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Research Article

Global Synchronization of Complex Networks with Discrete Time Delays on Time Scales

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This paper studies the global synchronization problem for a class of complex networks with discrete time delays. By using the theory of calculus on time scales, the properties of Kronecker product, and Lyapunov method, some sufficient conditions are obtained to ensure the global synchronization of the complex networks with delays on time scales. These sufficient conditions are formulated in terms of linear matrix inequalities (LMIs). The main contribution of the result is that the global synchronization problems with both discrete time and continuous time are unified under the same framework.

1. Introduction

As well known, complex dynamical networks have been a subject of high importance and increasing interest within the science and technology communities. The synchronization is one of the most typical phenomena in complex networks, which is ubiquitous in the real world, such as secure communication, chaos generators design, and harmonic oscillation generation, ([1–8], and references cited therein).

During the past many years, the synchronization of complex networks has received increasing research attention. There are lots of the papers studying the continuous time and the discrete time dynamical systems. However, most of the investigations are restricted to the continuous or discrete systems, respectively, [9–22]. For avoiding this trouble, it is meaningful to study this problem on time scales which can unify the continuous and discrete dynamical systems under the unified framework.

The theory of time scale calculus was initiated by Hilger in 1988, developed, and consummated by Bohner and Peterson [23–25], which has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in physics, population dynamics, biotechnology, economics, and so on [26, 27]. This novel and fascinating type of mathematics is more general and versatile than the traditional theories of differential and difference equations as it can, under one framework, mathematically describe continuous and discrete hybrid processes and hence is the optimal way forward for accurate and malleable mathematical modeling. The field of dynamic equations on time scales contains, links, and extends the classical theory of differential and difference equations.

However, to the best of our knowledge, there are few works investigating the synchronization problem of complex networks with delays on time scales.

Notations. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n\times m$ real matrices, respectively. $\mathbb{T}$ is a time scale, which is an arbitrary nonempty closed subset of the real number $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$, and assume that $0 \in \mathbb{T}$, and $\mathbb{T}$ is unbounded above, that is, $\sup \mathbb{T} = \infty$. Set $[a,b]_{\mathbb{T}} := \{t \in \mathbb{T}, a \le t \le b\}$. $\mathbb{T}^+ = \{t \in \mathbb{T}, t \ge 0\}$. $P > 0$ means that matrix P is real, symmetric, and positive definite. I and O denote the identity matrix and the zero matrix with compatible dimensions, respectively; and $\operatorname{diag}\{\cdots\}$ stands for a block-diagonal matrix. The superscript "T" stands for a matrix transposition. The Kronecker product of matrices $Q \in \mathbb{R}^{m\times n}$ and $R \in \mathbb{R}^{p\times q}$ is a matrix in $\mathbb{R}^{mp\times nq}$ and denoted as $Q \otimes R$. Let $\tau > 0$ and $C([-\tau,0]_{\mathbb{T}};\mathbb{R}^n)$ denote the family of continuous functions φ from $[-\tau,0]_{\mathbb{T}}$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\le\theta\le 0}\|\varphi(\theta)\|$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$.

The rest of this paper is organized as follows. In Section 2, some preliminaries on time scale are briefly outlined. In Section 3, by utilizing the approach of the Lyapunov functional method on time scale and the LMI [28], our main result for ensuring the global synchronization is derived. In Section 4, an example is given to illustrate the effectiveness of our main result. Finally, in Section 5, this paper is concluded.

2. Preliminaries

In this paper, the global synchronization problem is investigated for a class of complex networks with discrete time delays which is described by the following dynamic equation on time scale $\mathbb{T}$:

$$x_i^{\Delta}(t) = Ax_i(t) + Bf(x_i(t)) + B_\tau f(x_i(t-\tau_1)) + \sum_{j=1}^{N} G_{ij}\Gamma x_j(t-\tau_2) + \Lambda_i, \quad i = 1,2,\ldots,N, \tag{2.1}$$

where $t \in \mathbb{T}$, $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of the ith network at time t. A denotes a known connection matrix, B and B_τ denote the connection weight matrices, $\Gamma \in \mathbb{R}^{n\times n}$ is the matrix describing the innercoupling between the subsystems at time t, $G = (G_{ij})_{N\times N}$ is the outercoupling configuration matrix representing the coupling strength and the topological structure of the complex networks, and Λ_i is the external input. The constants τ_1 and τ_2 stand for the constant time delays, and $f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t)))^T$ is an unknown but sector-bounded nonlinear function.

The initial conditions associated with system (2.1) are given by

$$x_i(s) = \varphi_i(s), \quad s \in [-h,0]_{\mathbb{T}},\ i = 1,2,\ldots,N, \tag{2.2}$$

where $\varphi_i(s)$ is rd-continuous and the corresponding state trajectory is denoted as $x_i(t,\varphi_i)$.

Throughout this paper, the following assumptions are needed.

Assumption 2.1. The outer-coupling configuration matrix of the complex networks (2.1) satisfies

$$G_{ij} = G_{ji} \geq 0 \quad (i \neq j), \qquad G_{ii} = -\sum_{j=1, j\neq i}^{N} G_{ij} \quad (i, j = 1, 2, \ldots, N). \tag{2.3}$$

Assumption 2.2. For all $u, v \in \mathbb{R}^n$, the nonlinear function $f(\cdot)$ is assumed to satisfy the following sector-bounded condition:

$$\left(f(u) - f(v) - L_f(u - v)\right)^T \left(f(u) - f(v) - L^f(u - v)\right) \leq 0, \tag{2.4}$$

where L_f and L^f are real constant matrices with $L^f - L_f$ being symmetric and positive definite.

Assumption 2.3. $0 \leq \tau_i < h$ $(i = 1, 2)$.

Remark 2.4. System (2.1) is a general model of a class of complex networks. Its one special case with continuous time system is the following:

$$\begin{aligned} \frac{dx_i(t)}{dt} &= Ax_i(t) + Bf(x_i(t)) + B_\tau f(x_i(t - \tau_1)) \\ &\quad + \sum_{j=1}^{N} G_{ij}\Gamma x_j(t - \tau_2) + \Lambda_i, \quad i = 1, 2, \ldots, N \end{aligned} \tag{2.5}$$

for $t \in [t_0, +\infty)$, and its another special case with discrete time system is the following:

$$\begin{aligned} \Delta x_i(t) &= Ax_i(t) + Bf(x_i(t)) + B_\tau f(x_i(t - \tau_1)) \\ &\quad + \sum_{j=1}^{N} G_{ij}\Gamma x_j(t - \tau_2) + \Lambda_i, \quad i = 1, 2, \ldots, N \end{aligned} \tag{2.6}$$

for $t \in N$, where $\Delta x_i(t) = x_i(t + 1) - x_i(t)$ is the forward difference operator.

