3.1. *The semitrivial periodic solution of system* (1.1) *with* $r = 0$ *is locally stable if the condition*

$$-1 < q < q_0$$

(3.4)

holds, where $q_0 = (1 - p)^{-D/a}((a - bh(1 - p))/(a - bh))^{D/a - e/b} - 1$.

Proof. We already discussed the existence of the semitrivial periodic solution $(\varphi(t), 0)$. It follows from (2.8) that

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} a - 2b\varphi(t) & -c\varphi(t) \\ 0 & -D + e\varphi(t) \end{pmatrix}\Phi(t), \qquad \Phi(0) = I_2.$$

(3.5)

Let $\Phi(t) = \begin{pmatrix} w_1(t) & w_2(t) \\ w_3(t) & w_4(t) \end{pmatrix}$. Then we can infer from (3.5) that, for $0 < t < T = (1/a)\ln((1 - (1 - p)h)/(1 - h)(1 - p))$,

$$w_1'(t) = (a - 2b\varphi(t))w_1(t) - c\varphi(t)w_3(t), \quad w_1(0) = 1,$$
$$w_2'(t) = (a - 2b\varphi(t))w_2(t) - c\varphi(t)w_4(t), \quad w_2(0) = 0,$$
$$w_3'(t) = (-D + e\varphi(t))w_3(t), \quad w_3(0) = 0,$$
$$w_4'(t) = (-D + e\varphi(t))w_4(t), \quad w_4(0) = 1.$$

(3.6)

Since $u_0 = u(0) = 0$ and $g_2(t) = 0$, we obtain that $v_1 = F_q(v_0) = (1 + q)[v(T) - g_2(T)u(T)/g_1(T)] = (1 + q)w_4(T)v_0$. Thus it is only necessary to calculate $w_4(t)$. From the fourth equation of (3.6), we obtain $w_4(t) = \overline{w}\exp(\int -D + e\varphi(t)dt)$. Since $\int \varphi(t)dt = (1/b)\ln(\beta + b\exp(at))$ and $w_4(0) = 1$, so we obtain $w_4(T) = ((\beta + b\exp(aT))/(\beta + b))^{e/b}\exp(-DT)$. Therefore,

$$v_1 = (1 + q)(1 - p)^{D/a}\left(\frac{a - bh(1 - p)}{a - bh}\right)^{e/b - D/a}v_0.$$

(3.7)

Note that v_0 is a fixed point of $F_q(v_0)$ and

$$D_{v_0} F_q(0) = (1+q)(1-p)^{D/a} \left(\frac{a - bh(1-p)}{a - bh} \right)^{e/b - D/a}. \tag{3.8}$$

Under condition (3.4), we get $0 < D_{v_0} F_q(0) < 1$. So system (1.1) with $r = 0$ has a stable semitrivial periodic solution. □

Remark 3.2. From the proof of **Theorem 3.1**, we note that $D_{v_0} F_q(0) > 1$ if $q > q_0$. It means that the semitrivial periodic solution system (1.1) with $r = 0$ is unstable if $q > q_0$.

Now, we discuss the existence of a positive periodic solution of the system (3.1) with $r = 0$.

Theorem 3.3. *System* (1.1) *with $r = 0$ has a positive period-one solution if the condition*

$$q > q_0 \tag{3.9}$$

holds, where $q_0 = (1-p)^{-D/a}((a - bh(1-p))/(a - bh))^{D/a - e/b} - 1$.

Proof. It follows from **Theorem 3.1** that the semitrivial periodic solution passing through the points $A((1-p)h, 0)$ and $B(h, 0)$ is stable if $-1 < q < q_0$, where $q_0 = (1-p)^{-D/a}((a - bh(1-p))/(a - bh))^{D/a - e/b} - 1$. Now, define $G(x) = F(q, 0, x) - x$, where F is the Poincaré map. From now on, we will show that there exist two positive numbers ϵ_1 and w_0 such that $G(\epsilon_1) > 0$ and $G(w_0) \le 0$ by following two steps.

Step 1. We will show that $G(\epsilon_1) > 0$ for some $\epsilon_1 > 0$. First, consider the trajectory starting with the point $A_1 = ((1-p)h, \epsilon)$ for a sufficiently small number $\epsilon > 0$. This trajectory meets the Poincaré section $\sum_2$ at the point $B_1 = (h, \epsilon_1)$ and then jumps to the point $A_2 = ((1-p)h, (1+q)\epsilon_1)$ and reaches the point $B_2 = (h, \epsilon_2)$. Since $q > q_0$, the semitrivial solution is unstable by **Remark 5.4**. So we can choose an $\bar{\epsilon}$ such that $(1+q)\epsilon_1 > \epsilon$ for $q > q_0 + \bar{\epsilon}$. Thus the point B_2 is above the point B_1. So we have $\epsilon_1 < \epsilon_2$. From (2.12), we know that

$$\epsilon_1 - F(q, 0, \epsilon_1) = \epsilon_1 - \epsilon_2 < 0. \tag{3.10}$$

Thus we know that $G(\epsilon_1) > 0$.

