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Research Article

Global Dynamics for a Novel Differential Infectivity Epidemic Model with Stage Structure

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A novel differential infectivity epidemic model with stage structure is formulated and studied. Under biological motivation, the stability of equilibria is investigated by the global Lyapunov functions. Some novel techniques are applied to the global dynamics analysis for the differential infectivity epidemic model. Uniform persistence and the sharp threshold dynamics are established; that is, the reproduction number determines the global dynamics of the system. Finally, numerical simulations are given to illustrate the main theoretical results.

1. Introduction

Mathematical model that reflects the characteristics of an epidemic to some extent can help us to understand better how the disease spreads in the community and can investigate how changes in the various assumptions and parameter values affect the course of epidemic. In [1], Hyman et al. proposed a differential infectivity model that accounted for differences in infectiousness between individuals during the chronic stages and the correlation between viral loads and rates of developing AIDS. They assumed that the susceptible population was homogeneous and neglected variations in susceptibility, risk behavior, and many other factors associated with the dynamics of HIV spread. Ma et al. [2] presented several differential infectivity epidemic models under different assumptions.

In the real world, some epidemics, such as malaria, dengue, fever, gonorrhea, and bacterial infections, may have a different ability to transmit the infections in different ages. For example, measles and varicella always occur in juveniles, while it is reasonable to consider the disease transmission in adult population such as typhus and diphtheria. In recent years, epidemic models with stage structure have been studied in many papers [3–10].

In this paper, we formulate a differential infectivity epidemic model with stage structure. The proof of global stability of the endemic equilibrium utilizes a graph-theoretical approach [11–22] to the method of Lyapunov functions. Let S_1 and S_2 denote the immature susceptible and mature susceptible populations, respectively. The infectious population I was subdivided into n subgroups $I_1, I_2, \dots, I_n$. p_{1k} and p_{2k} denote the probabilities of an immature infectious individual and a mature infectious individual enter subgroup k, respectively, where $\sum_{k=1}^{n} p_{1k} = \sum_{k=1}^{n} p_{2k} = 1$. The disease incidence in the kth subgroup can be calculated as $\sum_{i=1}^{2} p_{ik} \sum_{j=1}^{n} \beta_{ij} S_i G_j(I_j)$, where β_{ij} is the transmission coefficient between compartments S_i and I_j. $G_j(I_j)$ includes some special incidence functions in the literature. For instance, $G_j(I_j) = I_j/(1 + \alpha_j I_j)$ (saturation effect). Since we do not assume that recovered individuals return into the susceptible class, the recovered class does not need to be explicitly modeled. Then, we obtain the following model:

$$\begin{aligned}
\dot{S}_1 &= \varphi(S_1) - \sum_{j=1}^{n} \beta_{1j} S_1 G_j(I_j) - a S_1, \\
\dot{S}_2 &= a S_1 - \sum_{j=1}^{n} \beta_{2j} S_2 G_j(I_j) - d_2 S_2, \\
\dot{I}_k &= \sum_{i=1}^{2} p_{ik} \sum_{j=1}^{n} \beta_{ij} S_i G_j(I_j) - m_k I_k, \quad k = 1, 2, \dots, n,
\end{aligned} \tag{1}$$

where $\varphi(S_1) = b - d_1 S_1$ with b being the recruitment constant and d_1 being the natural death rate. a is the conversion rate from an immature individual to a mature individual. d_2 is the natural death rate of the mature susceptible class. $m_k = d_k^I + \gamma_k$, where d_k^I is the death rate of I population in subgroup k and γ_k is the recovery rate in the kth subgroup. All parameter values are assumed to be nonnegative and $b, a, m_k, d_1, d_2 > 0$.

The organization of this paper is as follows. In Section 2, we prove some preliminary results for system (1). In Section 3, the main theorem of this paper is stated and proved. In Section 4, numerical simulations which support our theoretical analysis are given.

2. Preliminaries

We assume the following.