The continuous-time system (2.5) and the discrete-time system (2.6) are unified as system (2.1). The main objective of this paper is to study the synchronization problem of system (2.1) under the same framework.

In order to obtain the main results, some preliminary results are presented in this section.

Definition 2.5 (see [23]). A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$; $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$; $\mu(t) := \sigma(t) - t$. We put $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. A point t is said to be left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then we define $\mathbb{T}^k$ to be $\mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 2.6 (see [23]). A function $f : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous provided it is continuous at right-dense point of $\mathbb{T}$ and the left side limit exists (finite) at left-dense point of $\mathbb{T}$. The set of all right-dense continuous functions on $\mathbb{T}$ is defined by $C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

Definition 2.7 (see [23]). For a function $f : \mathbb{T} \to \mathbb{R}, t \in \mathbb{T}^k$, the delta derivative of $f(t)$, $f^{\Delta}(t)$, is the number (if it exists) with the property that, for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left| [f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] \right| < \varepsilon |\sigma(t) - s| \tag{2.7}$$

for all $s \in U$.

If t is right-scattered and f is continuous at t, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}. \tag{2.8}$$

If t is right-dense, then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{2.9}$$

For all $t \in \mathbb{T}^k$, one can get

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.10}$$

If $f^{\Delta} \geq 0$, then f is not decreasing on time scale.

Lemma 2.8 (see [23]). *If f and g are two differentiable functions, then the product rule for the derivative of product $f \cdot g$ is that*

$$(f \cdot g)^{\Delta} = f^{\Delta} \cdot g + f^{\sigma} \cdot g^{\Delta} = f^{\Delta} \cdot g^{\sigma} + f \cdot g^{\Delta}. \tag{2.11}$$

Definition 2.9 (see [23]). A function $F : \mathbb{T}^k \to \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta} = f$ holds for all $t \in \mathbb{T}^k$. In this case, we define the integral of f by

$$\int_a^t f(s)\Delta s = F(t) - F(a), \quad \text{for } t \in \mathbb{T}, \tag{2.12}$$

and we have the following formula:

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t) f(t), \quad \text{for } t \in \mathbb{T}^k. \tag{2.13}$$

Definition 2.10 (see [23]). A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive provided

$$1 + \mu(t) f(t) \neq 0, \quad \forall\, t \in \mathbb{T}. \tag{2.14}$$

The addition "$\oplus$" is defined by $p \oplus q := p + q + \mu pq$. The set of all regressive functions on a time scale $\mathbb{T}$ forms an Abelian group under the addition "$\oplus$". The additive inverse in this group is denoted by $\ominus p := -p/(1 + \mu p)$. Then the subtraction $\ominus$ on the set of regressive functions is defined by $p \ominus q := p \oplus (\ominus p)$. It can be shown easily that $p \ominus q = -(p - q)/(1 + \mu q)$. The set of all regressive and right-dense continuous functions will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

We denote that $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} : 1 + \mu(t) f(t) > 0 \text{ for all } t \in \mathbb{T}\}$. Obviously $\mathcal{R}^+$ is the set of all positively regressive elements of $\mathcal{R}$. One can easily verify that if $f \in \mathcal{R}^+$, then $\ominus f \in \mathcal{R}^+$.

Lemma 2.11 (see [23]). *Assume $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\overline{\mathbb{T}} = v(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and v is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t) v^{\Delta}(t) \Delta t = \int_{v(a)}^{v(b)} \left(f \circ v^{-1}(s)\right) \Delta s. \tag{2.15}$$

Definition 2.12. Let A be an $m \times n$-matrix-valued function on $\mathbb{T}$. We say that A is rd-continuous on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$, and the class of all such rd-continuous $m \times n$-matrix-valued function on $\mathbb{T}$ is denoted, similar to the scalar case, by $C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{m\times n})$.

We say that A is differentiable on $\mathbb{T}$ provided each entry of A is differentiable on $\mathbb{T}$. In this case, we put

$$A^{\Delta} = \left(a_{ij}^{\Delta}\right)_{1\le i\le m,\ 1\le j\le n}, \tag{2.16}$$

where $A = (a_{ij})_{1\le i\le m,\ 1\le j\le n}$. And denote that $A^{\sigma} = (a_{ij}^{\sigma})$.

Lemma 2.13 (see [23]). *Suppose A and B are differentiable $n \times n$-matrix-valued functions. Then*

(1) $(A + B)^{\Delta} = A^{\Delta} + B^{\Delta}$;

(2) $(\alpha A)^{\Delta} = \alpha A^{\Delta}$ *if α is a constant;*

(3) $(AB)^{\Delta} = A^{\Delta} B^{\sigma} + A B^{\Delta}$.

Let

$$\begin{gathered} x(t) = \left(x_1^T(t), x_2^T(t), \ldots, x_N^T(t)\right)^T, \qquad \Lambda = \left(\Lambda_1^T, \Lambda_2^T, \ldots, \Lambda_N^T\right)^T, \\ F(x(t)) = \left(f^T(x_1(t)), f^T(x_2(t)), \ldots, f^T(x_N(t))\right)^T, \end{gathered} \tag{2.17}$$

together with the Kronecker product "⊗" for matrices, system (2.1) can be recasted into

$$x^{\Delta}(t) = (I_N \otimes A)x(t) + (I_N \otimes B)F(x(t)) + (I_N \otimes B_\tau)F(x(t-\tau_1)) + (G \otimes \Gamma)x(t-\tau_2) + \Lambda. \tag{2.18}$$

Definition 2.14. The complex system (2.18) is said to be globally synchronized, if

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \tag{2.19}$$

hold for all $\varphi_i(s),\ \varphi_j(s) \in C_{rd}([-h,0]_{\mathbb{T}}, \mathbb{R}^n),\ i,j = 1,2,\dots N$.