Step 2. We will show that $G(w_0) \le 0$ for some $w_0 > 0$. To do this, suppose that the line $bx + cy - a = 0$ meets $\sum_1$ at $A_3 = ((1-p)h, (a - b(1-p)h)/c)$. The trajectory of system (1.1) with the initial point A_3 meets the line $\sum_2$ at $B_3 = (h, w_0)$ then jumps to the point $A_3^+ = ((1-p)h, (1+q)w_0)$ and then reaches the point $B_4 = (h, \overline{w}_0)$ on the Poincaré section $\sum_2$ again. However, for any $q > 0$, the point B_4 is not above the point B_3 in view of the vector field of system (1.1). Thus $\overline{w}_0 \le w_0$. So we have only to consider the following two cases.

Case (i): If $\overline{w}_0 = w_0$, that is, $G(w_0) = 0$, then system (1.1) has a positive period-one solution.

Case (ii): If $\overline{\omega}_0 < \omega_0$, then

$$\omega_0 - F(q, 0, \omega_0) = \omega_0 - \overline{\omega}_0 > 0, \quad \text{that is, } G(\omega_0) < 0. \tag{3.11}$$

Thus, it follows from (3.10) and (3.11) that the Poincaré map F has a fixed point, which corresponds to a positive period-one solution for system (1.1) with $r = 0$. Thus we complete the proof. $\qquad\square$

Remark 3.4. Under the condition $r = 0$, we show that the semitrivial periodic solution of system (1.1) is stable when $-1 < q < q_0$ and there exists a positive period-one solution of system (1.1). Since $D_{v_0} F_{q_0}(0) = 1$, a fold bifurcation takes place at $q = q_0$. Furthermore, from the proof of Theorem 3.3, we know that system (1.1) with $r = 0$ has a positive period-one solution $(\theta(t), \psi(t))$ passing through the points $L^+ = ((1 - p)h, (1 + q)\psi(0))$ and $L = (h, \psi(0))$ and satisfying the condition $(a - b(1 - p)h)/c = (1 + q_1)\psi(0)$ for some $q_1 > q_0$.

Now we discuss the stability of the positive periodic solution of system (1.1).

Theorem 3.5. *Assume that $r = 0$. Let $(\theta(t), \psi(t))$ be the positive period-one solution of system (1.1) with period τ passing through the points $M^+ = ((1 - p)h, (1 + q)\psi(0))$ and $M = (h, \psi(0))$. Then the positive periodic solution is orbitally asymptotically stable if the condition*

$$q_0 < q < q_2 \tag{3.12}$$

holds, where $g(q_2) = -1$ and $g(u) = ((a - b(1 - p)h - c((1 + u)\psi(0)))/(a - bh - c\psi(0))) \exp(\int_0^\tau -b\theta(t)dt)$.

Proof. In order to discuss the stability of the positive periodic solution $(\theta(t), \psi(t))$ of system (1.1), we will use the Lemma 2.3. First, we note that

$$P(x, y) = x(t)(a - bx(t) - cy(t)), \qquad Q(x, y) = y(t)(-D + ex(t)),$$

$$\alpha(x, y) = -px(t), \qquad \beta(x, y) = qy(t), \qquad \phi(x, y) = x(t) - h, \tag{3.13}$$

$$(\theta(\tau), \psi(\tau)) = (h, \psi(0)), \qquad (\theta(\tau^+), \psi(\tau^+)) = ((1 - p)h, (1 + q)\psi(0)).$$

Since

$$\frac{\partial P}{\partial x} = a - 2bx(t) - cy(t), \qquad \frac{\partial Q}{\partial y} = -D + ex(t), \qquad \frac{\partial \phi}{\partial x} = 1, \qquad \frac{\partial \phi}{\partial y} = 0,$$

$$\frac{\partial \alpha}{\partial x} = p, \qquad \frac{\partial \alpha}{\partial y} = 0, \qquad \frac{\partial \beta}{\partial x} = 0, \qquad \frac{\partial \beta}{\partial y} = q,$$

$$\tag{3.14}$$

we obtain that

$$
\begin{aligned}
\Delta_1 &= \frac{P_+\left(\theta(\tau^+), \psi(\tau^+)\right)(1+q)}{P(\theta(\tau), \psi(\tau))} \\
&= \frac{(1-p)(a - b(1-p)h - c((1+q)\psi(0)))(1+q)}{a - bh - c\psi(0)},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\tau \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} dt &= \int_0^\tau a - 2b\theta(t) - c\psi(t) - D + e\theta(t) dt \qquad (3.15)\\
&= \int_0^\tau \frac{\dot\theta(t)}{\theta(t)} + \frac{\dot\psi(t)}{\psi(t)}(-b\theta(t)) dt = \int_0^\tau d\ln(\theta(t)\psi(t)) + \int_0^\tau (-b\theta(t)) dt \\
&= \ln\left(\frac{1}{(1-p)(1+q)}\right) + \int_0^\tau (-b\theta(t)) dt.
\end{aligned}
$$

