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Research Article

Oscillation Properties for Second-Order Partial Differential Equations with Damping and Functional Arguments

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Using an integral averaging method and generalized Riccati technique, by introducing a parameter $\beta \geq 1$, we derive new oscillation criteria for second-order partial differential equations with damping. The results are of high degree of generality and sharper than most known ones.

1. Introduction

Consider the second-order partial delay differential equation

$$\frac{\partial}{\partial t}\left(r(t)\frac{\partial}{\partial t}u(x,t)\right)+p(t)\frac{\partial u(x,t)}{\partial t}=a(t)\Delta u(x,t)+\sum_{k=1}^{s}a_k(t)\Delta u\big(x,t-\rho_k(t)\big)-q(x,t)f(u(x,t))$$
$$-\sum_{j=1}^{m}q_j(x,t)f_j\big(u(x,t-\sigma_j)\big),\quad (x,t)\in\Omega\times R_+\equiv G, \tag{1.1}$$

where Δ is the Laplacian in R^N, $R_+=[0,\infty)$ and Ω is a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$.

Throughout this paper, we assume that

(H$_1$) $r(t)\in C^1(R_+,(0,\infty))$, $p(t)\in C(R_+,R)$;

(H$_2$) $q(x,t), q_j(x,t)\in C(\overline{G},R_+)$, $q(t)=\min_{x\in\overline{G}}q(x,t)$, $q_j(t)=\min_{x\in\overline{G}}q_j(x,t)$, $j\in I_m=\{1,2,\ldots,m\}$;

(H_3) $a(t), a_k(t), \rho_k(t) \in C(R_+, R_+)$, $\lim_{t\to\infty}(t-\rho_k(t)) = \infty$, and σ_j are nonnegative constants, $j \in I_m$, $k \in I_s = \{1, 2, \dots, s\}$;

(H_4) $f(u) \in C^1(R, R)$, $f_j(u) \in C(R, R)$ are convex in R_+ with $uf_j(u) > 0$, $uf(u) > 0$, and $f'(u) \geq \mu > 0$, $(u \neq 0)$.

We say that a continuous function $H(t,s)$ belongs to the function class ω, denoted by $H \in \omega$, if $H \in C(D, R_+)$, where $D = \{(t,s) : -\infty < s \leq t < +\infty\}$, satisfy

$$H(t,t) = 0, \quad H(t,s) > 0, \quad -\infty < s < t < +\infty. \tag{1.2}$$

Furthermore, the continuous partial derivative $\partial H/\partial S$ exists on D, and there is $h \in L_{\text{loc}}(D, R)$, such that

$$\frac{\partial H}{\partial s} = -h(t,s)\sqrt{H(t,s)}. \tag{1.3}$$

Various results on the oscillation for the partial functional differential equation have been obtained recently. We refer the reader to [1–3] for parabolic equations and to [4–11] for hyperbolic equations.

Recently, Li and Cui [12] studied the equation of the form

$$\begin{aligned}\frac{\partial}{\partial t}\left[p(t)\frac{\partial}{\partial t}\left(u(x,t) + \sum_{i=1}^{l}\lambda_i(t)u(x,t-\tau_i)\right)\right] &= a(t)\Delta u(x,t) + \sum_{k=1}^{s}a_k(t)\Delta u(x,t-\rho_k(t)) - q(t)u(x,t) \\ &\quad - \sum_{j=1}^{m}q_j(x,t)u(x,t-\sigma_j), \quad (x,t) \in \Omega \times R_+ \equiv G\end{aligned} \tag{1.4}$$

with Robin boundary condition

$$\frac{\partial u(x,t)}{\partial \gamma} + g(x,t)u(x,t) = 0, \quad (x,t) \in \partial\Omega \times R_+, \tag{1.5}$$

where γ is the unit exterior vector to $\partial\Omega$ and $g(x,t)$ is a nonnegative continuous function on $\partial\Omega \times R_+$ and obtained the following result.

