Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 190821, 8 pages
http://dx.doi.org/10.1155/2014/190821

Research Article

The Hierarchical Minimax Inequalities for Set-Valued Mappings

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Received 16 December 2013; Accepted 14 March 2014; Published 10 April 2014

Academic Editor: Ngai-Ching Wong

We study the minimax inequalities for set-valued mappings with hierarchical process and propose two versions of minimax inequalities in topological vector spaces settings. As applications, we discuss the existent results of solutions for set equilibrium problems. Some examples are given to illustrate the established results.

1. Introduction and Preliminaries

Let X be a nonempty set in a Hausdorff topological vector space, Z a Hausdorff topological vector space, and $C \subset Z$ a closed convex and pointed cone with apex at the origin with $\operatorname{int} C \neq \emptyset$; that is, C is properly closed with $\operatorname{int} C \neq \emptyset$ and satisfies $\lambda C \subseteq C$, for all $\lambda > 0$; $C + C \subseteq C$; and $C \cap (-C) = \{0\}$. The scalar hierarchical minimax inequalities are stated as follows: for given mappings $F, G : X \times X \rightrightarrows \mathbb{R}$, under some suitable conditions, the following inequality holds:

$$\min \bigcup_{x \in X} \max \bigcup_{y \in X} F(x, y) \leq \max \bigcup_{x \in X} G(x, x). \tag{s-Hi}$$

For given mappings $F, G : X \times X \rightrightarrows Z$, the first version of hierarchical minimax theorems states that under some suitable conditions, the following inequality holds:

$$\operatorname{Max} \bigcup_{x \in X} F(x, x) \subset \operatorname{Min} \left(\operatorname{co} \left(\bigcup_{x \in X} \operatorname{Max}_w \bigcup_{y \in X} F(x, y) \right) \right) + C. \tag{Hi-1}$$

The second version of hierarchical minimax theorems states that under some suitable conditions, the following inequality holds:

$$\operatorname{Max} \bigcup_{x \in X} G(x, x) \subset \operatorname{Min} \bigcup_{x \in X} \operatorname{Max}_w \bigcup_{y \in X} F(x, y) + C. \tag{Hi-2}$$

These versions, (Hi-1) and (Hi-2), arise naturally from some minimax theorems in the vector or real-valued settings. We refer to [1–4] and the references therein.

The notations we use in the above relations are as follows.

Definition 1 (see [1, 3]). Let A be a nonempty subset of Z. A point $z \in A$ is called a

(a) *minimal point* of A if $A \cap (z - C) = \{z\}$; $\operatorname{Min} A$ denotes the set of all minimal points of A;

(b) *maximal point* of A if $A \cap (z + C) = \{z\}$; $\operatorname{Max} A$ denotes the set of all maximal points of A;

(c) *weakly minimal point* of A if $A \cap (z - \operatorname{int} C) = \emptyset$; $\operatorname{Min}_w A$ denotes the set of all weakly minimal points of A;

(d) *weakly maximal point* of A if $A \cap (z + \operatorname{int} C) = \emptyset$; $\operatorname{Max}_w A$ denotes the set of all weakly maximal points of A.

We note that, for a nonempty compact set A, both sets $\operatorname{Max} A$ and $\operatorname{Min} A$ are nonempty. Furthermore, $\operatorname{Min} A \subset \operatorname{Min}_w A$, $\operatorname{Max} A \subset \operatorname{Max}_w A$, $A \subset \operatorname{Min} A + C$, and $A \subset \operatorname{Max} A - C$. Following [3], we denote both Max and Max_w by max (both Min and Min_w by min) in $\mathbb{R}$ since both Max and Max_w (both Min and Min_w) are the same in $\mathbb{R}$.

We present some fundamental concepts which will be used in the following.

Definition 2 (see [5, 6]). Let U, V be Hausdorff topological spaces. A set-valued map $F : U \rightrightarrows V$ with nonempty values is said to be

(a) *upper semicontinuous at* $x_0 \in U$ if for every $x_0 \in U$ and for every open set N containing $F(x_0)$ there exists a neighborhood M of x_0 such that $F(M) \subset N$;

(b) *lower semicontinuous at* $x_0 \in U$ if for any net $\{x_\nu\} \subset U$, $x_\nu \to x_0$, $y_0 \in T(x_0)$ implies that there exists net $y_\nu \in T(x_\nu)$ such that $y_\nu \to y_0$;

(c) *continuous at* $x_0 \in U$ if F is upper semicontinuous as well as lower semicontinuous at x_0.

We note that if T is upper semicontinuous at x_0 and $T(x_0)$ is compact, then for any net $\{x_\nu\} \subset U$, $x_\nu \to x_0$, and for any net $y_\nu \in T(x_\nu)$ for each ν there exists $y_0 \in T(x_0)$ and a subnet $\{y_{\nu_\alpha}\}$ such that $y_{\nu_\alpha} \to y_0$. We refer to [5, 6] for more details.

Definition 3 (see [3, 7]). Let $k \in \operatorname{int} C$ and $v \in Z$. The *Gerstewitz function* $\xi_{kv} : Z \to \mathbb{R}$ is defined by

$$\xi_{kv}(u) = \min\{t \in \mathbb{R} : u \in v + tk - C\}. \tag{1}$$

Some fundamental properties for the Gerstewitz function are as follows.