Thus we have $\mu_2 = ((a - b(1-p)h - c((1+q)\psi(0)))/(a - bh - c\psi(0))) \exp(\int_0^\tau (-b\theta(t)) dt) \equiv g(q)$. By Remark 3.4, for $q = q_1$, we have $(1+q_1)\psi(0) = (a - b(1-p)h)/c$, and so we get $\mu_2 = 0$ when $q = q_1$ which means that this periodic solution is stable. In addition, for $q = q_0$, we know $\mu_2 = 1$ due to $\psi(0) = 0$ and $\tau = (1/a)\ln((a - bh(1-p))/(a - bh)(1-p))$. Since the derivative $d\mu_2/dq$ with respect to q is negative, so we know that $0 < \mu_2 < 1$ when $q_0 < q < q_1$. Further, we can find $q_2 > q_1$ such that $\mu_2 = g(q_2) = -1$. Therefore, if the condition (3.12) holds, then we obtain $-1 < \mu_2 < 1$, which implies from Lemma 2.3 that the positive periodic solution $(\theta(t), \psi(t))$ is orbitally asymptotically stable. $\qquad\square$

Remark 3.6. System (1.1) has a stable periodic semitrivial solution and a stable positive period-1 solution if $0 < q < q_0$ and $q_0 < q < q_2$, respectively. We already know from Remark 3.4 that a fold bifurcation occurs at $q = q_0$. Thus, from the facts, we can suppose that a flip (period-doubling) bifurcation occurs at $q = q_2$. Moreover, we can figure out that system (1.1) might have a chaotic solution via a cascade of period doubling.

4. The Existence and Stability of a Positive Periodic Solution When $r > 0$

In this section we will take into account the existence and stability of positive periodic solutions in the two cases of $h < D/e$ and $D/e < h$. In fact, under the condition $h < D/e$, the trajectories starting from any initial point (x_0, y_0) with $x_0 < h$ intersects the section $\sum_2$ infinite times. However, under the condition $D/e < h$, the trajectories starting from any initial point (x_0, y_0) with $x_0 < h$ do not intersect the section $\sum_2$.

4.1. The Case of $h < D/e$

Theorem 4.1. *Assume that $h \leq D/e$, $q > -1$, and $r > 0$. Then the system (1.1) has a positive period-one solution. Moreover, if this periodic solution $(\varphi(t), \zeta(t))$ has a period λ and passes through*

the points $M^+ = ((1 - p)h, (1 + q)\zeta(0) + r)$ and $M = (h, \zeta(0))$, then it is asymptotically orbitally stable provided with

$$q^* < q < q^{**}, \tag{4.1}$$

where $\gamma(q^*) = 1$ and $\gamma(q^{**}) = -1$ and $\gamma(q) = (a - b(1 - p)h - c((1 + q)\zeta(0) + r))/(a - bh - c\zeta(0)) \exp(\int_0^\lambda b\zeta(t)dt)$.

Proof. We will use the similar method to Theorem 3.3 to prove the existence of a periodic solution of system (1.1).

Firstly, in order to show $F(q, r, \bar{r}_1) > \bar{r}_1$ for some $\bar{r}_1 > 0$, let $U_1 = ((1 - p)h, r_1)$ be in the Poincaré section $\sum_1$, where r_1 is small enough such that $0 < r_1 < r$. The trajectory of system (1.1) with the initial point U_1 intersects the point $V_1 = (h, \bar{r}_1)$ on the Poincaré section $\sum_2$, then jumps to the point $U_2 = ((1 - p)h, (1 + q)\bar{r}_1 + r)$, and then reaches the point $V_2 = (h, r_2)$ on $\sum_2$ again. From the choice of the value r_1, we know that $(1 + q)\bar{r}_1 + r > r_1$ and hence the points U_2 and V_2 are above the points U_1 and V_1, respectively. Thus we have $\bar{r}_1 < r_2$. It follows from (2.12) that

$$\bar{r}_1 - F(q, r, \bar{r}_1) = \bar{r}_1 - r_2 < 0. \tag{4.2}$$

Secondly, to find a positive number m_0 such that $m_0 - F(q, r, m_0) \geq 0$ suppose that the line $bx + cy - a = 0$ meets $\sum_1$ at $A = ((1 - p)h, (a - b(1 - p)h)/c)$. The trajectory of system (1.1) with the initial point A meets the line $\sum_2$ at $B = (h, m_0)$ then jumps to the point $A^+ = ((1 - p)h, (1 + q)m_0 + r)$ and then reaches the point $B_1 = (h, \overline{m}_0)$ on the line $\sum_2$ again. Suppose that there exists a $q_0 > 0$ such that $(1 + q_0)m_0 + r = (a - b(1 - p)h)/c$. Then the point A^+ is just the point A if $q = q_0$. The point A^+ lies above the point A if $q > q_0$, while it lies under A if $q < q_0$. However, for any $q > 0$, the point B_1 is not above the point B in view of the vector field of the system (1.1). Thus $m_0 \geq \overline{m}_0$ and hence $m_0 - F(q, r, m_0) \geq 0$.