Theorem A (see [12, Theorem 2.2]). *Suppose that $H \in \omega$, let*

(C_1) $0 < \inf_{s\geq t_0}\{\liminf_{t\to\infty}(H(t,s)/H(t,t_0))\} \leq \infty$, *suppose that there exists some $j_0 \in I_m$ and there exist two functions $\phi \in C^1[t_0,\infty)$, $A \in C[t_0,\infty)$ satisfying,*

(C_2) $\limsup_{t\to\infty}(1/H(t,t_0))\int_{t_0}^{t} p(s-\sigma_{j_0})\phi(s)h^2(t,s)ds < \infty$,

(C_3) $\int_{t_0}^{\infty}(A_+^2(s)/p(s-\sigma_{j_0})\phi(s))ds = \infty$, *and for every $t_1 \geq t_0$,*

(C_4) $\limsup_{t\to\infty}(1/H(t,t_1))\int_{t_1}^{t}[H(t,s)\psi(s) - (1/4)\phi(s)p(s-\sigma_{j_0})h^2(t,s)]ds \geq A(t_1)$,

where $\phi(s) = \exp\{-2\int^{s}\phi(\xi)d\xi\}$, $A_+(s) = \max\{A(s),0\}$, and $\psi(s) = \phi(s)\{\alpha_{j_0}q_{j_0}(s)\ [1-\sum_{i=1}^{l}\lambda_i(s-\sigma_{j_0})] + p(s-\sigma_{j_0})\phi^2(s) - [p(s-\sigma_{j_0})\phi(s)]'\}$. Then every solution $u(x,t)$ of the problem (1.4), (1.5) *is oscillatory in G.*

In 2008, Rogovchenko and Tuncay [13] established new oscillation criteria for second-order nonlinear differential equations with damping term

$$\left(r(t)x'(t)\right)' + p(t)x'(t) + q(t)f(x(t)) = 0, \tag{1.6}$$

without an assumption that has been required in related results reported in the literature over the last two decades. Motivated by the ideas in [12, 13], by introducing a Parameter $\beta \geq 1$, we will further improve Theorems A and derive new interval criteria for oscillation of (1.1). We suggest two different approaches which allow one to remove condition (C_2) in Theorem A. A modified integral averaging technique enables one to simplify essentially the proofs of oscillation criteria.

2. Main Results

Theorem 2.1. *Suppose that there exists a function $y \in C^1[t_0, \infty)$ such that for some $\beta \geq 1$ and for some $H \in \omega$,*

$$\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left(H(t, s)\varphi(s) - \frac{\beta}{4\mu} v(s) r(s) h^2(t, s) \right) ds = \infty, \quad t \geq 0, \tag{2.1}$$

where

$$v(t) = \exp\left(-2 \int^{t} \left(\mu y(s) - \frac{p(s)}{2r(s)} \right) ds \right), \tag{2.2}$$

$$\varphi(t) = v(t)\left[q(t) + \mu r(t) y^2(t) - p(t) y(t) - \left(r(t) y(t) \right)' \right], \tag{2.3}$$

then every solution $u(x, t)$ of the problem (1.1), (1.5) *is oscillatory in G.*

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x, t)$ of the problem (1.1), (1.5) which has no zero on $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality, we assume that $u(x, t) > 0, u(x, t - \rho_k(t)) > 0$ and $u(x, t - \sigma_j) > 0$ in $\Omega \times [t_1, \infty), t_1 \geq t_0, k \in I_s, j \in I_m$. Integrating (1.1) with respect to x over the domain Ω, we have

$$\begin{aligned} &\frac{d}{dt}\left(r(t) \frac{d}{dt} \int_{\Omega} u(x, t) dx \right) + p(t) \frac{d}{dt} \int_{\Omega} u(x, t) dx \\ &\quad = a(t) \int_{\Omega} \Delta u(x, t) dx + \sum_{k=1}^{s} a_k(t) \int_{\Omega} \Delta u(x, t - \rho_k(t)) dx \\ &\qquad - \int_{\Omega} q(x, t) f(u(x, t)) dx - \sum_{j=1}^{m} \int_{\Omega} q_j(x, t) f_j(u(x, t - \sigma_j)) dx, \quad t \geq t_1. \end{aligned} \tag{2.4}$$

From Green's formula and the boundary condition (1.5), we have

$$\begin{aligned}
\int_{\Omega} \Delta u(x,t)dx &= \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial \gamma} ds = -\int_{\partial\Omega} g(x,t)u(x,t)ds \le 0, \\
\int_{\Omega} \Delta u(x,t-\rho_k(t))dx &= \int_{\partial\Omega} \frac{\partial u(x,t-\rho_k(t))}{\partial \gamma} ds \\
&= -\int_{\partial\Omega} g(x,t-\rho_k(t))u(x,t-\rho_k(t))ds \le 0, \quad t \ge t_1,\ k \in I_s,
\end{aligned} \tag{2.5}$$

where ds denotes the surface element on $\partial\Omega$. Moreover, from (H_2), (H_4) and Jensen's inequality, we have

$$\begin{aligned}
\int_{\Omega} q(x,t)f(u(x,t))dx &\ge q(t)\int_{\Omega} f(u(x,t))dx \ge |\Omega| q(t) f\left(\frac{1}{|\Omega|}\int_{\Omega} u(x,t)dx\right), \\
\int_{\Omega} q_j(x,t)f_j(u(x,t-\sigma_j))dx &\ge q_j(t)\int_{\Omega} f_j(u(x,t-\sigma_j))dx \\
&\ge |\Omega| q_j(t) f_j\left(\frac{1}{|\Omega|}\int_{\Omega} u(x,t-\sigma_j)\right)dx,
\end{aligned} \tag{2.6}$$

where $|\Omega| = \int_{\Omega} dx$.