Lemma 2.15 (see [29]). *The Kronecker product has the following properties:*

(1) $(\alpha A) \otimes B = A \otimes (\alpha B)$,

(2) $(A+B) \otimes C = A \otimes C + B \otimes C$,

(3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$,

(4) $(A \otimes B)^T = A^T \otimes B^T$.

In the mean of time scale, Schwarz inequality holds similarly.

Lemma 2.16. *Suppose $f, g : \mathbb{T} \to R$ are two rd-continuous functions, and $0,\ t \in \mathbb{T}, t > 0$. Then*

$$\left(\int_0^t f(s)g(s)\Delta s\right)^2 \leq \int_0^t f^2(s)\Delta s \int_0^t g^2(s)\Delta s. \tag{2.20}$$

Remark 2.17. One can prove Lemma 2.16 easily by using analysis knowledge. The proof is not given here for the purpose of space saving.

Corollary 2.18. *Suppose $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and $0,\ t \in \mathbb{T},\ t > 0$. Then*

$$\left(\int_0^t f(s)\Delta s\right)^2 \leq t\int_0^t f^2(s)\Delta s. \tag{2.21}$$

Lemma 2.19. *Assume $0, t \in \mathbb{T},\ t > 0$. $M \in \mathbb{R}^{n\times n}$ is a symmetric and positive semidefinite matrix, and $f : [0,t]_{\mathbb{T}} \to \mathbb{R}^n$ is a vector function. If the integrations concerned are well defined, then the following inequality holds:*

$$\left(\int_0^t f(s)\Delta s\right)^T M\left(\int_0^t f(s)\Delta s\right) \leq t\int_0^t f^T(s)Mf(s)\Delta s. \tag{2.22}$$

Proof. Since matrix M is symmetric and positive semidefinite, there exists a reversible orthogonal matrix $P = (p_{ij}) \in \mathbb{R}^{n\times n}$, such that

$$M = P^{-1}\widetilde{\Lambda}P = P^T\widetilde{\Lambda}P, \tag{2.23}$$

where $\widetilde{\Lambda} = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\lambda_i \geq 0$ $(i = 1, 2, \ldots, n)$ are the eigenvalues of M. Then, one has

$$\begin{aligned}
&\left(\int_0^t f(s)\Delta s\right)^T M\left(\int_0^t f(s)\Delta s\right) = \left(\int_0^t f(s)\Delta s\right)^T P^T\widetilde{\Lambda}P\left(\int_0^t f(s)\Delta s\right)\\
&\quad = \left(P\int_0^t f(s)\Delta s\right)^T\widetilde{\Lambda}\left(P\int_0^t f(s)\Delta s\right) = \left(\int_0^t Pf(s)\Delta s\right)^T\widetilde{\Lambda}\left(\int_0^t Pf(s)\Delta s\right)\\
&\quad = \left(\int_0^t \begin{bmatrix} \sum_{i=1}^n p_{1i}f_i(s) \\ \sum_{i=1}^n p_{2i}f_i(s) \\ \vdots \\ \sum_{i=1}^n p_{ni}f_i(s) \end{bmatrix}\Delta s\right)^T \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \left(\int_0^t \begin{bmatrix} \sum_{i=1}^n p_{1i}f_i(s) \\ \sum_{i=1}^n p_{2i}f_i(s) \\ \vdots \\ \sum_{i=1}^n p_{ni}f_i(s) \end{bmatrix}\Delta s\right)\\
&\quad = \lambda_1\left(\int_0^t \sum_{i=1}^n p_{1i}f_i(s)\Delta s\right)^2 + \lambda_2\left(\int_0^t \sum_{i=1}^n p_{2i}f_i(s)\Delta s\right)^2 + \cdots + \lambda_n\left(\int_0^t \sum_{i=1}^n p_{ni}f_i(s)\Delta s\right)^2.
\end{aligned} \tag{2.24}$$

By using Corollary 2.18, one can obtain

$$\left(\int_0^t \sum_{i=1}^n p_{ji}f_i(s)\Delta s\right)^2 \leq t\int_0^t \left(\sum_{i=1}^n p_{ji}f_i(s)\right)^2\Delta s, \quad j = 1, 2, \ldots, n; \tag{2.25}$$

then

$$\begin{aligned}
&\left(\int_0^t f(s)\Delta s\right)^T M\left(\int_0^t f(s)\Delta s\right)\\
&\quad \leq \lambda_1 t\int_0^t \left(\sum_{i=1}^n p_{1i}f_i(s)\right)^2\Delta s + \lambda_2 t\int_0^t \left(\sum_{i=1}^n p_{2i}f_i(s)\right)^2\Delta s + \cdots + \lambda_n t\int_0^t \left(\sum_{i=1}^n p_{ni}f_i(s)\right)^2\Delta s\\
&\quad = t\int_0^t \left[\left(\sum_{i=1}^n p_{1i}f_i(s)\right)\lambda_1\left(\sum_{i=1}^n p_{1i}f_i(s)\right) + \cdots + \left(\sum_{i=1}^n p_{ni}f_i(s)\right)\lambda_n\left(\sum_{i=1}^n p_{ni}f_i(s)\right)\right]\Delta s\\
&\quad = t\int_0^t (Pf(s))^T\widetilde{\Lambda}(Pf(s))\Delta s = t\int_0^t f^T(s)\left(P^T\widetilde{\Lambda}P\right)f(s)\Delta s\\
&\quad = t\int_0^t f^T(s)Mf(s)\Delta s.
\end{aligned} \tag{2.26}$$

The proof is completed. □

Lemma 2.20 (see [30]). *Let $U = (\alpha_{ij})_{N\times N}$, $P \in \mathbb{R}^{n\times n}$, $x = (x_1^T, x_2^T, \ldots, x_N^T)^T$ where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n$ and $y = (y_1^T, y_2^T, \ldots, y_N^T)^T$ where $y_i = (y_{i1}, y_{i2}, \ldots, y_{in})^T \in \mathbb{R}^n$ ($i = 1, 2, \ldots, N$). If $U = U^T$ and each row sum of U is zero, then*

$$x^T (U \otimes P) y = -\sum_{1\le i<j\le N} \alpha_{ij} (x_i - x_j)^T P (y_i - y_j). \tag{2.27}$$

3. Main Results and Proofs

In this section, the main results for global synchronization criteria of the delayed complex networks on time scales are presented.