Therefore, we have a periodic solution by the similar method to Theorem 3.3. Further, the stability condition for this period-one solution can be obtained by using the same method used in the proof of Theorem 3.5. Thus we complete the proof.   □

4.2. The Case of $D/e < h$

Theorem 4.2. *Assume that $D/e < h$, $q > -1$, and $r > 0$. Then there exists $r_0 > 0$ such that system (1.1) has a stable positive solution of period 1 or 2 if $r > r_0$, where r_0 depends on the value h. Moreover, system (1.1) has no periodic solutions of period k ($k \geq 3$).*

Proof. First, assume that the orbit, which just touches $\sum_2$ at the point $B_0 = (h, \bar{y}_1)$ with $\bar{y}_1 = (a - bh)/c$, meets $\sum_1$ at the two points $B = ((1 - p)h, \bar{y}_2)$ and $B_1 = ((1 - p)h, \bar{y}_3)$, where $\bar{y}_3 < (a - b(1 - p)h)/c < \bar{y}_2$. We will prove this theorem by the following five steps.

Step 1. We will show that if $r > \bar{y}_2$, then any trajectory of system (1.1) intersects with $\sum_2$ infinite times. Note that every trajectory passing through the point $((1 - p)h, y)$ with $y \in (\bar{y}_3, \bar{y}_2)$ cannot intersect with $\sum_1$ as time goes to infinite and tends to the focus $(D/e, (ae - bD)/ce)$ eventually. Therefore, if all trajectories of system (1.1) pass through the points $((1 - p)h, y)$ with $y \in (\bar{y}_3, \bar{y}_2)$ after finite times impulsive effects on $\sum_2$, they all tend to the focus

and there are no positive periodic solutions. From this fact, we know that the condition $r > \overline{y}_2$, in which $\overline{y}_2$ depends on the value h as a function $g(h)$, is a sufficient condition for a trajectory of system (1.1) which intersects with $\sum_2$ infinite times in view of the impulsive effects $\Delta x = -px$ and $\Delta y = qy + r$.

From now on, let the condition $r > \overline{y}_2$ hold.

Step 2. Next, we will show that $y_{j+1} < y_{m+1}$ for $y_m < y_j$, where (h, y_{k+1}) is the next point of (h, y_k) that touches $\sum_2$. Note that for any point (h, y) with $0 < y < (a - b(1 - p)h)/c$, the point $((1 - p)h, (1 + q)y + r)$ is above the point B. Thus, for any two points $E_m(h, y_m)$ and $E_j(h, y_j)$, where $0 < y_m < y_j < (a - bh)/c$, the points $E_m^+((1 - p)h, (1 + q)y_m + r)$ and $E_j^+((1 - p)h, (1 + q)y_j + r)$ lie above the point B and, further, it follows from the vector field of the system (1.1) that $0 < y_{j+1} < y_{m+1} < (a - bh)/c$, that is,

$$y_{j+1} < y_{m+1} \quad \text{for } y_m < y_j. \tag{4.3}$$

Thus, from the Poincaré map and $r > \overline{y}_2$, we obtain $y_1 = F(q, r, y_0)$, $y_2 = F(q, r, y_1)$, and $y_{n+1} = F(q, r, y_n)$ $(n = 3, 4, \ldots)$ for given $y_0 \in (0, (a - bh)/c)$. Therefore, we have only to consider three cases as follows:

Case (i): $y_0 = y_1$,

Case (ii): $y_0 \neq y_1$,

Case (iii): $y_i \neq y_j$ $(0 \leq i < j \leq k - 1, k \geq 3)$.

Step 3. In order to show the existence of a positive solution of period 1 or 2, consider the Cases (i) and (ii). First, if Case (i) is satisfied, then it is easy to see that system (1.1) has a positive period-one solution. Now, suppose that Case (ii) is satisfied. Then without loss of generality, we can say that $y_1 < y_0$. It follows form (4.3) that $y_2 > y_1$. Furthermore, if $y_2 = y_0$, then there exists a positive period-two solution of system (1.1).