Proposition 4 (see [3, 7]). *Let $k \in \operatorname{int} C$ and $v \in Z$. The Gerstewitz function $\xi_{kv} : Z \to \mathbb{R}$ has the following properties:*

(a) $\xi_{kv}(u) > r \Leftrightarrow u \notin v + rk - C$;

(b) $\xi_{kv}(u) \geq r \Leftrightarrow u \notin v + rk - \operatorname{int} C$;

(c) $\xi_{kv}(\cdot)$ *is a convex, continuous, and increasing function.*

We also need the following different kinds of cone-convexities for set-valued mappings.

Definition 5 (see [1]). Let X be a nonempty convex subset of a topological vector space. A set-valued mapping $F : X \rightrightarrows Z$ is said to be

(a) *above-C-convex* (resp., *above-C-concave*) on X if, for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$,

$$F(\lambda x_1 + (1-\lambda) x_2) \subset \lambda F(x_1) + (1-\lambda) F(x_2) - C$$
$$(\text{resp.}, \lambda F(x_1) + (1-\lambda) F(x_2) \subset F(\lambda x_1 + (1-\lambda) x_2) - C); \tag{2}$$

(b) *above-naturally C-quasiconvex* on X if, for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$,

$$F(\lambda x_1 + (1-\lambda) x_2) \subset \operatorname{co}\{F(x_1) \cup F(x_2)\} - C, \tag{3}$$

where co A denotes the convex hull of a set A;

(c) *above-C-convex-like* (resp., *above-C-concave-like*) on X (X is not necessary convex) if, for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$, there is an $x' \in X$ such that

$$F(x') \subset \lambda F(x_1) + (1-\lambda) F(x_2) - C$$
$$(\text{resp.}, \lambda F(x_1) + (1-\lambda) F(x_2) \subset F(x') - C). \tag{4}$$

We note that whenever F is a scalar function and $C = \mathbb{R}_+$, the mappings in Definition 5 reduce to the classical ones. The following theorem is a special case of the scalar hierarchical minimax theorem by Lin [8].

Theorem 6. *Let X be a nonempty compact convex subset of real Hausdorff topological vector space. Let the set-valued mappings $F, G, H : X \times X \rightrightarrows \mathbb{R}$ such that $F(x, y) \subset G(x, y) \subset H(x, y)$ for all $(x, y) \in X \times X$; $\bigcup_{y \in X} F(x, y)$ and $\bigcup_{x \in X} H(x, y)$ are compact for each $x \in X$ and for each $y \in X$ and satisfy the following conditions:*

(i) *$x \mapsto F(x, y)$ is lower semicontinuous on X for each $y \in X$ and $y \mapsto F(x, y)$ is above-$\mathbb{R}_+$-concave on X for each $x \in X$;*

(ii) *$x \mapsto G(x, y)$ is above-naturally $\mathbb{R}_+$-quasiconvex for each $y \in X$, and $y \mapsto G(x, y)$ is lower semicontinuous on X for each $x \in X$;*

(iii) *$x \mapsto H(x, y)$ is lower semicontinuous on X for each $y \in X$, $y \mapsto H(x, y)$ is above-$\mathbb{R}_+$-concave on X for each $x \in X$, and $y \mapsto H(x, y)$ is lower semicontinuous for each $x \in X$.*

Then one has

$$\min \bigcup_{x \in X} \max \bigcup_{y \in X} F(x, y) \leq \max \bigcup_{y \in X} \min \bigcup_{x \in X} H(x, y). \tag{5}$$

Lemma 7. *Let $F : X \rightrightarrows \mathbb{R}$ be such that $\max \bigcup_{x \in X} F(x)$, $\max \bigcup_{x \in X} \max F(x)$, and $\max F(x)$ exist for all $x \in X$. Then*

$$\max \bigcup_{x \in X} F(x) = \max \bigcup_{x \in X} \max F(x). \tag{6}$$

Proof. By using the similar technique of Lemma 3.3 [9], we can show that the conclusion is valid. □

2. Scalar Hierarchical Minimax Inequalities

We first state the following scalar hierarchical minimax inequalities.

Theorem 8. *Let X be a nonempty compact (not necessarily convex) subset of a real Hausdorff topological space. Let the set-valued mappings $F, S, T, G : X \times X \rightrightarrows \mathbb{R}$ with nonempty compact values such that*

(i) *$(x, y) \mapsto F(x, y)$ and $(x, y) \mapsto G(x, y)$ are upper semicontinuous on $X \times X$;*

(ii) *$x \mapsto \max S(x, y)$ is convex-like for each $y \in X$, and $y \mapsto \max T(x, y)$ is concave-like on Y for each $x \in X$;*

(iii) *for all $(x, y) \in X \times X$, $\max F(x, y) \leq \max S(x, y) \leq \max T(x, y) \leq \max G(x, y)$.*

Then the relation (s-Hi) holds.