Theorem 3.1. *Suppose Assumptions 2.1 and 2.2 hold. The global synchronization of system* (2.18) *is achieved if there exist $n \times n$ positive matrices $P > 0$, $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, $R > 0$, and matrices M_1, M_2, S, and positive scalars ε_1, ε_2, such that the following LMI holds for all $1 \le i < j \le N$:*

$$\mathbb{D}_{ij} = D_1 + D_2 + D_2^T + D_3 + D_3^T < 0, \tag{3.1}$$

where

$$\begin{aligned}
D_1 = \operatorname{diag}\Big\{ & 2PA + Q_1 + Q_2 + 2M_1 + 2M_2 - \varepsilon_1 \widehat{L}, 0_{n\times n}, -Q_1 - \varepsilon_2 \widehat{L}, -Q_2, R - 2\varepsilon_1 I_n, \\
& -R - 2\varepsilon_2 I_n, hP_1 + hP_2 - S - S^T, -\frac{1}{h}P_1, -\frac{1}{h}P_2 \Big\},
\end{aligned}$$

$$D_2 = \begin{bmatrix} O_{n\times n} \;\; PA \;\; -M_1 \;\; -M_2 \;\; O_{n\times n} \;\; O_{n\times n} & A^T S^T & -M_1 \;\; -M_2 \\ & O_{2n\times n} & \\ & -NG_{ij}\Gamma^T S^T & \\ O_{8n\times 6n} & B^T S^T & O_{8n\times 2n} \\ & B_\tau^T S^T & \\ & O_{3n\times n} & \end{bmatrix}, \tag{3.2}$$

$$D_3 = \begin{bmatrix} & -NG_{ij}(P\Gamma) & PB + \varepsilon_1 \check{L} & PB_\tau & \\ & -NG_{ij}(P\Gamma) & PB & PB_\tau & \\ O_{9n\times 3n} & O_{n\times n} & O_{n\times n} & \varepsilon_2 \check{L} & O_{9n\times 3n} \\ & O_{6n\times n} & O_{6n\times n} & O_{6n\times n} & \end{bmatrix}$$

with

$$\widehat{L} = L_f^T L^f + {L^f}^T L_f, \qquad \check{L} = L_f^T + {L^f}^T. \tag{3.3}$$

Proof. Letting

$$y(t) = (I_N \otimes A)x(t) + (I_N \otimes B)F(x(t\ )) + (I_N \otimes B_\tau)F(x(t-\tau_1)) + (G \otimes \Gamma)x(t-\tau_2), \tag{3.4}$$

system (2.18) becomes

$$x^\Delta(t) = y(t) + \Lambda. \tag{3.5}$$

Based on the theory of calculus on time scales, we choose the following Lyapunov functional candidate:

$$V(t) = 2V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t) + V_6(t), \tag{3.6}$$

where

$$\begin{gathered} V_1(t) = x^T(t)(U \otimes P)x(t), \qquad V_2(t) = \int_{-\tau_1}^{0}\int_{t+\theta}^{t} y^T(s)(U \otimes P_1)y(s)\Delta s\Delta\theta, \\ V_3(t) = \int_{-\tau_2}^{0}\int_{t+\theta}^{t} y^T(s)(U \otimes P_2)y(s)\Delta s\Delta\theta, \qquad V_4(t) = \int_{t-\tau_1}^{t} x^T(s)(U \otimes Q_1)x(s)\Delta s, \\ V_5(t) = \int_{t-\tau_2}^{t} x^T(s)(U \otimes Q_2)x(s)\Delta s, \qquad V_6(t) = \int_{t-\tau_1}^{t} F^T(x(s))(U \otimes R)F(x(s))\Delta s, \\ U = \begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & N-1 \end{bmatrix}_{N\times N}. \end{gathered} \tag{3.7}$$

Note that $UG = GU = NG$. For any matrix H with appropriate dimension, one obtains

$$(U \otimes H)(G \otimes \Gamma) = (UG) \otimes (H\Gamma) = (NG) \otimes (H\Gamma). \tag{3.8}$$

Calculating the delta derivative $V^\Delta(t)$ along the trajectories of the network (2.1) (or (2.18)), one has

$$\begin{aligned} V_1^\Delta(t) &= x^T(t)(U \otimes P)x^\Delta(t) + x^{\sigma^T}(t)(U \otimes P)^T x^\Delta(t) \\ &= [x(t) + x^\sigma(t)]^T (U \otimes P)x^\Delta(t) \\ &= [x(t) + x^\sigma(t)]^T (U \otimes P) \\ &\quad \times [(I_N \otimes A)x(t) + (I_N \otimes B)F(x(t)) + (I_N \otimes B_\tau)F(x(t-\tau_1)) + (G \otimes \Gamma)x(t-\tau_2) + \Lambda] \end{aligned}$$

$$
\begin{aligned}
&= [x(t) + x^{\sigma}(t)]^T [(U \otimes (PA))x(t) + (U \otimes (PB))F(x(t)) \\
&\qquad + (U \otimes (PB_\tau))F(x(t-\tau_1)) + ((NG) \otimes (P\Gamma))x(t-\tau_2) + (U \otimes P)\Lambda] \\
&= \sum_{1 \le i < j \le N} \Big\{ \left(x_i(t) - x_j(t)\right)^T (PA)\left(x_i(t) - x_j(t)\right) + \left(x_i(t) - x_j(t)\right)^T (PB) \\
&\qquad \times \left(f(x_i(t)) - f\left(x_j(t)\right)\right) + \left(x_i(t) - x_j(t)\right)^T (PB_\tau)\left(f(x_i(t-\tau_1)) - f\left(x_j(t-\tau_1)\right)\right) \\
&\qquad - \left(x_i(t) - x_j(t)\right)^T \left(NG_{ij}(P\Gamma)\right)\left(x_i(t-\tau_2) - x_j(t-\tau_2)\right) \Big\} \\
&\quad + \sum_{1 \le i < j \le N} \Big\{ \left(x_i^{\sigma}(t) - x_j^{\sigma}(t)\right)^T (PA)\left(x_i(t) - x_j(t)\right) + \left(x_i^{\sigma}(t) - x_j^{\sigma}(t)\right)^T (PB) \\
&\qquad \times \left(f(x_i(t)) - f\left(x_j(t)\right)\right) + \left(x_i^{\sigma}(t) - x_j^{\sigma}(t)\right)^T (PB_\tau) \\
&\qquad \times \left(f(x_i(t-\tau_1)) - f\left(x_j(t-\tau_1)\right)\right) \\
&\qquad - \left(x_i^{\sigma}(t) - x_j^{\sigma}(t)\right)^T \left(NG_{ij}(P\Gamma)\right)\left(x_i(t-\tau_2) - x_j(t-\tau_2)\right) \Big\}.
\end{aligned}
\tag{3.9}
$$