Step 4. Now, we will prove that system (1.1) cannot have periodic solutions of period k ($k \geq 3$) if Case (iii) holds. For this, assume that $y_0 = y_k$, which means that system (1.1) has a positive period-k solution. However, we will show that this is impossible. If $y_0 < y_1$, then from (4.3), we obtain that $y_1 < y_2$ and then $y_2 < y_0 < y_1$ or $y_0 < y_2 < y_1$. If $y_0 > y_1$, then from (4.3), we have $y_1 < y_2$ and then $y_1 < y_2 < y_0$ and $y_1 < y_0 < y_2$. So the relation of y_0, y_1, and y_2 is one of the following:

(a) $y_2 < y_0 < y_1$,   (b) $y_0 < y_2 < y_1$,   (c) $y_1 < y_2 < y_0$,   (d) $y_1 < y_0 < y_2$.   (4.4)

(a) If $y_2 < y_0 < y_1$, then from (4.3), we have $y_2 < y_1 < y_3$. It is also true that $y_2 < y_0 < y_1 < y_3$. We again obtain $y_4 < y_2 < y_1 < y_3$ and then $y_4 < y_2 < y_0 < y_1 < y_4$. By means of induction, we have

$$0 < \cdots < y_{2k} < \cdots < y_4 < y_2 < y_0 < y_1 < y_3 < y_5 < \cdots < y_{2k+1} < \cdots < 1. \tag{4.5}$$

Similar to (a), for Cases (b), (c), and (d), we obtain

(b) $0 < y_0 < y_2 < y_4 < \cdots < y_{2k} < \cdots < y_{2k+1} < \cdots < y_5 < y_3 < y_1 < 1$,

(c) $0 < y_1 < y_3 < y_5 < \cdots < y_{2k+1} < \cdots < y_{2k} < \cdots < y_4 < y_2 < y_0 < 1$, (4.6)

(d) $0 < \cdots < y_{2k+1} < \cdots < y_5 < y_3 < y_1 < y_2 < y_4 < y_6 < \cdots < y_{2k} < \cdots < 1$,

respectively. If there exists a positive period-k solution ($k \geq 3$) in the system (1.1), then $y_i \neq y_j$, ($0 \leq i < j \leq k-1$), $y_k = y_0$ which is a contradiction to (4.5)–(4.6). Thus there is no positive period-k solution ($k \geq 3$) if $r > \overline{y}_2$.

Step 5. From Step 4, we can show that there exists a stable period-1 or-2 solution in these cases. In fact, it follows from (4.5) that

$$\lim_{k \to \infty} y_{2k} = y_0^*, \qquad \lim_{k \to \infty} y_{2k+1} = y_1^*, \tag{4.7}$$

where $0 < y_0^* < y_1^* < (a - bh)/c$. Therefore, $y_1^* = F(q, r, y_0^*)$ and $y_2^* = F(q, r, y_1^*)$. Thus system (1.1) has a positive period-2 solution in the case (a). Moreover, it is easily proven from (4.5) and (4.7) that this positive period-2 solution is local stable. Similarly, we have system (1.1) has a stable positive period-1 solution in cases (b) and (c) and has a stable positive period-2 solution in case (d). □

5. Numerical Examples

In this section, we will present some numerical examples to discuss the various dynamical aspects of system (1.1) and to testify the validity of our theoretical results obtained in the previous sections.

Example 5.1. In order to exhibit the dynamical complexity as q varies, let $r = 0$ and fix the other parameters as follows:

$$a = 0.4, \quad b = 0.8, \quad c = 0.8, \quad D = 0.4, \quad e = 0.8, \quad h = 0.1, \quad p = 0.35. \tag{5.1}$$