Set

$$U(t) = \frac{1}{|\Omega|}\int_{\Omega} u(x,t)dx, \quad t \ge t_1. \tag{2.7}$$

In view of (2.5)–(2.7), (2.4) yields that

$$(r(t)U'(t))' + p(t)U'(t) + q(t)f(U(t)) + \sum_{j=1}^{m} q_j(t)f_j(U(t-\sigma_j)) \le 0, \quad t \ge t_1. \tag{2.8}$$

Note that (H_4), (2.8) yields that

$$(r(t)U'(t))' + p(t)U'(t) + q(t)f(U(t)) \le 0, \quad t \ge t_1. \tag{2.9}$$

Put

$$w(t) = v(t)r(t)\left[\frac{U'(t)}{f(U(t))} + y(t)\right], \quad t \ge t_1, \tag{2.10}$$

where $v(t)$ is given by (2.2), then

$$\begin{aligned}
w'(t) &= \left[-2\mu y(t) + \frac{p(t)}{r(t)}\right]w(t) + v(t)\left[\frac{(r(t)U'(t))'}{f(U(t))} - \frac{r(t)f'(U(t))(U'(t))^2}{f^2(U(t))} + (r(t)y(t))'\right] \\
&\le \left[-2\mu y(t) + \frac{p(t)}{r(t)}\right]w(t) + v(t)\left[\frac{(r(t)U'(t))'}{f(U(t))} - \frac{\mu r(t)(U'(t))^2}{f^2(U(t))} + (r(t)y(t))'\right] \\
&\le \left[-2\mu y(t) + \frac{p(t)}{r(t)}\right]w(t) - v(t)\left[\frac{p(t)U'(t)}{f(U(t))} + q(t) + \mu r(t)\left(\frac{U'(t)}{f(U(t))}\right)^2 - (r(t)y(t))'\right] \\
&= \left[-2\mu y(t) + \frac{p(t)}{r(t)}\right]w(t) - v(t)\left[p(t)\left(\frac{w(t)}{v(t)r(t)} - y(t)\right) + q(t)\right. \\
&\qquad\qquad \left. + \mu r(t)\left[\frac{w(t)}{v(t)r(t)} - y(t)\right]^2 - (r(t)y(t))'\right] \\
&= \left[-2\mu y(t) + \frac{p(t)}{r(t)}\right]w(t) \\
&\quad - v(t)\left[\frac{p(t)}{v(t)r(t)}w(t) - p(t)y(t) + q(t)\right. \\
&\qquad \left. + \mu r(t)\frac{w^2(t)}{v^2(t)r^2(t)} - 2\mu\frac{w(t)y(t)}{v(t)} + \mu r(t)y^2(t) - (r(t)y(t))'\right] \\
&= -2\mu y(t)w(t) + \frac{p(t)}{r(t)}w(t) - \frac{p(t)}{r(t)}w(t) - v(t)\left[-p(t)y(t) + q(t) + \mu r(t)y^2(t) - (r(t)y(t))'\right] \\
&\quad + 2\mu y(t)w(t) - \mu\frac{w^2(t)}{v(t)r(t)} \\
&= -v(t)\left[-p(t)y(t) + q(t) + \mu r(t)y^2(t) - (r(t)y(t))'\right] - \mu\frac{w^2(t)}{v(t)r(t)} \\
&= -\varphi(t) - \mu\frac{w^2(t)}{v(t)r(t)},
\end{aligned} \tag{2.11}$$

that is,

$$\varphi(t) \le -w'(t) - \mu\frac{w^2(t)}{v(t)r(t)}, \tag{2.12}$$

where $\psi(t)$ is defined by (2.3). Multiplying (2.12) by $H(t,s)$ and integrating from T to t, we have, for some $\beta \geq 1$ and for all $t \geq T \geq t_1$,