And synchronously,

$$
\begin{aligned}
V_1^{\Delta}(t) &= x^{\Delta^T}(t)(U \otimes P)^T x(t) + x^{\Delta^T}(t)(U \otimes P)x^{\sigma}(t) \\
&= x^{\Delta^T}(t)(U \otimes P)[x(t) + x^{\sigma}(t)] \\
&= \Big[x^T(t)(I_N \otimes A)^T + F(x(t))^T (I_N \otimes B)^T + F(x(t-\tau_1))^T (I_N \otimes B_\tau)^T \\
&\qquad + x^T(t-\tau_2)(G \otimes \Gamma)^T + \Lambda^T\Big](U \otimes P)[x(t) + x^{\sigma}(t)] \\
&= \Big[x^T(t)(U \otimes (PA)) + F(x(t))^T (U \otimes (PB)) + F(x(t-\tau_1))^T (U \otimes (PB_\tau)) \\
&\qquad + x^T(t-\tau_2)((NG) \otimes (P\Gamma)) + \Lambda^T (U \otimes P)\Big][x(t) + x^{\sigma}(t)] \\
&= \sum_{1 \le i < j \le N} \Big\{ \left(x_i(t) - x_j(t)\right)^T (PA)\left(x_i(t) - x_j(t)\right) + \left(f(x_i(t)) - f\left(x_j(t)\right)\right)^T \\
&\qquad \times (PB)\left(x_i(t) - x_j(t)\right) + \left(f(x_i(t-\tau_1)) - f\left(x_j(t-\tau_1)\right)\right)^T \\
&\qquad \times (PB_\tau)\left(x_i(t) - x_j(t)\right) \\
&\qquad - \left(x_i(t-\tau_2) - x_j(t-\tau_2)\right)^T \left(NG_{ij}(P\Gamma)\right)\left(x_i(t) - x_j(t)\right) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{1\le i<j\le N}\Big\{\left(x_i(t)-x_j(t)\right)^T(P\mathrm{A})\left(x_i^\sigma(t)-x_j^\sigma(t)\right)+\left(f\left(x_i(t)\right)-f\left(x_j(t)\right)\right)^T\\
&\qquad\times(PB)\left(x_i^\sigma(t)-x_j^\sigma(t)\right)+\left(f\left(x_i(t-\tau_1)\right)-f\left(x_j(t-\tau_1)\right)\right)^T\\
&\qquad\times(PB_\tau)\left(x_i^\sigma(t)-x_j^\sigma(t)\right)\\
&\qquad-\left(x_i(t-\tau_2)-x_j(t-\tau_2)\right)^T\left(NG_{ij}(P\Gamma)\right)\left(x_i^\sigma(t)-x_j^\sigma(t)\right)\Big\},
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
V_2^\Delta(t)&=\tau_1 y^T(t)(U\otimes P_1)y(t)-\int_{t-\tau_1}^t y^T(s)(U\otimes P_1)y(s)\Delta s\\
&\le hy^T(t)(U\otimes P_1)y(t)-\int_{t-\tau_1}^t y^T(s)(U\otimes P_1)y(s)\Delta s\\
&\le hy^T(t)(U\otimes P_1)y(t)-\frac{1}{\tau_1}\left(\int_{t-\tau_1}^t y(s)ds\right)^T(U\otimes P_1)\left(\int_{t-\tau_1}^t y(s)\Delta s\right)\\
&\le h\sum_{1\le i<j\le N}\left(y_i(t)-y_j(t)\right)^T P_1\left(y_i(t)-y_j(t)\right)\\
&\quad-\frac{1}{h}\sum_{1\le i<j\le N}\left(\int_{t-\tau_1}^t\left(y_i(s)-y_j(s)\right)\Delta s\right)^T P_1\left(\int_{t-\tau_1}^t\left(y_i(s)-y_j(s)\right)\Delta s\right)\\
&=h\sum_{1\le i<j\le N}\left(y_i(t)-y_j(t)\right)^T P_1\left(y_i(t)-y_j(t)\right)\\
&\quad-\frac{1}{h}\sum_{1\le i<j\le N}\left(Y_i^{(1)}-Y_j^{(1)}\right)^T P_1\left(Y_i^{(1)}-Y_j^{(1)}\right),
\end{aligned}
\tag{3.11}
$$

where

$$
Y_i^{(1)}(t)=\int_{t-\tau_1}^t y_i(s)\Delta s. \tag{3.12}
$$

Similarly, one has

$$
\begin{aligned}
V_3^\Delta(t)&=\tau_2 y^T(t)(U\otimes P_2)y(t)-\int_{t-\tau_2}^t y^T(s)(U\otimes P_2)y(s)\Delta s\\
&\le h\sum_{1\le i<j\le N}\left(y_i(t)-y_j(t)\right)^T P_2\left(y_i(t)-y_j(t)\right)\\
&\quad-\frac{1}{h}\sum_{1\le i<j\le N}\left(Y_i^{(2)}-Y_j^{(2)}\right)^T P_2\left(Y_i^{(2)}-Y_j^{(2)}\right),
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{aligned}
Y_i^{(2)}(t) &= \int_{t-\tau_2}^{t} y_i(s)\Delta s, \\
V_4^{\Delta}(t) &= x^T(t)(U\otimes Q_1)x(t) - x^T(t-\tau_1)(U\otimes Q_1)x(t-\tau_1) \\
&= \sum_{1\le i<j\le N} \left(x_i(t)-x_j(t)\right)^T Q_1\left(x_i(t)-x_j(t)\right) \\
&\quad - \sum_{1\le i<j\le N} \left(x_i(t-\tau_1)-x_j(t-\tau_1)\right)^T Q_1\left(x_i(t-\tau_1)-x_j(t-\tau_1)\right).
\end{aligned}
\tag{3.14}
$$