In this example, we set an initial value as $(0.05, 0.1)$. It is from Theorem 3.1 that the periodic semitrivial solution is stable if $-1 < q < q_0 = (1-p)^{-D/a}((a - bh(1-p))/(a - bh))^{D/a - e/b} - 1 \approx 0.4286$ (see Figures 1 and 2). We display the bifurcation diagram in Figure 2(a). From the Remark 3.4, we know that a fold bifurcation takes place at $q = q_0$. Figure 2(a) shows that a positive period-one solution bifurcates from the periodic semitrivial solution at $q = q_0 \approx 0.4286$ and a positive period-two solution bifurcates from the positive period-one solution via a flip bifurcation at $q = q_2 \approx 6.25$, which leads to the period-doubling bifurcation and then chaos (see Figures 2(b) and 3). It follows from Theorem 4.2 that system with $r > 0$ cannot have positive period-3 solution under some conditions. However, if $r = 0$, a period-3 solution can exist (see Figure 4).
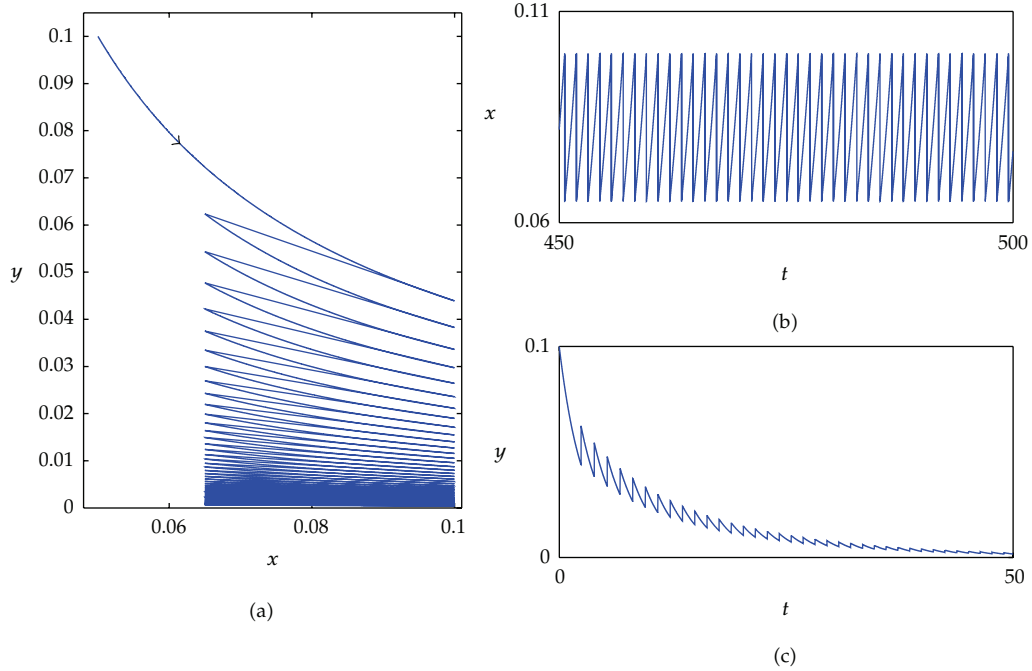
Figure 1: (a) The trajectory of system (1.1) with $r = 0$ when $q = 0.42$. (b-c) Time series.
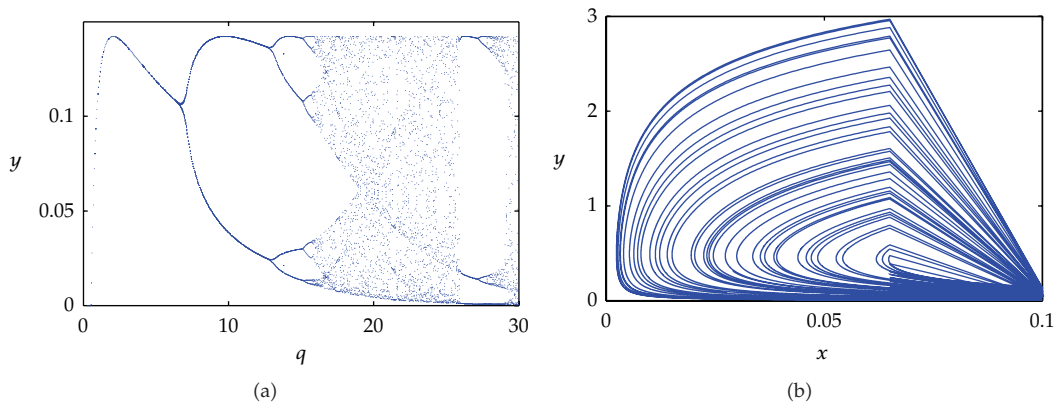

Figure 2: (a) The bifurcation diagram of system (1.1) $r = 0$. (b) A chaotic solution of system (1.1) with $r = 0$ when $q = 20$.

Example 5.2. Under the condition $r > 0$, we know that there is no semitrivial solution in system (1.1). In this case, set the parameters as follows:

$$a = 1.0, \qquad b = 0.6, \qquad c = 0.8, \qquad D = 0.4, \qquad e = 0.8, \qquad p = 0.2, \qquad r = 0.1. \qquad (5.2)$$

Throughout this example, we regard the point $(0.1, 0.2)$ as an initial value. Figure 5(a) shows the bifurcation diagrams of system (1.1) with q as a bifurcation parameter when $h = 0.3 < D/e$. It follows from Theorem 4.1 that there exists a period-one solution for any
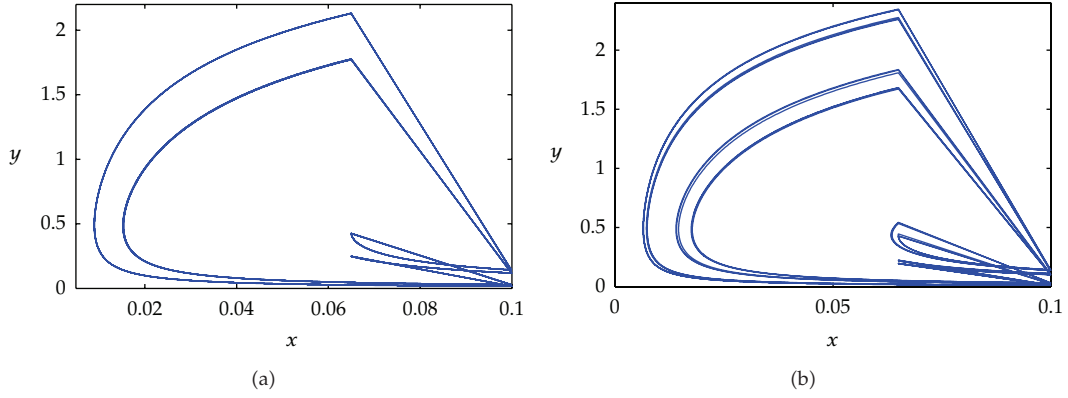
(a)