Proof. From (i), we know that both sides of (s-Hi) exist. For any $r \in \mathbb{R}$,

$$r > \max \bigcup_{x \in X} G(x, x). \tag{7}$$

Define $M : X \rightrightarrows X$ by

$$M(x) = \{y \in X : \max F(x, y) \geq r\} \tag{8}$$

for all $x \in X$. By (i), the set $M(x)$ is closed for all $x \in X$. We claim that the whole intersection

$$\bigcap_{x\in X} M(x) \tag{9}$$

is empty. Indeed, if not, there exists $y_0 \in \bigcap_{x\in X} M(x)$ such that, for all $x \in X$, $\max F(x, y_0) \geq r$. In particular, we choose $x = y_0$; then $\max F(y_0, y_0) \geq r$ which, with the aid of condition (iii), contradicts the choice of r. Hence, by the compactness of X, there exist $x_1, x_2, \ldots, x_m \in X$ such that

$$X \subset \bigcup_{i=1}^{m} (X \setminus M(x_i)). \tag{10}$$

Let

$$N(x) = \{y \in X : \max T(x, y) \geq r\} \tag{11}$$

for all $x \in X$. Then, by (iii), we have

$$X \subset \bigcup_{i=1}^{m} (X \setminus N(x_i)). \tag{12}$$

This implies that, for each $y \in X$, there is $x_{i_0} \in \{x_1, x_2, \ldots, x_m\}$ such that

$$\max T(x_{i_0}, y) < r. \tag{13}$$

Define two sets as follows:

$$\begin{aligned} L_1 :&= \operatorname{co}\{(\max T(x_1, y), \max T(x_2, y), \\ &\ldots, \max T(x_m, y)) \in \mathbb{R}^m : y \in X\}, \\ L_2 :&= \{(z_1, z_2, \ldots, z_m) \in \mathbb{R}^m : z_i \geq r \\ &\forall i = 1, 2, \ldots, m\}. \end{aligned} \tag{14}$$

By the concave-like property of T, we can see that these two sets are disjoint. For each $y \in X$, by the separation theorem, there exists nonzero vector $(\tau_1, \tau_2, \ldots, \tau_m) \in \mathbb{R}^m$ such that

$$\sum_{i=1}^{m} \tau_i \max T(x_i, y) \leq \sum_{i=1}^{m} \tau_i z_i, \tag{15}$$

for all $(z_1, z_2, \ldots, z_m) \in L_2$. Then, $\sum_{i=1}^{m} \tau_i > 0$ and $\tau_i > 0$ for all $i = 1, 2, \ldots, m$. Let $\delta_i = \tau_i / \sum_{i=1}^{m} \tau_i$ for all $i = 1, 2, \ldots, m$. Then we have

$$\sum_{i=1}^{m} \delta_i \max T(x_i, y) \leq \sum_{i=1}^{m} \delta_i z_i. \tag{16}$$

For each $i = 1, 2, \ldots, m$, by taking $z_i = r$ and noting $\max S(x_i, y) \leq \max T(x_i, y)$, we have

$$\sum_{i=1}^{m} \delta_i \max S(x_i, y) \leq r, \tag{17}$$

for all $y \in X$. Since the mapping $x \mapsto \max S(x, y)$ is convex-like for each $y \in Y$, there is $x_0 \in X$ such that

$$\max S(x_0, y) \leq r. \tag{18}$$

Since $\max F(x, y) \leq \max S(x, y)$ for all $(x, y) \in X \times X$, we have

$$\max F(x_0, y) \leq r, \tag{19}$$

for all $y \in X$. By Lemma 7, we know that

$$\begin{aligned} \min \bigcup_{x\in X} \max \bigcup_{y\in Y} F(x, y) &\leq \max \bigcup_{y\in X} F(x_0, y) \\ &= \max \bigcup_{y\in X} \max F(x_0, y) \leq r. \end{aligned} \tag{20}$$

Therefore, the relation (s-Hi) holds. □

Theorem 9. *Let X be a nonempty compact convex subset of a real Hausdorff topological vector space. Let the set-valued mappings $F, G : X \times X \rightrightarrows \mathbb{R}$ with nonempty compact values such that*

(i) *$(x, y) \mapsto F(x, y)$ and $(x, y) \mapsto G(x, y)$ are upper semicontinuous on $X \times X$;*

(ii) *$y \mapsto \max F(x, y)$ is quasiconcave for each $x \in X$; that is, for each $x \in X$, the set $\{y \in X : \max F(x, y) \geq r\}$ is convex in X;*

(iii) *for all $(x, y) \in X \times X$, $\max F(x, y) \leq \max G(x, y)$.*

Then the relation (s-Hi) holds.

Proof. By (i), we know that both sides of (s-Hi) exist. Choose any $r \in \mathbb{R}$ satisfies

$$r > \max \bigcup_{x\in X} G(x, x). \tag{21}$$

Define $W : X \rightrightarrows X$ by

$$W(x) = \{y \in X : \max F(x, y) \geq r\}, \tag{22}$$

for all $x \in X$. By (ii), the set $W(x)$ is convex for all $x \in X$. By (iii), we have

$$\max F(x, x) \leq \max G(x, x) \leq \max \bigcup_{x\in X} G(x, x) < r. \tag{23}$$

Hence,

$$x \notin W(x), \tag{24}$$

for all $x \in X$. By the upper semicontinuity of F, we know that the mapping $x \mapsto \max F(x, y)$ is upper semicontinuous for each $x \in X$. Thus, for each $x \in X$, $W(x)$ is closed; hence it is compact. In order to claim that the mapping $x \mapsto W(x)$ is upper semicontinuous on X, we only need to show that the mapping $x \mapsto W(x)$ has a closed graph. Since, for any net $\{(x_\alpha, y_\alpha)\} \subset \operatorname{Graph}(W)$ we have the net $\{(x_\alpha, y_\alpha)\}$. converges to some point (x_0, y_0). Then, for each α,