Similarly, one has

$$
\begin{aligned}
V_5^{\Delta}(t) &= x^T(t)(U\otimes Q_2)x(t) - x^T(t-\tau_2)(U\otimes Q_2)x(t-\tau_2) \\
&= \sum_{1\le i<j\le N} \left(x_i(t)-x_j(t)\right)^T Q_2\left(x_i(t)-x_j(t)\right) \\
&\quad - \sum_{1\le i<j\le N} \left(x_i(t-\tau_2)-x_j(t-\tau_2)\right)^T Q_2\left(x_i(t-\tau_2)-x_j(t-\tau_2)\right),
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
V_6^{\Delta}(t) &= F^T(x(t))(U\otimes R)F(x(t)) - F^T(x(t-\tau_1))(U\otimes R)F(x(t-\tau_1)) \\
&= \sum_{1\le i<j\le N} \left(f(x_i(t))-f(x_j(t))\right)^T R\left(f(x_i(t))-f(x_j(t))\right) \\
&\quad - \sum_{1\le i<j\le N} \left(f(x_i(t-\tau_1))-f(x_j(t-\tau_1))\right)^T R\left(f(x_i(t-\tau_1))-f(x_j(t-\tau_1))\right) .
\end{aligned}
\tag{3.16}
$$

By formula (2.18), for any matrices M_1, the following equality is satisfied

$$
2x^T(t)(U\otimes M_1)\left[x(t)-x(t-\tau_1)-\int_{t-\tau_1}^{t} y(s)\Delta s\right] = 0,
\tag{3.17}
$$

which can be rewritten as

$$
2\sum_{1\le i<j\le N} \left(x_i(t)-x_j(t)\right)^T M_1\left[\left(x_i(t)-x_j(t)\right)-\left(x_i(t-\tau_1)-x_j(t-\tau_1)\right)-\left(Y_i^{(1)}-Y_j^{(1)}\right)\right] = 0.
\tag{3.18}
$$

Similarly, for any matrices M_2, one has

$$
2\sum_{1\le i<j\le N} \left(x_i(t)-x_j(t)\right)^T M_2\left[\left(x_i(t)-x_j(t)\right)-\left(x_i(t-\tau_2)-x_j(t-\tau_2)\right)-\left(Y_i^{(2)}-Y_j^{(2)}\right)\right] = 0.
\tag{3.19}
$$

In addition, for any matrix S, the following equality is always true:

$$\begin{aligned}
&2y^T(t)(U \otimes S)[(I_N \otimes A)x(t) + (I_N \otimes B)F(x(t)) \\
&\qquad +(I_N \otimes B_\tau)F(x(t-\tau_1)) + (G \otimes \Gamma)x(t-\tau_2) - y(t)] \\
&= 2y^T(t)[(U \otimes (SA))x(t) + (U \otimes (SB))F(x(t)) \\
&\qquad +(\mathrm{U} \otimes (SB_\tau))F(x(t-\tau_1)) + (NG \otimes (S\Gamma))x(t-\tau_2) - (U \otimes S)y(t)\big] = 0,
\end{aligned} \tag{3.20}$$

that is,

$$\begin{aligned}
&\sum_{1\le i<j\le N}\Big[\left(x_i(t)-x_j(t)\right)^T\left(2A^TS^T\right) + \left(f(x_i(t)) - f(x_j(t))\right)^T\left(2B^TS^T\right) \\
&\quad + \left(f(x_i(t-\tau_1)) - f(x_j(t-\tau_1))\right)^T\left(2B_\tau^TS^T\right) - \left(x_i(t-\tau_2) - x_j(t-\tau_2)\right)^T\left(2NG_{ij}\Gamma^TS^T\right) \\
&\quad -\left(y_i(t) - y_j(t)\right)^T\left(S+S^T\right)\Big]\left(y_i(t) - y_j(t)\right) = 0.
\end{aligned} \tag{3.21}$$

Moreover, from Assumption 2.2, for $\varepsilon_1 > 0$, one obtains

$$\varepsilon_1\begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix}^T \begin{bmatrix} \widehat{L} & -\check{L} \\ -\check{L}^T & 2I \end{bmatrix}\begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix} \le 0, \tag{3.22}$$

where $\widehat{L} = L_f^TL^f + L^{f^T}L_f$ and $\check{L} = L_f^T + L^{f^T}$.

Applying $\sum_{1\le i<j\le N}$ on both sides of the above inequality, the following formula can be obtained

$$\begin{aligned}
&\sum_{1\le i<j\le N}\Big\{\left(x_i(t)-x_j(t)\right)^T\Big[2(\varepsilon_1\check{L})\left(f(x_i(t)) - f(x_j(t))\right) - \left(\varepsilon_1\widehat{L}\right)\left(x_i(t)-x_j(t)\right)\Big] \\
&\qquad -\left(f(x_i(t)) - f(x_j(t))\right)^T(2\varepsilon_1 I_n)\left(f(x_i(t)) - f(x_j(t))\right)\Big\} \ge 0.
\end{aligned} \tag{3.23}$$

Similarly, for $\varepsilon_2 > 0$, one has

$$\varepsilon_2\begin{bmatrix} x_i(t-\tau_1) - x_j(t-\tau_1) \\ f(x_i(t-\tau_1)) - f(x_j(t-\tau_1)) \end{bmatrix}^T \begin{bmatrix} \widehat{L} & -\check{L} \\ -\check{L}^T & 2I \end{bmatrix}\begin{bmatrix} x_i(t-\tau_1) - x_j(t-\tau_1) \\ f(x_i(t-\tau_1)) - f(x_j(t-\tau_1)) \end{bmatrix} \le 0, \tag{3.24}$$

and then

$$\sum_{1\le i<j\le N}\Big\{\left(x_i(t-\tau_1)-x_j(t-\tau_1)\right)^T \\ \times\left[2(\varepsilon_2\check{L})\left(f(x_i(t-\tau_1))-f\left(x_j(t-\tau_1)\right)\right)-\left(\varepsilon_2\widehat{L}\right)\left(x_i(t-\tau_1)-x_j(t-\tau_1)\right)\right] \\ -\left(f(x_i(t-\tau_1))-f\left(x_j(t-\tau_1)\right)\right)^T(2\varepsilon_2 I_n)\left(f(x_i(t-\tau_1))-f\left(x_j(t-\tau_1)\right)\right)\Big\}\ge 0. \tag{3.25}$$