(b)

Figure 3: (a) A period-4 solution when $r = 0$ and $q = 14$. (b) A period-8 solution when $r = 0$ and $q = 15.5$.
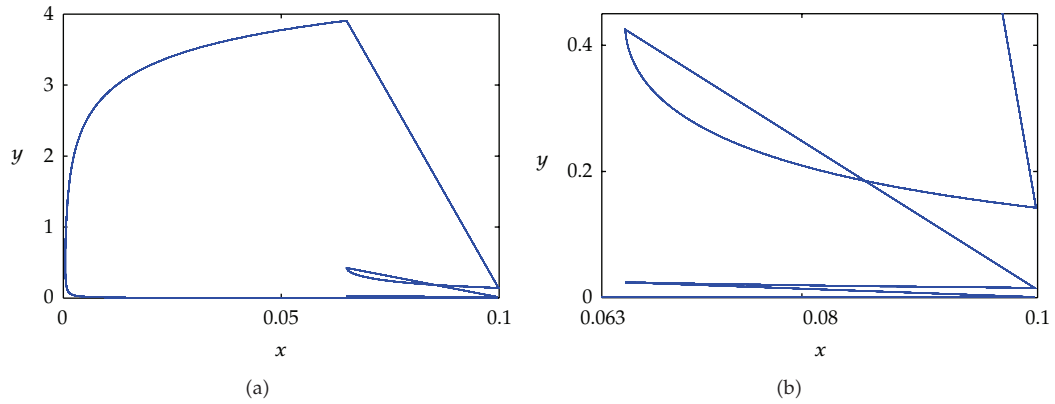

(a)

(b)

Figure 4: (a) A period-3 solution when $r = 0$ and $q = 26.5$. (b) The enlarged part of (a) for $0.063 \leq x \leq 1$.
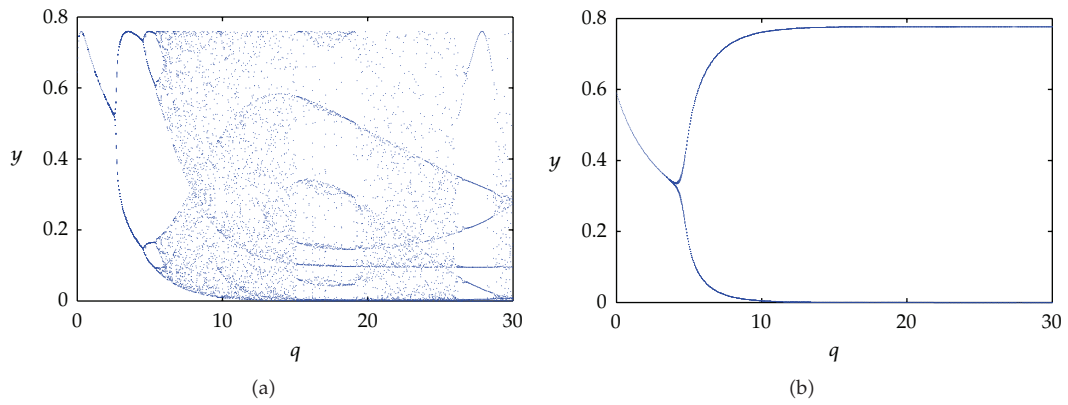

(a)

(b)

Figure 5: (a) The bifurcation diagram of system (1.1) with $h = 0.3$. (b) The bifurcation diagram of system (1.1) with $h = 0.52$.
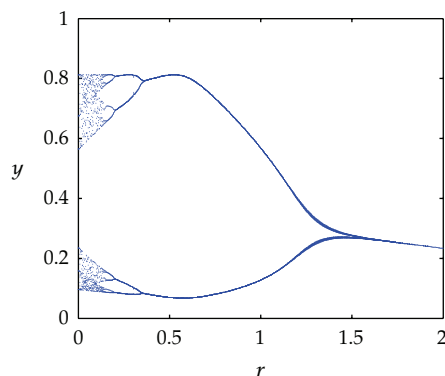
Figure 6: The bifurcation diagram of system (1.1) with $h = 0.3$ and $q = 5$ with respect to $r > 0$.