$\max F(x_\alpha, y_\alpha) \geq r$. Since the mapping $(x, y) \mapsto \max F(x, y)$ is upper semicontinuous, we have

$$\max F(x_0, y_0) \geq \limsup_{\alpha} \max F(x_\alpha, y_\alpha) \geq r. \tag{25}$$

Thus, $(x_0, y_0) \in \mathrm{Graph}(W)$. Suppose that $W(x) \neq \emptyset$ for all $x \in X$. Then, by Kakutani fixed point theorem, the mapping W has a fixed point which is a contradiction to (24). Hence, there is an $x_0 \in X$ such that $W(x_0) = \emptyset$. From this, we know that

$$r > \min \bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y). \tag{26}$$

This implies that the relation (s-Hi) holds. □

The following examples illustrate Theorems 8 and 9.

Example 10. Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ and $f(x) = x^2, g(y) = 1 - y^2$ for all $x, y \in X$. Define $F, S, T, G : X \times X \rightrightarrows \mathbb{R}$ by $F(x, y) = [0, f(x)g(y)]$, $S(x, y) = [-1, f(x)g(y) + 1]$, $T(x, y) = [2, f(x)g(y) + 2]$, and $G(x, y) = [3, f(x)g(y) + 3]$. Obviously, all conditions of Theorem 8 hold. Hence the relation (s-Hi) holds. Indeed, by simple calculation, we can see that

$$\min \bigcup_{x \in X} \max \bigcup_{y \in X} F(x, y) = 0,$$
$$\max \bigcup_{x \in X} G(x, x) = \frac{13}{4}. \tag{27}$$

Example 11. Let $X = [0, 1]$. The mappings f, g, F, and G are the same as in Example 10. Then, all conditions of Theorem 9 hold. We can see that both values of $\min \bigcup_{x \in X} \max \bigcup_{y \in X} F(x, y)$ and $\max \bigcup_{x \in X} G(x, x)$ are the same as those in Example 10. Hence the relation (s-Hi) holds.

3. Hierarchical Minimax Inequalities

In this section, we will present two versions of hierarchical minimax inequalities. The following theorem is the first result satisfies the relation (Hi-1).

Theorem 12. *Let X be a nonempty compact convex subset of a real Hausdorff topological vector space. Let the set-valued mappings $F, G, H : X \times X \rightrightarrows Z$ with nonempty compact values such that $F(x, y) \subset G(x, y) \subset H(x, y)$ for all $(x, y) \in X \times Y$ satisfy the following conditions:*

(i) *$(x, y) \mapsto F(x, y)$ is upper semicontinuous, $y \mapsto F(x, y)$ is above-C-concave on Y for each $x \in X$, and $x \mapsto F(x, y)$ is lower semicontinuous on X for each $y \in Y$;*

(ii) *$x \mapsto G(x, y)$ is above-naturally C-quasiconvex for each $y \in Y$, and $y \mapsto G(x, y)$ is lower semicontinuous on Y for each $x \in X$;*

(iii) *$y \mapsto H(x, y)$ is lower semicontinuous and above-C-concave on Y for each $x \in X$, and $x \mapsto H(x, y)$ is lower semicontinuous on X for each $y \in Y$;*

(iv) *for each $y \in Y$,*

$$\operatorname*{Max}_{w} \bigcup_{x \in X} F(x, x) \subset \operatorname*{Min}_{w} \bigcup_{x \in X} H(x, y) + C. \tag{28}$$

Then the relation (Hi-1) is valid.

Proof. Let $\Gamma(x) := \mathrm{Max}_w \bigcup_{y \in Y} F(x, y)$ for all $x \in X$. From Lemma 2.4 and Proposition 3.5 in [1], the mapping $x \mapsto \Gamma(x)$ is upper semicontinuous with nonempty compact values on X. Hence $\bigcup_{x \in X} \Gamma(x)$ is compact and so is $\mathrm{co}(\bigcup_{x \in X} \Gamma(x))$. Then $\mathrm{co}(\bigcup_{x \in X} \Gamma(x)) + C$ is a closed convex set with nonempty interior. Suppose that $v \notin \mathrm{co}(\bigcup_{x \in X} \Gamma(x)) + C$. By separation theorem, there is a $k \in \mathbb{R}$, $\epsilon > 0$, and a nonzero continuous linear functional $\xi : Z \mapsto \mathbb{R}$ such that

$$\xi(v) \leq k - \epsilon < k \leq \xi(u + c), \tag{29}$$

for all $u \in co(\bigcup_{x \in X} \Gamma(x))$ and $c \in C$. From this we can see that $\xi \in C^*$, where $C^* = \{g : Z \mapsto \mathbb{R} : g(c) \geq 0 \ \forall c \in C\}$, and $\xi(v) < \xi(u)$ for all $u \in \mathrm{co}(\bigcup_{x \in X} \Gamma(x))$. By Proposition 3.14 of [1], for any $x \in X$, there is a $y_x^\star \in Y$ and $f(x, y_x^\star) \in F(x, y_x^\star)$ with $f(x, y_x^\star) \in \Gamma(x)$ such that