From (3.9)–(3.23), we have

$$\begin{aligned}V^{\Delta}(t)\le\sum_{1\le i<j\le N}\Big\{&\left(x_i(t)-x_j(t)\right)^T\left[2(PA)Q_1+Q_2+2M_1+2M_2-\left(\varepsilon_1\widehat{L}\right)\right]\left(x_i(t)-x_j(t)\right)\\
&+\left(x_i(t)-x_j(t)\right)^T\left[(PB)+2(\varepsilon_1\check{L})\right]\left(f(x_i(t))-f\left(x_j(t)\right)\right)\\
&+\left(x_i(t)-x_j(t)\right)^T(PB_\tau)\left(f(x_i(t-\tau_1))-f\left(x_j(t-\tau_1)\right)\right)\\
&-\left(x_i(t)-x_j(t)\right)^T\left[(NG_{ij}(P\Gamma))+2M_2\right]\left(x_i(t-\tau_2)-x_j(t-\tau_2)\right)\\
&+\left(x_i^\sigma(t)-x_j^\sigma(t)\right)^T(PA)\left(x_i(t)-x_j(t)\right)+\left(x_i^\sigma(t)-x_j^\sigma(t)\right)^T(PB)\\
&\times\left(f(x_i(t))-f\left(x_j(t)\right)\right)\\
&+\left(x_i^\sigma(t)-x_j^\sigma(t)\right)^T(PB_\tau)\left(f(x_i(t-\tau_1))-f\left(x_j(t-\tau_1)\right)\right)\\
&-\left(x_i^\sigma(t)-x_j^\sigma(t)\right)^T(NG_{ij}(P\Gamma))\left(x_i(t-\tau_2)-x_j(t-\tau_2)\right)\\
&+\left(f(x_i(t))-f\left(x_j(t)\right)\right)^T(PB)\left(x_i(t)-x_j(t)\right)\\
&+\left(f(x_i(t-\tau_1))-f\left(x_j(t-\tau_1)\right)\right)^T(PB_\tau)\left(x_i(t)-x_j(t)\right)\\
&-\left(x_i(t-\tau_2)-x_j(t-\tau_2)\right)^T(NG_{ij}(P\Gamma))\left(x_i(t)-x_j(t)\right)\\
&+\left(x_i(t)-x_j(t)\right)^T(PA)\left(x_i^\sigma(t)-x_j^\sigma(t)\right)+\left(f(x_i(t))-f\left(x_j(t)\right)\right)^T(PB)\\
&\times\left(x_i^\sigma(t)-x_j^\sigma(t)\right)+\left(f(x_i(t-\tau_1))-f\left(x_j(t-\tau_1)\right)\right)^T(PB_\tau)\left(x_i^\sigma(t)-x_j^\sigma(t)\right)\\
&-\left(x_i(t-\tau_2)-x_j(t-\tau_2)\right)^T(NG_{ij}(P\Gamma))\left(x_i^\sigma(t)-x_j^\sigma(t)\right)\\
&+\left(y_i(t)-y_j(t)\right)^T\left(hP_1+hP_2-S-S^T\right)\left(y_i(t)-y_j(t)\right)\\
&-\frac{1}{h}\left(Y_i^{(1)}-Y_j^{(1)}\right)^TP_1\left(Y_i^{(1)}-Y_j^{(1)}\right)-\frac{1}{h}\left(Y_i^{(2)}-Y_j^{(2)}\right)^TP_2\left(Y_i^{(2)}-Y_j^{(2)}\right)\end{aligned}$$

$$
\begin{aligned}
& - \left(x_i(t-\tau_1) - x_j(t-\tau_1)\right)^T \left[Q_1 + \left(\varepsilon_2 \widehat{L}\right)\right] \left(x_i(t-\tau_1) - x_j(t-\tau_1)\right) \\
& - \left(x_i(t-\tau_2) - x_j(t-\tau_2)\right)^T Q_2 \left(x_i(t-\tau_2) - x_j(t-\tau_2)\right) \\
& + \left(f(x_i(t)) - f(x_j(t))\right)^T [R - 2\varepsilon_1 I_n] \left(f(x_i(t)) - f(x_j(t))\right) \\
& - \left(f(x_i(t-\tau_1)) - f(x_j(t-\tau_1))\right)^T [R + 2\varepsilon_2 I_n] \\
& \times \left(f(x_i(t-\tau_1)) - f(x_j(t-\tau_1))\right) \\
& - 2(x_i(t) - x_j(t))^T M_1 \left[(x_i(t-\tau_1) - x_j(t-\tau_1)) + \left(Y_i^{(1)} - Y_j^{(1)}\right)\right] \\
& - 2\left(x_i(t) - x_j(t)\right)^T M_2 \left(Y_i^{(2)} - Y_j^{(2)}\right) \\
& + \left[\left(x_i(t) - x_j(t)\right)^T \left(2A^T S^T\right) + \left(f(x_i(t)) - f(x_j(t))\right)^T \left(2B^T S^T\right)\right. \\
& \qquad + \left(f(x_i(t-\tau_1)) - f(x_j(t-\tau_1))\right)^T \left(2B_\tau^T S^T\right) \\
& \qquad \left. - \left(x_i(t-\tau_2) - x_j(t-\tau_2)\right)^T \left(2NG_{ij}\Gamma^T S^T\right)\right] \left(y_i(t) - y_j(t)\right) \\
& \left. + \left(x_i(t-\tau_1) - x_j(t-\tau_1)\right)^T (2\varepsilon_2 \check{L}) \left(f(x_i(t-\tau_1)) - f(x_j(t-\tau_1))\right)\right\} \\
= & \sum_{1 \le i < j \le N} \sigma_{ij}^T(t) \mathbb{D}_{ij} \sigma_{ij}(t) < 0,
\end{aligned}
\tag{3.26}
$$

where

$$
\sigma_{ij}(t) = \begin{bmatrix} x_i(t) - x_j(t) \\ x_i^\sigma(t) - x_j^\sigma(t) \\ x_i(t-\tau_1) - x_j(t-\tau_1) \\ x_i(t-\tau_2) - x_j(t-\tau_2) \\ f(x_i(t)) - f(x_j(t)) \\ f(x_i(t-\tau_1)) - f(x_j(t-\tau_1)) \\ y_i(t) - y_j(t) \\ Y_i^{(1)}(t) - Y_j^{(1)}(t) \\ Y_i^{(2)}(t) - Y_j^{(2)}(t) \end{bmatrix}, \tag{3.27}
$$

and $\mathbb{D}_{ij}$ is as defined in (3.1).