(A1) G_k is continuous and Lipschitz on $[0, +\infty)$, $G_k(x)/x$ is nonincreasing on $(0, +\infty)$, and

$$\delta_k = \lim_{x\to 0} \frac{G_k(x)}{x} > 0 \quad \text{exists.} \tag{2}$$

From our assumptions, it is clear that system (1) has a unique solution for any given initial data $(S_1(0), S_2(0), I_1(0), \ldots, I_n(0))$ with $S_1(0) > 0$, $S_2(0) > 0$, and $I_k(0) > 0$ for $k = 1, 2, \ldots, n$ and the solution remains nonnegative. We see that system (1) exits in a disease-free equilibrium $P_0 = (S_1^0, S_2^0, 0, \ldots, 0)$, where $S_2^0 = aS_1^0/d_2$. Let $\underline{m} = \min\{d_1, d_2, m_1, \ldots, m_n\}$. Then, we derive from (1) that the region,

$$\Gamma = \left\{ (S_1, S_2, I_1, \ldots, I_n) \in \mathbb{R}_+^{n+2} : S_1 \le S_1^0,\ S_2 \le S_2^0,\ S_1 + S_2 + \sum_{k=1}^{n} I_k \le \frac{b}{\underline{m}} \right\}, \tag{3}$$

is a forward invariant compact absorbing set with respect to (1). Also let Γ° denote the interior of Γ. The next generation matrix for system (1) is

$$\mathbf{Q} := \left[\frac{\sum_{i=1}^{2} p_{ik}\beta_{ij} S_i^0 \delta_j}{m_k} \right]_{n\times n} = \begin{bmatrix} \dfrac{\sum_{i=1}^{2} p_{i1}\beta_{i1} S_i^0 \delta_1}{m_1} & \cdots & \dfrac{\sum_{i=1}^{2} p_{i1}\beta_{in} S_i^0 \delta_n}{m_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\sum_{i=1}^{2} p_{in}\beta_{i1} S_i^0 \delta_1}{m_n} & \cdots & \dfrac{\sum_{i=1}^{2} p_{in}\beta_{in} S_i^0 \delta_n}{m_n} \end{bmatrix}. \tag{4}$$

Then, we define the basic reproduction number as the spectral radius of Q, $R_0 = \rho(\mathbf{Q})$. A square matrix is said to be reducible, if there is a permutation matrix P, such that $P'AP$ is a block upper triangular matrix; otherwise it is irreducible.

3. Main Results

In the section, we will study the global asymptotical stability of equilibria of system (1).

Theorem 1. *Assume that (A1) holds and $\mathbf{B} = [\sum_{i=1}^{2} p_{ik}\beta_{ij}]$ is irreducible.*

(1) *If $R_0 \le 1$, then P_0 is globally asymptotically stable in Γ.*

(2) *If $R_0 > 1$, then P_0 is unstable and system (1) admits at least one endemic equilibrium in Γ°.*

Proof. Let $\mathbf{S} = (S_1, S_2)$, $\mathbf{S^0} = (S_1^0, S_2^0)$, $\mathbf{I} = (I_1, I_2, \ldots, I_n)$, and $\mathbf{Q}(\mathbf{S}, \mathbf{I}) = (\sum_{i=1}^{2}(p_{ik}\beta_{ij}S_i G_j(I_j))/m_k I_j)_{n\times n}$. Notice that $\mathbf{B}$ is irreducible, and then $\mathbf{Q}$ is also irreducible. Hence, there exists $\omega_k > 0$, $k = 1, 2, \ldots, n$, such that

$$(\omega_1, \omega_2, \ldots, \omega_n)\, \rho(\mathbf{Q}) = (\omega_1, \omega_2, \ldots, \omega_n)\, \mathbf{Q}. \tag{5}$$

Define $L = \sum_{k=1}^{n} (\omega_k I_k / m_k)$. Then

$$\begin{aligned} \dot{L} &= \sum_{k=1}^{n} \omega_k \left[\frac{\sum_{i=1}^{2} p_{ik} \sum_{j=1}^{n} \beta_{ij} S_i G_j(I_j)}{m_k} - I_k \right] \\ &= (\omega_1, \omega_2, \ldots, \omega_n) \left[\mathbf{Q}(\mathbf{S}, \mathbf{I})\, \mathbf{I}^T - \mathbf{I}^T \right] \\ &\le (\omega_1, \omega_2, \ldots, \omega_n) \left[\mathbf{Q}\mathbf{I}^T - \mathbf{I}^T \right] \\ &= \left[\rho(\mathbf{Q}) - 1 \right] (\omega_1, \omega_2, \ldots, \omega_n)\, \mathbf{I}^T. \end{aligned} \tag{6}$$

We see that the only compact invariant subset of the set where $\dot{L} = 0$ is the singleton $\{P_0\}$. By LaSalle's Invariance Principle, P_0 is globally asymptotically stable in Γ if $R_0 \le 1$.