$$\begin{aligned}
\int_T^t H(t,s)\psi(s)ds &\leq -\int_T^t H(t,s)w'(s)ds - \int_T^t H(t,s)\frac{\mu}{v(s)r(s)}w^2(s)ds \\
&= H(t,T)w(T) - \int_T^t \left(h(t,s)\sqrt{H(t,s)}w(s) + H(t,s)\frac{\mu w^2(s)}{v(s)r(s)} \right) ds \\
&= H(t,T)w(T) - \int_T^t \left[\sqrt{\frac{\mu H(t,s)}{\beta v(s)r(s)}}w(s) + \sqrt{\frac{\beta v(s)r(s)}{4\mu}}h(t,s) \right]^2 ds \\
&\quad + \frac{\beta}{4\mu}\int_T^t v(s)r(s)h^2(t,s)ds - \int_T^t \frac{(\beta-1)\mu}{\beta v(s)r(s)}H(t,s)w^2(s)ds.
\end{aligned} \tag{2.13}$$

Writing the latter inequality in the form

$$\begin{aligned}
&\int_T^t \left[H(t,s)\psi(s) - \frac{\beta}{4\mu}v(s)r(s)h^2(t,s) \right] ds \\
&\quad \leq H(t,T)w(T) - \int_T^t \left[\sqrt{\frac{\mu H(t,s)}{\beta v(s)r(s)}}w(s) + \sqrt{\frac{\beta v(s)r(s)}{4\mu}}h(t,s) \right]^2 ds \\
&\quad\quad - \int_T^t \frac{(\beta-1)\mu}{\beta v(s)r(s)}H(t,s)w^2(s)ds.
\end{aligned} \tag{2.14}$$

Using the properties of $H(t,s)$, we have

$$\int_{t_1}^t \left[H(t,s)\psi(s) - \frac{\beta}{4\mu}v(s)r(s)h^2(t,s) \right] ds \leq H(t,t_1)|w(t_1)| \leq H(t,t_0)|w(t_1)|, \quad t \geq t_1, \tag{2.15}$$

and for all $t \geq t_1 \geq t_0$,

$$\int_{t_0}^t \left[H(t,s)\psi(s) - \frac{\beta}{4\mu}v(s)r(s)h^2(t,s) \right] ds \leq H(t,t_0)\left[\int_{t_0}^{t_1} \left|\psi(s)\right| ds + |w(t_1)| \right]. \tag{2.16}$$

By (2.16),

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left(H(t,s)\psi(s) - \frac{\beta}{4\mu}v(s)r(s)h^2(t,s) \right) ds \leq \int_{t_0}^{t_1} \left|\psi(s)\right| ds + |w(t_1)| < \infty, \tag{2.17}$$

which contradicts (2.1). This proves Theorem 2.1. □

Consider a Kamenev-type function $H(t,s)$ defined by $H(t,s) = (t-s)^{n-1}$, $(t,s) \in D$, where $n > 2$ is an integer. Obviously, H belongs to the class ω, and $h(t,s) = (n-1)(t-s)^{(n-3)/2}$, $(t,s) \in D$. Then, we can get the following results.

Corollary 2.2. *Suppose that there exists a function $y(t) \in C^1([t_0,\infty);R)$ such that for some integer $n > 2$ and some $\beta \geq 1$,*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-3} \left[(t-s)^2 \psi(s) - \frac{\beta(n-1)^2}{4\mu} v(s) r(s) \right] ds = \infty, \tag{2.18}$$

where $v(t)$ and $\psi(t)$ are as defined in Theorem 2.1. Then every solution $u(x,t)$ of the problem (1.1), (1.5) *is oscillatory in G.*

Theorem 2.3. *Suppose that*

$$0 < \inf_{s \geq t_0} \left(\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right) \leq \infty. \tag{2.19}$$

Assume that there exist functions $f \in C^1([t_0,\infty);R)$ and $\phi \in C([t_0,\infty);R)$ such that, for all $t \geq T \geq t_0$ and for some $\beta > 1$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left(H(t,s)\psi(s) - \frac{\beta}{4\mu} v(s) r(s) h^2(t,s) \right) ds \geq \phi(T), \tag{2.20}$$

where $v(t), \psi(t)$ are as defined in Theorem 2.1 and suppose further that

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{\phi_+^2(s)}{v(s) r(s)} = \infty, \tag{2.21}$$

where $\phi_+(t) = \max(\phi(t), 0)$. Then every solution $u(x,t)$ of the problem (1.1), (1.5) *is oscillatory in G.*

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1.1), (1.5) which has no zero on $\Omega \times [t_1,\infty)$ for some $t_1 > t_0$, without loss of generality, we assume that $u(x,t) > 0, u(x,t-\rho_k(t)) > 0$ and $u(x,t-\sigma_j) > 0$ in $\Omega \times [t_1,\infty)$, $t \geq t_1 \geq t_0$, $k \in I_s$, $j \in I_m$.