From condition (3.1), it is guaranteed that all the subsystems in (2.1) are globally synchronized for any fixed time delays $\tau_i \in (0, h]$ $(i = 1, 2)$. The proof is completed. □

Specially, in the case of system (2.5) with continuous time, the following corollary can be obtained.

Corollary 3.2. *Suppose Assumptions 2.1 and 2.2 hold. The global synchronization of system* (2.5) *is achieved if there exist $n \times n$ positive matrices $P > 0$, $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, $R > 0$, and matrices M_1, M_2, S, and positive scalars ε_1, ε_2, such that the following LMI holds for all $1 \le i < j \le N$:*

$$\mathbb{D}_{ij} = D_1 + D_2 + D_2^T + D_3 + D_3^T < 0, \tag{3.28}$$

where

$$\begin{gathered}
D_1 = \operatorname{diag}\Big\{2PA + Q_1 + Q_2 + 2M_1 + 2M_2 - \varepsilon_1 \widehat{L}, -Q_1 - \varepsilon_2 \widehat{L}, -Q_2, R - 2\varepsilon_1 I_n, \\
-R - 2\varepsilon_2 I_n, hP_1 + hP_2 - S - S^T, -\frac{1}{h}P_1, -\frac{1}{h}P_2\Big\}, \\
D_2 = \begin{bmatrix} O_{n\times n} & -M_1 & -M_2 & O_{n\times n} & O_{n\times n} & A^T S^T & -M_1 & -M_2 \\ & & & & & O_{n\times n} & & \\ & & & & & -NG_{ij}\Gamma^T S^T & & \\ & & O_{7n\times 5n} & & & B^T S^T & O_{7n\times 2n} & \\ & & & & & B_\tau^T S^T & & \\ & & & & & O_{3n\times n} & & \end{bmatrix}, \\
D_3 = \begin{bmatrix} & -NG_{ij}(P\Gamma) & PB + \varepsilon_1 \check{L} & PB_\tau & \\ O_{8n\times 2n} & O_{n\times n} & O_{n\times n} & \varepsilon_2 \check{L} & O_{8n\times 3n} \\ & O_{6n\times n} & O_{6n\times n} & O_{6n\times n} & \end{bmatrix}
\end{gathered} \tag{3.29}$$

with

$$\widehat{L} = L_f^T L^f + {L^f}^T L_f, \qquad \check{L} = L_f^T + {L^f}^T. \tag{3.30}$$

4. A Numerical Example

In this part, a numerical example is given to verify the theoretical result.

Consider the following complex networks (4.1) with time delays on time scale $\mathbb{T}$:

$$x_i^\Delta(t) = Ax_i(t) + Bf(x_i(t)) + B_\tau f(x_i(t - \tau_1)) + \sum_{j=1}^{10} G_{ij}\Gamma x_j(t - \tau_2) + \Lambda_i, \quad i = 1, 2, \dots, 10, \tag{4.1}$$

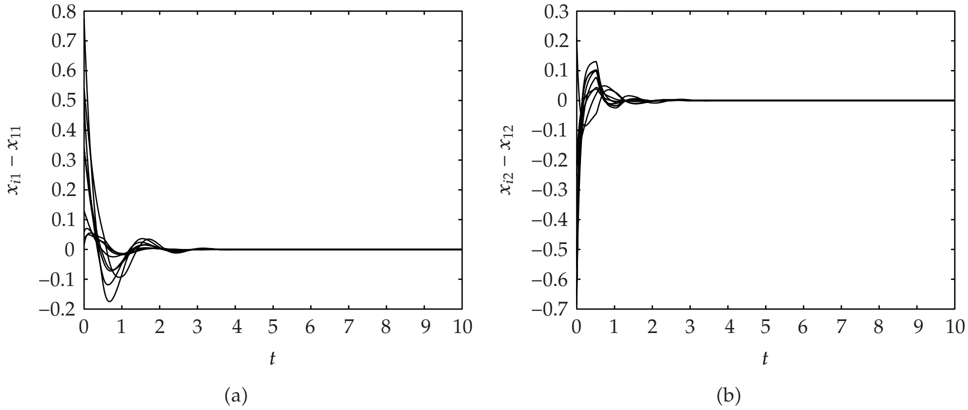

Figure 1: Synchronization error of system (4.1).

where $x_i(t) = (x_{i1}(t), x_{i2}(t))^T$ $(i = 1, 2, \ldots, 10)$ is the state vector of the ith subsystem. Choose the coupling matrix G and the linking matrix Γ as

$$G = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -9 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 0.35 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}, \qquad \Lambda_i = (1, 1)^T. \tag{4.2}$$

The other parameters are as follows:

$$A = \begin{bmatrix} -12 & 0.2 \\ 0.2 & -10 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -0.1 \\ -5 & 3 \end{bmatrix}, \qquad B_\tau = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{bmatrix}, \quad \tau_1 = \tau_2 = 2. \tag{4.3}$$

The nonlinear function is given by $f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)))^T$, with $f_j(x_{ij}(t)) = \tan h(x_{ij}(t))(j = 1, 2)$. It is easy to see that the nonlinear functions satisfy all the assumptions. By using the Matlab LMI Toolbox, LMI (3.1) is feasible. According to Theorem 3.1, one concludes that the complex networks (4.1) with delays on time scale $\mathbb{T}$ can achieve global synchronization. The synchronization errors of the complex networks (4.1) are plotted in Figure 1.

5. Conclusions

In this paper, we have investigated the global synchronization of a kind of delayed complex networks on time scales. Utilizing the theory of calculus on time scales and the properties of Kronecker product, the synchronization conditions have been derived through a suitably Lyapunov functional. The obtained synchronization criterion which is expressed in the form of LMIs can be easily verified by the standard numerical software such as Matlab LMI toolbox. The obtained results are novel since there are few works about the synchronization of delayed complex networks on time scales. In addition, the approach utilized in this paper can be considered as a universal framework for the study of other complex systems on time scales.

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