$$\xi f(x, y_x^\star) = \max \bigcup_{y \in Y} \xi F(x, y). \tag{30}$$

Let us choose $c = 0$ and $u = f(x, y_x^\star)$ in (29); we have

$$\xi(v) < \xi(f(x, y_x^\star)) = \max \bigcup_{y \in Y} \xi F(x, y) \tag{31}$$

for all $x \in X$. Therefore,

$$\xi(v) < \min \bigcup_{x \in X} \max \bigcup_{y \in Y} \xi F(x, y). \tag{32}$$

From conditions (i)–(iii), by applying Proposition 3.9 and Proposition 3.13 in [1], all conditions of Theorem 6 hold. Hence, we have

$$\xi(v) < \max \bigcup_{y \in Y} \min \bigcup_{x \in X} \xi H(x, y). \tag{33}$$

Since Y is compact, there is $y' \in Y$ such that

$$\xi(v) < \min \bigcup_{x \in X} \xi H(x, y'). \tag{34}$$

Thus,

$$v \notin \bigcup_{x \in X} H(x, y') + C, \tag{35}$$

and, hence,

$$v \notin \operatorname*{Min}_{w} \bigcup_{x \in X} H(x, y') + C. \tag{36}$$

Therefore,

$$\operatorname*{Min}_{w} \bigcup_{x \in X} H(x, y') + C \subset \mathrm{co}\left(\bigcup_{x \in X} \Gamma(x)\right) + C. \tag{37}$$

By taking into account condition (iv), we know that

$$\operatorname*{Max}_{w}\bigcup_{x\in X}F(x,x)\subset\operatorname*{Min}_{w}\bigcup_{x\in X}H(x,y')+C. \tag{38}$$

Hence, the relation (Hi-1) is valid. □

The following example illustrates that Theorem 12 is valid.

Example 13. Let $X=[0,1]$, $C=\mathbb{R}^2_+$, and $f:X\rightrightarrows\mathbb{R}$ define

$$f(y)=\begin{cases}[-1,0], & y=0,\\ \{0\}, & y\neq 0,\end{cases} \tag{39}$$

and $F,G,H:X\times X\rightrightarrows\mathbb{R}^2$ define

$$\begin{aligned}F(x,y)&=\left\{\sin\left(\frac{x\pi}{2}\right)\right\}\times f(y),\\ G(x,y)&=\left[0,\sin\left(\frac{x\pi}{2}\right)\right]\times[y-1,0],\\ H(x,y)&=\left[0,\sin\left(\frac{x\pi}{2}\right)\right]\times\left[y^2-1,0\right],\end{aligned} \tag{40}$$

for all $(x,y)\in X\times X$.

We can easily see that $F(x,y)\subset G(x,y)\subset H(x,y)$ for all $(x,y)\in X\times Y$ and conditions (i)–(iii) of Theorem 12 are valid. Now we claim that condition (iv) holds. Indeed,

$$\begin{aligned}\bigcup_{x\in X}F(x,x)&=\left(\bigcup_{x\in(0,1]}\left\{\sin\left(\frac{x\pi}{2}\right)\right\}\times\{0\}\right)\bigcup(\{0\}\times[-1,0])\\ &=\left(([0,1]\times\{0\})\bigcup(\{0\}\times[-1,0])\right).\end{aligned} \tag{41}$$

Hence, $\operatorname{Max}_w\bigcup_{x\in X}F(x,x)=\{0\}\times[0,1]$.

On the other hand, $\bigcup_{x\in X}H(x,y)=[0,1]\times[y^2-1,0]$. Hence,

$$\operatorname*{Min}_{w}\bigcup_{x\in X}H(x,y)=(\{0\}\times[0,1])\bigcup\left(\{1\}\times\left[y^2-1,0\right]\right). \tag{42}$$

Thus, condition (iv) of Theorem 12 holds. By Theorem 12, the relation (Hi-1) is valid. Indeed,

$$\begin{aligned}\operatorname{Max}\bigcup_{x\in X}F(x,x)&=\{(1,0)\},\\ \operatorname*{Max}_{w}\bigcup_{y\in X}F(x,y)=\bigcup_{y\in X}F(x,y)&=\left\{\sin\left(\frac{x\pi}{2}\right)\right\}\times[-1,0].\end{aligned} \tag{43}$$

Hence,

$$\begin{aligned}\operatorname{co}\bigcup_{x\in X}\operatorname*{Max}_{w}\bigcup_{y\in X}F(x,y)&=\bigcup_{x\in X}\operatorname*{Max}_{w}\bigcup_{y\in X}F(x,y)\\ &=[0,1]\times[-1,0].\end{aligned} \tag{44}$$

Thus,

$$\operatorname{Min}\left(\operatorname{co}\bigcup_{x\in X}\operatorname*{Max}_{w}\bigcup_{y\in X}F(x,y)\right)=\{(0,-1)\}, \tag{45}$$

and hence the conclusion of Theorem 12 is valid.