As in the proof of Theorem 2.1, (2.14) holds for all $t \geq T \geq t_1$, we have

$$\begin{aligned}
&\frac{1}{H(t,T)}\int_T^t \left[H(t,s)\varphi(s) - \frac{\beta}{4\mu}v(s)r(s)h^2(t,s)\right]ds \\
&\quad\leq w(T) - \frac{1}{H(t,T)}\int_T^t \left[\sqrt{\frac{\mu H(t,s)}{\beta v(s)r(s)}}w(s) + \sqrt{\frac{\beta v(s)r(s)}{4\mu}}h(t,s)\right]^2 ds \\
&\qquad - \frac{1}{H(t,T)}\int_T^t \frac{(\beta-1)\mu}{\beta v(s)r(s)}H(t,s)w^2(s)ds \\
&\quad\leq w(T) - \frac{1}{H(t,T)}\int_T^t \frac{(\beta-1)\mu}{\beta v(s)r(s)}H(t,s)w^2(s)ds.
\end{aligned} \tag{2.22}$$

Therefore, for $t > T \geq T_0$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_T^t \left[H(t,s)\varphi(s) - \frac{\beta}{4\mu}v(s)r(s)h^2(t,s)\right]ds \\
&\quad\leq w(T) - \liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \frac{(\beta-1)\mu}{\beta v(s)r(s)}H(t,s)w^2(s)ds.
\end{aligned} \tag{2.23}$$

It follows from (2.20) that

$$w(T) \geq \phi(T) + \liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \frac{(\beta-1)\mu}{\beta v(s)r(s)}H(t,s)w^2(s)ds \tag{2.24}$$

for all $T \geq t_1$ and for any $\beta > 1$. Then, for all $T \geq t_1$,

$$w(T) \geq \phi(T), \tag{2.25}$$

$$\liminf_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^t \frac{H(t,s)}{v(s)r(s)}w^2(s)ds \leq \frac{\beta}{\mu(\beta-1)}\left(w(t_1) - \phi(t_1)\right). \tag{2.26}$$

Now, we claim that

$$\int_{t_1}^\infty \frac{w^2(s)}{v(s)r(s)}ds < \infty. \tag{2.27}$$

Suppose the contrary, that is,

$$\int_{t_1}^\infty \frac{w^2(s)}{v(s)r(s)}ds = \infty. \tag{2.28}$$

By (2.19), there is a positive constant M_1, satisfying

$$\inf_{s\geq t_0}\left(\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right) > M_1 > 0. \tag{2.29}$$

Let M be any arbitrary positive number, then from (2.28) we get that there exists a $T_1 > t_1$ such that, for all $t \geq T_1$,

$$\int_{t_1}^{t}\frac{w^2(s)}{v(s)r(s)}ds \geq \frac{M}{M_1}. \tag{2.30}$$

Using integration by parts, for all $t \geq T_1$, we get

$$\begin{aligned}\frac{1}{H(t,t_1)}\int_{t_1}^{t}H(t,s)\frac{w^2(s)}{v(s)r(s)}ds &= \frac{1}{H(t,t_1)}\int_{t_1}^{t}\left(-\frac{\partial H(t,s)}{\partial s}\right)\left(\int_{t_1}^{s}\frac{w^2(\tau)}{v(\tau)r(\tau)}d\tau\right)ds\\ &\geq \frac{1}{H(t,t_1)}\int_{T_1}^{t}\left(-\frac{\partial H(t,s)}{\partial s}\right)\left(\int_{t_1}^{s}\frac{w^2(\tau)}{v(\tau)r(\tau)}d\tau\right)ds\\ &\geq \frac{M}{M_1}\frac{1}{H(t,t_1)}\int_{T_1}^{t}\left(-\frac{\partial H(t,s)}{\partial s}\right)ds\\ &= \frac{M}{M_1}\frac{H(t,T_1)}{H(t,t_1)}.\end{aligned} \tag{2.31}$$

By (2.29), there exists a $T_2 > T_1$ such that, for all $t \geq T_2$,

$$\frac{H(t,T_1)}{H(t,t_1)} \geq M_1. \tag{2.32}$$

It follows from (2.31) that for all $t \geq T_2$,

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}H(t,s)\frac{w^2(s)}{v(s)r(s)}ds \geq M. \tag{2.33}$$

Since M is an arbitrary positive constant,

$$\liminf_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}H(t,s)\frac{w^2(s)}{v(s)r(s)}ds = \infty, \tag{2.34}$$

which contradicts (2.26). Consequently, (2.27) holds. And from (2.25), we obtain

$$\int_{t_1}^{\infty}\frac{\phi_+^2(s)}{v(s)r(s)}ds \leq \int_{t_1}^{\infty}\frac{w^2(s)}{v(s)r(s)}ds < \infty, \tag{2.35}$$

which contradicts (2.21). This completes the proof of Theorem 2.3. □

Choosing H as in Corollary 2.2, by Theorem 2.3, we can obtain the following corollary.