If $R_0 > 1$, by continuity, we obtain that $\dot{L} = (\omega_1, \omega_2, \ldots, \omega_n)[\mathbf{Q}(\mathbf{S}, \mathbf{I})\mathbf{I}^T - \mathbf{I}^T] > 0$ in a neighborhood of P_0 in Γ°. This implies that P_0 is unstable. From a uniform persistence result of [23] and a similar argument as in the proof of Proposition 3.3 of [24], we can deduce that the instability of P_0 implies the uniform persistence of system (1) in Γ°. This together with the uniform boundedness of solutions of system (1) in Γ° implies that system (1) has an endemic equilibrium in Γ° (see Theorem 2.8.6 of [25] or Theorem D.3 of [26]). The proof is completed. □

By Theorem 1, we have the idea that if $\mathbf{B} = [\sum_{i=1}^{2} p_{ik}\beta_{ij}]$ is irreducible, (A1) holds and $R_0 > 1$, and then system (1) exists in endemic equilibrium P^* in Γ°. Let $P^* = (S_1^*, S_2^*, I_1^*, \ldots, I_n^*)$, and then the components of P^* satisfy

$$\varphi(S_1^*) = \sum_{i=1}^{2}\sum_{j=1}^{n} \beta_{ij} S_i^* G_j(I_j^*) + d_2 S_2^*, \tag{7}$$

$$aS_1^* = \sum_{j=1}^{n} \beta_{2j} S_2^* G_j(I_j^*) + d_2 S_2^*, \tag{8}$$

$$\sum_{i=1}^{2} p_{ik} \sum_{j=1}^{n} \beta_{ij} S_i^* G_j(I_j^*) = m_k I_k^*. \tag{9}$$

Since φ is strictly decreasing on $[0, +\infty)$, we have

$$[\varphi(S_1) - \varphi(S_1^*)] \left(p_{1k} - \frac{p_{1k} S_1^*}{S_1} \right) \le 0. \tag{10}$$

For convenience of notations, set

$$\overline{\beta}_{kj} = \sum_{i=1}^{2} p_{ik}\beta_{ij}S_i^* G_j\left(I_j^*\right), \quad 1 \le k, j \le n,$$

$$\overline{B} = \begin{bmatrix} \sum_{l\neq 1}\overline{\beta}_{1l} & -\overline{\beta}_{21} & \cdots & -\overline{\beta}_{n1} \\ -\overline{\beta}_{12} & \sum_{l\neq 2}\overline{\beta}_{2l} & \cdots & -\overline{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{\beta}_{1n} & -\overline{\beta}_{2n} & \cdots & \sum_{l\neq n}\overline{\beta}_{nl} \end{bmatrix}. \tag{11}$$

Then, $\overline{B}$ is also irreducible. It follows from Lemma 2.1 of [11] that the solution space of linear system,

$$\overline{B}\mathbf{v} = 0, \tag{12}$$

has dimension 1, with a basis

$$\mathbf{v} := \left(v_1, v_2, \ldots, v_n\right)^T = \left(c_1, c_2, \ldots, c_n\right)^T, \tag{13}$$

where c_k denotes the cofactor of the kth diagonal entry of $\overline{B}$. Note that from (12) we have

$$\sum_{j=1}^{n}\overline{B}_{kj}v_k = \sum_{j=1}^{n}\overline{B}_{jk}v_j, \quad k = 1, 2, \ldots, n. \tag{14}$$