Theorem 14. *Let X be a nonempty compact convex subset of real Hausdorff topological vector space. Let the set-valued mappings $F,G,H:X\times X\rightrightarrows Z$ such that $F(x,y)\subset G(x,y)\subset H(x,y)$ for all $(x,y)\in X\times X$ and satisfy the following conditions:*

(i) *$(x,y)\mapsto F(x,y)$ is continuous with nonempty compact values, and $y\mapsto\xi_{kv}F(x,y)$ is above-$\mathbb{R}_+$-concave on X for each $x\in X$ and any Gerstewitz function ξ_{kv};*

(ii) *$x\mapsto G(x,y)$ is above-naturally C-quasiconvex for each $y\in X$, and $y\mapsto G(x,y)$ is lower semicontinuous on X for each $x\in X$;*

(iii) *$(x,y)\mapsto H(x,y)$ is upper semicontinuous with nonempty compact values, $y\mapsto\xi_{kv}H(x,y)$ is above-$\mathbb{R}_+$-concave on X for each $x\in X$, and $x\mapsto H(x,y)$ is lower semicontinuous on X for each $y\in X$ and any Gerstewitz function ξ_{kv};*

(iv) *for each $y\in Y$,*

$$\operatorname*{Max}_{w}\bigcup_{x\in X}G(x,x)\subset\operatorname*{Min}_{w}\bigcup_{x\in X}H(x,y)+C. \tag{46}$$

Then the relation (Hi-2) is valid.

Proof. Let $\Gamma(x)$ be defined the same as that in Theorem 12 for all $x\in X$. From the process in the proof of Theorem 12, we know that the set $\bigcup_{x\in X}\Gamma(x)$ is nonempty compact. Suppose that $v\notin\bigcup_{x\in X}\Gamma(x)+C$. For any $k\in\operatorname{int}C$, there is a Gerstewitz function $\xi_{kv}:Z\mapsto\mathbb{R}$ such that

$$\xi_{kv}(u)>0, \tag{47}$$

for all $u\in\bigcup_{x\in X}\Gamma(x)$. Then, for each $x\in X$, there is $y_x^\star\in X$ and $f(x,y_x^\star)\in F(x,y_x^\star)$ with $f(x,y_x^\star)\in\operatorname{Max}_w\bigcup_{y\in X}F(x,y)$ such that

$$\xi_{kv}(f(x,y_x^\star))=\max\bigcup_{y\in X}\xi_{kv}F(x,y). \tag{48}$$

Choosing $u=f(x,y_x^\star)$ in (47), we have

$$\max\bigcup_{y\in X}\xi_{kv}F(x,y)>0, \tag{49}$$

for all $x\in X$. Therefore,

$$\min\bigcup_{x\in X}\max\bigcup_{y\in X}\xi_{kv}F(x,y)>0. \tag{50}$$

By conditions (i)–(iii), we know that all conditions of Theorem 6 hold for the mappings $\xi_{kv}F(x,y)$, $\xi_{kv}G(x,y)$, and $\xi_{kv}H(x,y)$, and, hence, we have

$$\max\bigcup_{y\in X}\min\bigcup_{x\in X}\xi_{kv}H(x,y)>0. \tag{51}$$

Since X is compact, there is a $y' \in X$ such that

$$\min \bigcup_{x\in X} \xi_{kv} H\left(x, y'\right) > 0. \quad (52)$$

Thus,

$$v \notin \bigcup_{x\in X} H\left(x, y'\right) + C, \quad (53)$$

and, hence,

$$v \notin \operatorname*{Min}_{w} \bigcup_{x\in X} H\left(x, y'\right) + C. \quad (54)$$

If $v \in \operatorname{Max} \bigcup_{y\in X} \operatorname{Min}_w \bigcup_{x\in X} F(x, y)$, then, by (iv), we have

$$v \in \operatorname*{Min}_{w} \bigcup_{x\in X} H\left(x, y'\right) + C, \quad (55)$$

which contradicts (54). From this, we can deduce that the relation (Hi-2) is valid. □

4. Strong and Weak Solutions for SEP

In our previous work [10], we establish existence of solutions for set equilibrium problems (SEP, for short). Let Y be a Hausdorff topological vector space, and let K be a nonempty compact convex subset of a Hausdorff topological vector space. For a given mapping $T : K \rightrightarrows Y$ and a trimapping $F : TK \times K \times K \rightrightarrows Z$, a weak solution for $(\mathrm{SEP})_F$ is a point $\overline{x} \in K$ such that

$$F\left(\overline{s}, \overline{x}, y\right) \not\subset -\operatorname{int} C, \quad (56)$$

for all $y \in K$ and for some $\overline{s} \in T(\overline{x})$. A strong solution for $(\mathrm{SEP})_F$ is a point $\overline{x} \in K$ with some $\overline{s} \in T(\overline{x})$ such that

$$F\left(\overline{s}, \overline{x}, y\right) \not\subset -\operatorname{int} C, \quad (57)$$

for all $y \in K$. A strong solution is obviously a weak solution for (SEP) for the same mapping.

We recall that a set-valued mapping $\Omega : X \mapsto Z$ is called a KKM mapping if $\operatorname{co}\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^{n} \Omega(xi)$ for each finite subset $\{x_1, \dots, x_n\} \subset X$.

Fan Lemma (see [11]). *Let $\Omega : X \rightrightarrows Z$ be a KKM mapping with nonempty closed values. If there exists an $x_0 \in X$ such that $\Omega(x_0)$ is a compact set of Z, then $\bigcap_{x\in X} \Omega(x) \neq \emptyset$.*

We first state that the existent result of weak solution for (SEP) is as follows.

Theorem 15. *Let Z be a finite dimensional space and the set-valued mappings F and T are two upper semicontinuous mappings with nonempty compact values such that,*

(i) *for each $x \in K$, there is $s \in T(x)$ such that $F(s, x, x) \not\subset -\operatorname{int} C$;*

(ii) *for each $x \in K$, the sets $\{(s, y) \in TK \times K : F(s, x, y) \subset -\operatorname{int} C\}$ and $T(x)$ are convex.*

Then $(\mathrm{SEP})_F$ has a weak solution.