Corollary 2.4. *Let $v(t)$ and $\varphi(t)$ be as in Theorem 2.1, assume further that there exist functions $f \in C^1([t_0,\infty);R)$ and $\phi \in C([t_0,\infty);R)$ such that, for all $T \ge t_0$, for some integer $n > 2$, and for some $\beta > 1$,*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_T^t (t-s)^{n-3}\left[(t-s)^2\varphi(s) - \frac{\beta(n-1)^2}{4\mu} v(s)r(s)\right]ds \ge \phi(T) \tag{2.36}$$

and (2.21) *hold. Then every solution $u(x,t)$ of the problem* (1.1), (1.5) *is oscillatory in G.*

Theorem 2.5. *Suppose that there exists a function $f \in C^1([t_0,\infty);R)$ such that for some $\beta \ge 1$ and for some $H \in \omega$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\left(\overline{\varphi}(s) - \frac{p^2(s)\overline{v}(s)}{2\mu r(s)}\right) - \frac{\mu\beta}{2}\overline{v}(s)r(s)h^2(t,s)\right]ds = \infty, \tag{2.37}$$

where

$$\begin{gathered}\overline{v}(t) = \exp\left(-2\mu\int^t y(s)ds\right),\\ \overline{\varphi}(t) = \overline{v}(t)\Big(q(t) + \mu r(t)y^2(t) - p(t)y(t) - \big(r(t)y(t)\big)'\Big).\end{gathered} \tag{2.38}$$

Then every solution $u(x,t)$ of the problem (1.1), (1.5) *is oscillatory in G.*

Proof. As in Theorem 2.1, without loss of generality, we assume that a nonoscillatory solution $u(x,t)$ of the problem (1.1), (1.5) satisfies $u(x,t) > 0, u(x,t-\rho_k(t)) > 0$ and $u(x,t-\sigma_j) > 0$ in $\Omega\times[t_1,\infty)$, $t_1 \ge t_0$, $k \in I_s, j \in I_m$. Define a generalized Riccati transformation

$$\overline{w}(t) = \overline{v}(t)r(t)\left[\frac{U'(t)}{f(U(t))} + y(t)\right], \quad t \ge t_1, \tag{2.39}$$

where $\overline{v}(t)$ is given by (2.38). Then

$$\begin{aligned}\overline{w}'(t) &\le -2\mu y(t)\overline{w}(t) + \overline{v}(t)\left\{-q(t) + \big(r(t)y(t)\big)' - p(t)\left[\frac{\overline{w}(t)}{\overline{v}(t)r(t)} - y(t)\right] - \mu r(t)\left[\frac{\overline{w}(t)}{\overline{v}(t)r(t)} - y(t)\right]^2\right\}\\ &= -\overline{\varphi}(t) - \frac{p(t)}{r(t)}\overline{w}(t) - \mu\frac{1}{r(t)\overline{v}(t)}\overline{w}^2(t), \quad t \ge t_1.\end{aligned} \tag{2.40}$$

Using an elementary inequality

$$-ax^2 + bx \le -\frac{a}{2}x^2 + \frac{b^2}{2a}, \tag{2.41}$$

for all $a > 0$ and for all $b, x \in R$, we conclude from (2.40) that

$$\overline{\varphi}(t) - \frac{p^2(t)\overline{v}(t)}{2\mu r(t)} \leq -\overline{w}'(t) - \frac{\mu}{2\overline{v}(t)r(t)}\overline{w}^2(t), \quad t \geq t_1. \tag{2.42}$$

Multiplying (2.42) by $H(t,s)$ and integrating from $T < t$, we obtain, for some $\beta \geq 1$ and for all $t \geq T \geq t_1$,

$$\begin{aligned}\int_T^t H(t,s)\left(\overline{\varphi}(s) - \frac{p^2(s)\overline{v}(s)}{2\mu r(s)}\right)ds &\leq H(t,T)\overline{w}(T) - \int_T^t h(t,s)\sqrt{H(t,s)}\overline{w}(s)ds - \int_T^t \frac{\mu\overline{w}^2(s)}{2\overline{v}(s)r(s)}ds \\ &\leq H(t,T)\overline{w}(T) + \frac{\mu\beta}{2}\int_T^t \overline{v}(s)r(s)h^2(t,s)ds \\ &\quad - \mu\int_T^t \frac{(\beta-1)H(t,s)}{2\beta\overline{v}(s)r(s)}\overline{w}^2(s)ds \\ &\quad - \frac{\mu}{2}\int_T^t\left(\sqrt{\frac{H(t,s)}{\beta\overline{v}(s)r(s)}}\overline{w}(s) + \sqrt{\beta\overline{v}(s)r(s)}h(t,s)\right)^2 ds.\end{aligned} \tag{2.43}$$