From (14), we have

$$\begin{aligned}
\sum_{k=1}^{n} v_k \sum_{j=1}^{n}\sum_{i=1}^{2} p_{ik}\beta_{ij}S_i^* G_j\left(I_j\right) &= \sum_{k,j=1}^{n}\sum_{i=1}^{2} p_{ij}\beta_{ik}S_i^* v_j G_k\left(I_k\right) \\
&= \sum_{k=1}^{n}\left[\sum_{j=1}^{n}\sum_{i=1}^{2} p_{ij}\beta_{ik}S_i^* G_k\left(I_k^*\right)v_j\right]\frac{G_k\left(I_k\right)}{G_k\left(I_k^*\right)} \\
&= \sum_{k=1}^{n}\left[\sum_{j=1}^{n}\left(\overline{\beta}_{jk}v_j\right)\right]\frac{G_k\left(I_k\right)}{G_k\left(I_k^*\right)} \\
&= \sum_{k=1}^{n}\left[\sum_{j=1}^{n}\left(\overline{\beta}_{kj}v_k\right)\right]\frac{G_k\left(I_k\right)}{G_k\left(I_k^*\right)} \\
&= \sum_{k=1}^{n} v_k \sum_{j=1}^{n}\sum_{i=1}^{2} p_{ik}\beta_{ij}S_i^* G_j\left(I_j^*\right)\frac{G_k\left(I_k\right)}{G_k\left(I_k^*\right)}.
\end{aligned} \tag{15}$$

We further make the following assumption.

(A2) G_k is strictly increasing on $[0, +\infty)$, and

$$\frac{G_k\left(x_k\right)I_k}{G_k\left(I_k\right)x_k} + \frac{G_k\left(I_k\right)}{G_k\left(x_k\right)} - \frac{I_k}{x_k} \le 1, \quad k = 1, 2, \ldots, n, \tag{16}$$

where $x_k > 0$ is chosen in an arbitrary way and equality holds if $I_k = x_k$.

Theorem 2. *Assume that (A1) and (A2) hold, $\sum_{k=1}^{n} v_k(p_{1k} - p_{2k}) \ge 0$, and $\mathbf{B} = [\sum_{i=1}^{2} p_{ik}\beta_{ij}]$ is irreducible. If $R_0 > 1$, then P^* is globally asymptotically stable in Γ° and thus is the unique endemic equilibrium.*

Proof. Consider a Lyapunov functional

$$\begin{aligned}
V = \sum_{k=1}^{n} v_k &\left[\sum_{i=1}^{2} p_{ik}\left(S_i - S_i^* - S_i^* \ln\frac{S_i}{S_i^*}\right)\right. \\
&\left. + \int_{I_k^*}^{I_k}\frac{G_k(x) - G_k\left(I_k^*\right)}{G_k(x)}dx\right].
\end{aligned} \tag{17}$$

Differentiating V along the solution of system (1), we obtain

$$\begin{aligned}
\dot{V} = \sum_{k=1}^{n} v_k &\left\{ p_{1k}\left[\varphi\left(S_1\right) - aS_1\right] + p_{2k}\left[aS_1 - d_2S_2\right]\right. \\
&- m_kI_k - \frac{p_{1k}S_1^*}{S_1}\left[\varphi\left(S_1\right) - \sum_{j=1}^{n}\beta_{1j}S_1G_j\left(I_j\right) - aS_1\right] \\
&- \frac{p_{2k}S_2^*}{S_2}\left[aS_1 - \sum_{j=1}^{n}\beta_{2j}S_2G_j\left(I_j\right) - d_2S_2\right] \\
&\left.- \frac{G_k\left(I_k^*\right)}{G_k\left(I_k\right)}\left[\sum_{i=1}^{2} p_{ik}\sum_{j=1}^{n}\beta_{ij}S_iG_j\left(I_j\right) - m_kI_k\right]\right\} \\
= \sum_{k=1}^{n} v_k &\left\{\varphi\left(S_1\right)\left(p_{1k} - \frac{p_{1k}S_1^*}{S_1}\right)\right. \\
&+ d_2S_2^*\left(p_{2k} - \frac{p_{2k}S_2}{S_2^*}\right) + \sum_{i=1}^{2} p_{ik}\sum_{j=1}^{n}\beta_{ij}S_i^*G_j\left(I_j\right) \\
&+ \frac{m_kG_k\left(I_k^*\right)I_k}{G_k\left(I_k\right)} - m_kI_k \\
&+ aS_1^*\left[p_{1k} - \frac{p_{1k}S_1}{S_1^*} + \frac{p_{2k}S_1}{S_1^*} - \frac{p_{2k}S_1S_2^*}{S_1^*S_2}\right] \\
&\left.- \frac{G_k\left(I_k^*\right)}{G_k\left(I_k\right)}\left[\sum_{i=1}^{2} p_{ik}\sum_{j=1}^{n}\beta_{ij}S_iG_j\left(I_j\right)\right]\right\}.
\end{aligned} \tag{18}$$