Proof. Define $\Omega : K \rightrightarrows K$ by

$$\Omega\left(y\right) := \left\{x \in K : F\left(s, x, y\right) \not\subset -\operatorname{int} C \text{ for some } s \in T\left(x\right)\right\}, \quad (58)$$

for all $y \in K$. By (i), $y \in \Omega(y)$ for all $y \in K$. Hence the set $\Omega(y)$ is nonempty for all $y \in K$. Next, we claim that the set $\Omega(y)$ is closed for all $y \in K$. Let a net $\{x_\alpha\} \subset \Omega$ converge to some point x_0. Then there are $s_\alpha \in Tx_\alpha$ and $z_\alpha \in F(x_\alpha, x_\alpha, y)$ such that $z_\alpha \in Z \setminus (-\operatorname{int} C)$. By the upper semicontinuities of F and T, the sets TK and $F(TK \times K \times K)$ are compact. Hence, there is a convergent subnet $\{z_{\alpha_\beta}\}$ of $\{z_\alpha\}$ that converges to some point z_0. Furthermore, the net $\{s_{\alpha_\beta}\}$ has a convergent subnet $\{s_{\alpha_{\beta_\gamma}}\}$ which converges to some point s_0. Again, by the upper semicontinuities of F and T, we have $z_0 \in T(x_0)$ and $z_0 \in F(s_0, z_0, y)$. Since the set $Z \setminus (-\operatorname{int} C)$ is closed, $z_0 \in Z \setminus (-\operatorname{int} C)$. Hence, $x_0 \in \Omega(y)$, and, thus, $\Omega(y)$ is closed for all $y \in K$. We next claim that the mapping $\Omega : K \rightrightarrows K$ is a KKM mapping. Indeed, if not, there exist $x_1, x_2, \dots, x_n \in K$ and x_0 such that

$$x_0 \in \operatorname{co}\left\{x_1, x_2, \dots, x_n\right\} \not\subset \bigcup_{i=1}^{n} \Omega\left(x_i\right). \quad (59)$$

Then there is $x_0 = \sum_{i=1}^{n} \lambda_i x_i$ where $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$ for all $i = 1, 2, \dots, n$.

Since $x_0 \notin \bigcup_{i=1}^{n} \Omega(x_i)$, for all $i \in \{1, 2, \dots, n\}$, choose any $s_i \in T(x_0)$; we have

$$F\left(s_i, x_0, x_i\right) \subset -\operatorname{int} C. \quad (60)$$

By (ii),

$$\begin{gathered} s_0 \in T\left(x_0\right), \\ \left(s_0, x_0\right) \in \left\{(s, x) \in TK \times K : F\left(s, x_0, x\right) \subset -\operatorname{int} C\right\}. \end{gathered} \quad (61)$$

This implies that

$$F\left(s_0, x_0, x_0\right) \subset -\operatorname{int} C, \quad (62)$$

which contradicts (i). Thus, the mapping $\Omega : K \rightrightarrows K$ is a KKM mapping. By the Fan lemma, the whole intersection

$$\bigcap_{y\in Y} \Omega\left(y\right) \quad (63)$$

is nonempty. Any point in the whole intersection is a weak solution for $(\mathrm{SEP})_F$. □

For the existence of strong solution for (SEP), we propose the following results.

Theorem 16. *Under the framework of Theorem 15, in addition, the mappings $A, B, G : TK \times K \times K \rightrightarrows Z$ with nonempty compact values such that*

(i) *the mapping $s \mapsto G(s, x, y)$ is upper semicontinuous mappings for each $x, y \in K$;*

(ii) *both sets* $\bigcup_{s\in T(x)} F(s,x,y)$ *and* $\bigcup_{y\in K} G(s,x,y)$ *are compact for* $x, y \in K, s \in T(x)$;

(iii) *the mapping* $s \mapsto \max B(s,x,y)$ *is concave-like for each* $x, y \in K$, *and the mapping* $y \mapsto \max A(s,x,y)$ *is convex-like for each* $x, y \in K, s \in T(x)$;

(iv) *for each* $x, y \in K, s \in T(x)$, $\max F(s,x,y) \leq \max A(s,x,y) \leq \max B(s,x,y) \leq \max G(s,x,y)$;

(v) *for each* $y \in K$, *there is an* $\tilde{x} \in K$ *with* $\tilde{s}_y \in T(\tilde{x})$ *such that*

$$\max G\left(\tilde{s}_y, \tilde{x}, y\right) < \max \bigcup_{s\in T(\tilde{x})} \min \bigcup_{y\in K} G\left(s, \tilde{x}, y\right). \tag{64}$$

Then $(\text{SEP})_G$ *has a strong solution.*

Proof. According to Theorem 15, we know that $(\text{SEP})_F$ has a weak solution. That is, there is an $\overline{x} \in K$ such that

$$F\left(\overline{s}_x, \overline{x}, x\right) \not\subset -\operatorname{int} C, \tag{65}$$

for all $x \in K$ and for some $\overline{s}_x \in T(\overline{x})$. For any $k \in \operatorname{int} C$, from Proposition 4, the Gerstewitz function ξ_{k0} satisfies

$$\xi_{k0} F\left(\overline{s}_x, \overline{x}, x\right) \geq 0. \tag{66}$$

Hence, there is $\overline{x} \in K$ such that, for each $x \in K$,

$$\max \bigcup_{s\in T(\overline{x})} \xi_{k0} F\left(s, \overline{x}, x\right) \geq 0. \tag{67}$$