Therefore, for all $t \geq T \geq t_1$, we have

$$\begin{aligned}&\int_T^t \left[H(t,s)\overline{\varphi}(s) - H(t,s)\frac{p^2(s)\overline{v}(s)}{2\mu r(s)} - \frac{\mu\beta}{2}\overline{v}(s)r(s)h^2(t,s)\right]ds \\ &\quad\leq H(t,T)\overline{w}(T) - \mu\int_T^t \frac{(\beta-1)H(t,s)}{2\beta\overline{v}(s)r(s)}\overline{w}^2(s)ds \\ &\quad\quad - \frac{\mu}{2}\int_T^t\left(\sqrt{\frac{H(t,s)}{\beta\overline{v}(s)r(s)}}\overline{w}(s) + \sqrt{\beta\overline{v}(s)r(s)}h(t,s)\right)^2 ds.\end{aligned} \tag{2.44}$$

Following the same lines as in the proof of Theorem 2.1, we have

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t\left(H(t,s)\overline{\varphi}(s) - H(t,s)\frac{p^2(s)\overline{v}(s)}{2\mu r(s)} - \frac{\mu\beta}{2}\overline{v}(s)r(s)h^2(t,s)\right)ds \\ &\quad\leq \int_{t_0}^{t_1}\left|\overline{\varphi}(s)\right|ds + |\overline{w}(t_1)| < \infty\end{aligned} \tag{2.45}$$

which contradicts the assumption (2.19).

This completes the proof. □

Theorem 2.6. *Let* (2.19) *holds. Assume that there exist functions* $f \in C^1([t_0,\infty); R)$ *and* $\phi \in C([t_0,\infty), R)$ *such that, for all* $t \geq t_0$*, any* $T \geq t_0$*, and for some* $\beta > 1$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\left(\overline{\varphi}(s) - \frac{p^2(s)\overline{v}(s)}{2\mu r(s)} \right) - \frac{\mu\beta}{2}\overline{v}(s) r(s) h^2(t,s) \right] ds \geq \phi(T) \tag{2.46}$$

and (2.21) *holds, where* $\varphi(t), \overline{v}(t)$ *are defined as in Theorem 2.6 and* $\phi_+(t) = \max(\phi(t), 0)$*, then every solution* $u(x,t)$ *of the problem* (1.1), (1.5) *is oscillatory in* G.

Theorem 2.7. *Let all assumptions of Theorem 2.6 be satisfied except that condition* (2.46) *be replaced by*

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\left(\overline{\varphi}(s) - \frac{p^2(s)\overline{v}(s)}{2\mu r(s)} \right) - \frac{\mu\beta}{2}\overline{v}(s) r(s) h^2(t,s) \right] ds \geq \phi(T). \tag{2.47}$$

Then every solution $u(x,t)$ *of the problem* (1.1), (1.5) *is oscillatory in* G.

Remark 2.8. By introducing the parameter β in Theorem 2.3, we derive new oscillation criteria of the problem (1.1), (1.5) which are simpler than that in Theorem A; furthermore, modifications of the proofs through the refinement of the standard integral averaging method allowed us to shorten significantly the proofs of Theorem 2.3. We can also derive a number of oscillation criteria with the appropriate choice of the function H and ρ, here, we omit the details.

3. Examples

Now, we consider these following examples.

Example 3.1. Consider the partial differential equation

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\frac{1}{t}\frac{\partial}{\partial t}\left(u(x,t) + \frac{1}{2}u(x,t-\pi)\right)\right] + 2\cos t \frac{\partial u(x,t)}{\partial t} \\ &\quad = \Delta u(x,t) + \frac{1}{t^2}\Delta u\left(x, t - \frac{3}{2}\pi\right) \\ &\qquad - \left(\frac{2}{t^3} + t\cos^2 t + \sin t\right) f(u(x,t)) - \frac{1}{t^2} f_1(u(x,t-\pi)), \quad (x,t) \in (0,\pi)\times(0,\infty), \end{aligned} \tag{3.1}$$

with the boundary condition

$$u_x(0,t) = u_x(\pi,t) = 0, \quad t > 0, \tag{3.2}$$

where $f(u) = u^3 + u$, $f_1(u) = ue^u + u$.