From (7), we know that

$$\begin{aligned}
&\varphi\left(S_1^*\right)\left(p_{1k} - \frac{p_{1k}S_1^*}{S_1}\right) \\
&\quad = \left(\sum_{i=1}^{2}\sum_{j=1}^{n}\beta_{ij}S_i^*G_j\left(I_j^*\right) + d_2S_2^*\right)\left(p_{1k} - \frac{p_{1k}S_1^*}{S_1}\right).
\end{aligned} \tag{19}$$

It follows from (8), (9), and (19) that

$$\begin{aligned}
\dot{V} = \sum_{k=1}^{n} v_k \Bigg\{ & [\varphi(S_1) - \varphi(S_1^*)] \left(p_{1k} - \frac{p_{1k}S_1^*}{S_1} \right) \\
& + d_2 S_2^* \left(p_{2k} - \frac{p_{2k}S_2}{S_2^*} \right) + \sum_{i=1}^{2} p_{ik}\beta_{ij}S_{ij}^* G_j(I_j^*) \\
& \cdot \left(\frac{G_k(I_k^*) I_k}{G_k(I_k) I_k^*} + \frac{G_j(I_j)}{G_j(I_j^*)} - \frac{I_k}{I_k^*} \right) \\
& + \sum_{j=1}^{n} \beta_{1j} S_1^* G_j(I_j^*) \left(p_{1k} - \frac{p_{1k}S_1^*}{S_1} \right) \\
& + \left(d_2 S_2^* + \sum_{j=1}^{n} \beta_{2j} S_2^* G_j(I_j^*) \right) \\
& \cdot \left[2p_{1k} - \frac{p_{1k}S_1^*}{S_1} - \frac{p_{1k}S_1}{S_1^*} + \frac{p_{2k}S_1}{S_1^*} - \frac{p_{2k}S_1S_2^*}{S_1^*S_2} \right] \\
& - \frac{G_k(I_k^*)}{G_k(I_k)} \left[\sum_{i=1}^{2} p_{ik} \sum_{j=1}^{n} \beta_{ij} S_i G_j(I_j) \right] \Bigg\}.
\end{aligned} \tag{20}$$