Thus, we have

$$\min \bigcup_{x\in K} \max \bigcup_{s\in T(\overline{x})} \xi_{k0} F\left(s, \overline{x}, x\right) \geq 0. \tag{68}$$

By conditions (i)–(v), all conditions of Theorem 6 hold; hence we have

$$\max \bigcup_{s\in T(\overline{x})} \min \bigcup_{x\in K} \xi_{k0} G\left(s, \overline{x}, x\right) \geq 0. \tag{69}$$

Since $T(\overline{x})$ is compact, there is $\overline{s} \in T(\overline{x})$ such that

$$\min \bigcup_{x\in K} \xi_{k0} G\left(\overline{s}, \overline{x}, x\right) \geq 0. \tag{70}$$

This implies that

$$\xi_{k0} G\left(\overline{s}, \overline{x}, x\right) \geq 0 \tag{71}$$

or

$$G\left(\overline{s}, \overline{x}, x\right) \not\subset -\operatorname{int} C, \tag{72}$$

for all $x \in K$. Therefore, $(\text{SEP})_G$ has a strong solution. □

Finally, we give the following example to illustrate that Theorems 15 and 16 are valid.

Example 17. Let $K = [1,2]$, $C = \mathbb{R}_+$, $Z = \mathbb{R}$, and $T : K \rightrightarrows \mathbb{R}$ be defined by $T(x) = [0, 2x]$ for all $x \in K$. Then we define $F, A, B, G : TK \times K \times K \rightrightarrows \mathbb{R}$ which are defined by

$$\begin{aligned} F\left(s,x,y\right) &= \left\{x\left(y - \xi s x\right) : \xi \in \left[\frac{1}{2}, 1\right]\right\}, \\ A\left(s,x,y\right) &= \left\{x\left(y - \xi s x\right) + 1 : \xi \in \left[\frac{1}{2}, 1\right]\right\}, \\ B\left(s,x,y\right) &= \left\{x\left(y - \xi s x\right) + 2 : \xi \in \left[\frac{1}{2}, 1\right]\right\}, \\ G\left(s,x,y\right) &= \left\{x\left(y - \xi s x\right) + 3 : \xi \in \left[\frac{1}{2}, 1\right]\right\}, \end{aligned} \tag{73}$$

for all $(x, y) \in K \times K$.

Then, the set-valued mappings F and T are two upper semicontinuous mappings with nonempty compact. We can easily see that $F(s,x,x) \not\subset -\operatorname{int} C$ for all $x \in K$ and if we choose any $s \in T(x) \cap [0,1]$. So, condition (i) of Theorem 15 holds. It is obvious that condition (ii) of Theorem 15 holds since the mapping

$$(s, y) \longmapsto F\left(s, x, y\right) \tag{74}$$

is linear. Hence $(\text{SEP})_F$ has a weak solution by Theorem 15. Indeed, $\overline{x} = 2$ is a weak solution for $(\text{SEP})_F$ where we can choose $\overline{s} = 1/2$.

Next, we claim that $(\text{SEP})_G$ has a strong solution. We can easily deduce that conditions (i), (iii), and (iv) hold. The condition (ii) is valid since

$$\begin{aligned} \bigcup_{s\in T(x)} F\left(s,x,y\right) &= \bigcup_{s\in[0,2x]} \left\{x\left(y - \xi s x\right) : \xi \in \left[\frac{1}{2}, 1\right]\right\} \\ &= \left[xy - 2x^3, xy\right] \end{aligned} \tag{75}$$

is compact for $x, y \in K$ and so is

$$\begin{aligned} \bigcup_{y\in K} G\left(s,x,y\right) &= \bigcup_{y\in[1,2]} \left\{x\left(y - \xi s x\right) + 3 : \xi \in \left[\frac{1}{2}, 1\right]\right\} \\ &= \left[x - sx^2 + 3, 2x - \frac{sx^2}{2} + 3\right], \end{aligned} \tag{76}$$

for $x \in K, s \in T(x)$. Finally, condition (v) of Theorem 16 is valid, since, for each $y \in K$, we can choose an $\tilde{x} \in K$ with $\tilde{x} > \sqrt{y-1}$ and $\tilde{s}_y = 2\tilde{x}$ such that

$$\begin{aligned} \max G\left(\tilde{s}_y, \tilde{x}, y\right) &= \tilde{x}\left(y - \frac{\tilde{s}_y \tilde{x}}{2}\right) + 3 \\ &< \tilde{x} + 3 \\ &= \max \bigcup_{s\in T(\tilde{x})} \min \bigcup_{y\in K} G\left(s, \tilde{x}, y\right). \end{aligned} \tag{77}$$

Indeed, $\overline{x} = 3/2$ with $\overline{s} = 2$ is a strong solution for $(\text{SEP})_G$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

In this research, the first author was supported by Grant no. NSC102-2115-M-039-001- of the National Science Council of Taiwan (Taiwan). The second author was supported partly by National Science Council of the Republic of China.

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