Here, $N = 1$, $l = 1$, $s = 1$, $m = 1$, $\mu = 1$, $r(t) = 1/t$, $p(t) = 2\cos t$, $q(x,t) = q(t) = (2/t^3 + t\cos^2 t + \sin t)$, $q_1(x,t) = 1/t^2$, $f(u) = f_j(u) = u$, $a(t) = 1$, $a_1(t) = 1/t^2$, $\rho_1(t) = (3/2)\pi$, $\sigma_1 = \pi$, $\tau_1 = \pi$.

Let

$$y(t) = \frac{1}{t} + t\cos t, \tag{3.3}$$

then

$$v(t) = t^2, \qquad \varphi(t) = t^{-1}. \tag{3.4}$$

Let $n = 3$, for any $\beta \geq 1$,

$$\limsup_{t\to\infty} \frac{1}{t^2}\int_1^t \left[(t-s)^2 s^{-1} - \beta s^2 \frac{1}{s}\right] ds = \limsup_{t\to\infty} \frac{1}{t^2}\int_1^t \left[(t-s)^2 s^{-1} - \beta s\right] ds = \infty. \tag{3.5}$$

Therefore, Corollary 2.2 holds, then every solution $u(x,t)$ of the problem (3.1), (3.2) oscillates in $(0,\pi)\times(0,\infty)$.

Example 3.2. Consider the partial differential equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\left(1+\frac{1}{2t^3}\right)(2+\sin t)\frac{\partial u(x,t)}{\partial t}\right] + \frac{3}{t}\left(1+\frac{1}{2t^3}\right)(2+\sin t)\frac{\partial u(x,t)}{\partial t}\\
&\quad = 3\Delta u(x,t) + (2-\cos t)\Delta u\left(x, t-\frac{3}{2}\pi\right) - t^{-3}\left(\left(1-t^3+2t^2-6t\right)\sin t + 12t\right) f(u(x,t))\\
&\qquad - (2t+\sin t) f_1(u(x,t-\pi)) - 2 f_2\left(u\left(x,t-\frac{\pi}{2}\right)\right), \quad (x,t)\in(0,\pi)\times(0,\infty)
\end{aligned} \tag{3.6}$$

with the boundary condition (3.2), where $f(u) = u^5 + u$, $f_1(u) = u\sin^2 u$, $f_2(u) = u^3\cos^2 u$.

Here $N = 1$, $s = 1$, $m = 2$, $\mu = 1$, $r(t) = (1+1/2t^3)(2+\sin t)$, $p(t) = (3/t)(1+1/2t^3)(2+\sin t)$, $q(t) = t^{-3}[(1-t^3+2t^2-6t)\sin t + 12t]$, $a(t) = 3$, $a_1(t) = 2-\cos t$, $q_1(x,t) = 2+\sin t$, $q_2(x,t) = 2$, $\rho_1(t) = (3/2)\pi$, $\sigma_1 = \pi$, $\sigma_2 = \pi/2$.

Let $y(t) = 0$, then $v(t) = t^3$ and $\varphi(t) = v(t)q(t) = (1-t^3+2t^2-6t)\sin t + 12t$.

Choose $\beta = 2$, $n = 3$, a straightforward computation yields

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^2}\int_T^t \left[(t-s)^2\left(\left(1-s^3+2s^2-6s\right)\sin s + 12s\right) - \left(2s^3+1\right)(2+\sin s)\right] ds\\
&\quad = 16 - T^3\cos T + T^2(2\cos T - 6 + 3\sin T) - 4T\sin T - 3\cos T = \phi(T).
\end{aligned} \tag{3.7}$$

Let $\phi_+(t) = \max(\phi(t), 0)$. It is not difficult to see that

$$\limsup_{t\to\infty}\int_1^t \frac{\phi_+^2(s)}{(s^3+(1/2))(2+\sin s)} ds \geq \limsup_{t\to\infty}\int_1^t \frac{\phi_+^2(s)}{3(s^3+(1/2))} ds = \infty. \tag{3.8}$$

By Corollary 2.4, we obtain that every solution of problem (3.6), (3.2) oscillates in $(0,\pi)\times(0,\infty)$.

Note that in this example,

$$\limsup_{t\to\infty}\frac{1}{t^2}\int_1^t 4\left(s^3+\frac{1}{2}+(2+\sin s)\right)ds=\infty, \tag{3.9}$$

so the condition (C_2) would not have been satisfied with the same choices of $v(t)$.

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