$q > -1$ and this solution is stable when $q^* < q < q^{**} \approx 4.35$ as shown in Figure 5(a). It is easy to see that there are no fold bifurcations. However, at $q = q^{**} \approx 4.35$, a flip bifurcation occurs and the cascade of the flip bifurcation leads to chaotic solutions like the previous example. Thanks to Figure 5(a), we know that system (1.1) undergoes the complex dynamical behaviors including periodic doubling, chaotic behaviors, and periodic windows.

Example 5.3. It follows from Theorem 4.2 that if the value h satisfies the condition $D/e < h < a/b$, there exists some $r_0 > 0$ such that, for all $q > 0$, system (1.1) has a stable positive period-one or-two solution if $r > r_0$, but does not have period-k ($k \geq 3$) solutions. To substantiate these theoretical results by numerical simulation, let $h = 0.52$ and $r = 1.2$, and let the other parameters be the same as in Example 5.2. Then we obtain $D/e < h < b/a$. Figure 5(b) of the bifurcation diagram of system (1.1) numerically displays that there exist no period-k solutions ($k \geq 3$) except stable positive period-1 or-2 solutions. Thus the value r is also an important parameter in the dynamical aspects of system (1.1). For this reason, we investigate the effects of the parameter r on system (1.1). For this, let $q = 5$ and $h = 0.52$, and let r be a bifurcation parameter. It is easy to see from Figure 6 that the parameter r causes various dynamical behaviors of system (1.1) such as a cascade of reverse period-doubling bifurcations, also called period halving, period windows, chaotic regions, stable period-2 solutions, and so forth.

From a biological point of view, as mentioned in Section 1, the value r represents the amount of immigration or releasing of the predator. Particularly, from Figure 6, one can figure out that the number of the predator cannot be easily estimated when the amount of r is small due to chaotic behaviors of solutions to the system; on the contrary, if sufficient amount of the predator is released impulsively, then the number of the predator (eventually, the number of the prey) can be predictable due to periodic behaviors of solutions to the system.

Remark 5.4. Now, we will demonstrate the superiority of the state-dependent feedback control in comparison with the fixed-time control via an example. For this, assume that $a = 1.0$, $b = 0.6$, $c = 0.8$, $D = 0.4$, $e = 0.8$, $h = 0.3$, $p = 0.6$, $q = 4$, and $r = 0.1$ in system (1.1) with an initial value $(0.05, 4.1)$. Figure 7(b) shows that the prey population cannot be controlled below the threshold value if we take the impulsive control measure at fixed time $t = 6k$ ($k = 1, 2, \ldots$). However, it is seen from Figure 7(a) that only after several attempts of
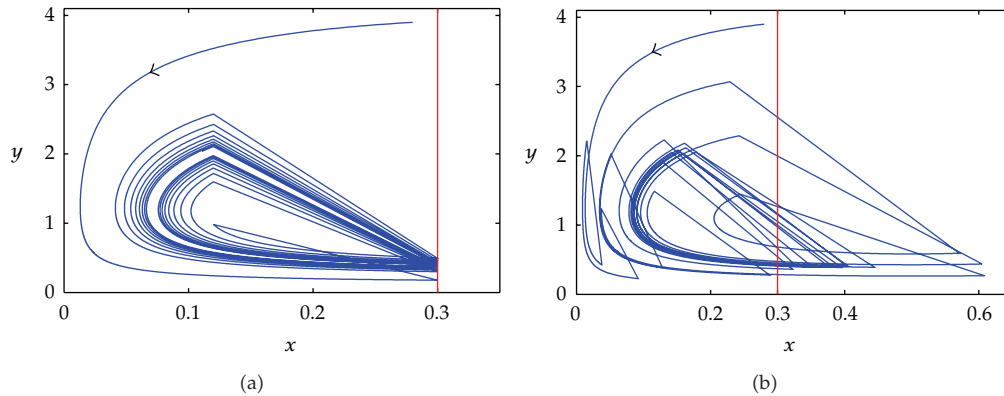
(a)   (b)

Figure 7: The trajectories of system (1.1) with $h = 0.3$ (a) under the state feedback control and (b) under the fixed time control when $t = 6k$ $(k = 1, 2 \ldots)$.

control does the solution approach the periodic solution. Thus this example shows that the impulsive state feedback measure is more effective in real biological control.

6. Conclusion

In this paper, a state-dependent impulsive dynamical system concerning control strategy has been proposed and analyzed. Particularly, a state feedback measure for controlling the prey population is taken when the amount of the prey reaches a threshold value. The dynamical behaviors have been investigated, including the existence of periodic solutions with period 1 and 2 and their stabilities. In addition, we have numerically shown that system (1.1) has various dynamical aspects including a chaotic behavior. Based on the main theorems of this paper, the amount of the prey population can be completely controlled below the threshold value by one, two, or at most finite number of applying impulsive effects. From a biological point of view, it will be very helpful and useful to control the prey population.

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