From (10), (15), and (16), we obtain

$$\begin{aligned}
\dot{V} \le \sum_{k=1}^{n} v_k \Bigg\{ & d_2 S_2^* \left(p_{2k} - \frac{p_{2k}S_2}{S_2^*} \right) + \sum_{i=1}^{2} p_{ik}\beta_{ij}S_{ij}^* G_j(I_j^*) \\
& + \sum_{j=1}^{n} \beta_{1j} S_1^* G_j(I_j^*) \left(p_{1k} - \frac{p_{1k}S_1^*}{S_1} \right) + \Bigg(d_2 S_2^* \\
& + \sum_{j=1}^{n} \beta_{2j} S_2^* G_j(I_j^*) \Bigg) \left[2p_{1k} - \frac{p_{1k}S_1^*}{S_1} - \frac{p_{1k}S_1}{S_1^*} \right. \\
& \left. + \frac{p_{2k}S_1}{S_1^*} - \frac{p_{2k}S_1S_2^*}{S_1^*S_2} \right] - \frac{G_k(I_k^*)}{G_k(I_k)} \Bigg[\sum_{i=1}^{2} p_{ik} \\
& \cdot \sum_{j=1}^{n} \beta_{ij} S_i G_j(I_j) \Bigg] \Bigg\} = \sum_{k=1}^{n} v_k \Bigg\{ d_2 S_2^* \Bigg(p_{2k} + 2p_{1k} \\
& - \frac{p_{1k}S_1^*}{S_1} - \frac{p_{1k}S_1}{S_1^*} + \frac{p_{2k}S_1}{S_1^*} - \frac{p_{2k}S_1S_2^*}{S_1^*S_2} - \frac{p_{2k}S_2}{S_2^*} \Bigg) \\
& + \sum_{j=1}^{n} \beta_{2j} S_2^* G_j(I_j^*) \\
& \cdot \left[2p_{1k} + p_{2k} - \frac{p_{1k}S_1^*}{S_1} - \frac{p_{1k}S_1}{S_1^*} + \frac{p_{2k}S_1}{S_1^*} - \frac{p_{2k}S_1S_2^*}{S_1^*S_2} \right] \\
& + \sum_{j=1}^{n} \beta_{1j} S_1^* G_j(I_j^*) \left(2p_{1k} - \frac{p_{1k}S_1^*}{S_1} \right) \\
& - \frac{G_k(I_k^*)}{G_k(I_k)} \left[\sum_{i=1}^{2} p_{ik} \sum_{j=1}^{n} \beta_{ij} S_i G_j(I_j) \right] \Bigg\} = -\sum_{k=1}^{n} v_k \Big(p_{1k} \\
& - p_{2k} \Big) \left(d_2 S_2^* + \sum_{j=1}^{n} \beta_{2j} S_2^* G_j(I_j^*) \right) \left(\frac{S_1^*}{S_1} + \frac{S_1}{S_1^*} \right) \\
& + \sum_{k=1}^{n} v_k \Bigg\{ d_2 S_2^* \Bigg[p_{2k} + 2p_{1k} \\
& - p_{2k} \left(\frac{S_1^*}{S_1} + \frac{S_1S_2^*}{S_1^*S_2} + \frac{S_2}{S_2^*} \right) \Bigg] + \sum_{j=1}^{n} \beta_{2j} S_2^* G_j(I_j^*) \Bigg[2p_{1k} \\
& + p_{2k} - p_{2k} \left(\frac{S_1^*}{S_1} + \frac{S_1S_2^*}{S_1^*S_2} + \frac{S_2 G_k(I_k^*) G_j(I_j)}{S_2^* G_k(I_k) G_j(I_j^*)} \right) \Bigg] \\
& + \sum_{j=1}^{n} \beta_{1j} S_1^* G_j(I_j^*) \\
& \cdot \left[2p_{1k} - p_{1k} \left(\frac{S_1^*}{S_1} + \frac{S_1 G_k(I_k^*) G_j(I_j)}{S_1^* G_k(I_k) G_j(I_j^*)} \right) \right] \Bigg\} =: B_1.
\end{aligned} \tag{21}$$

By $\sum_{k=1}^{n} v_k(p_{1k} - p_{2k}) \ge 0$ and the arithmetic-geometric mean, we easily see that

$$\begin{aligned}
B_1 \le & -\sum_{k=1}^{n} v_k (2p_{1k} - 2p_{2k}) \Bigg(d_2 S_2^* \\
& + \sum_{j=1}^{n} \beta_{2j} S_2^* G_j(I_j^*) \Bigg) + \sum_{k=1}^{n} v_k \Bigg\{ d_2 S_2^* [p_{2k} + 2p_{1k} \\
& - 3p_{2k}] + \sum_{j=1}^{n} \beta_{2j} S_2^* G_j(I_j^*) \\
& \cdot \left[2p_{1k} + p_{2k} - 3p_{2k} \left[\frac{G_k(I_k^*) G_j(I_j)}{G_k(I_k) G_j(I_j^*)} \right]^{1/3} \right] \\
& + \sum_{j=1}^{n} \beta_{1j} S_1^* G_j(I_j^*) \\
& \cdot \left[2p_{1k} - 2p_{1k} \left[\frac{G_k(I_k^*) G_j(I_j)}{G_k(I_k) G_j(I_j^*)} \right]^{1/2} \right] \Bigg\} \\
= & \sum_{k=1}^{n} v_k \Bigg\{ \sum_{j=1}^{n} \beta_{2j} S_2^* G_j(I_j^*) \Bigg[3p_{2k}
\end{aligned}$$

$$- 3p_{2k}\left[\frac{G_k(I_k^*)G_j(I_j)}{G_k(I_k)G_j(I_j^*)}\right]^{1/3}\Bigg] + \sum_{j=1}^{n}\beta_{1j}S_1^*G_j(I_j^*) \cdot \left[2p_{1k} - 2p_{1k}\left[\frac{G_k(I_k^*)G_j(I_j)}{G_k(I_k)G_j(I_j^*)}\right]^{1/2}\right]\Bigg\} =: B_2. \tag{22}$$

We can rewrite B_2 as

$$\begin{aligned} B_2 = \sum_{k=1}^{n} v_k \Bigg\{ & 3\sum_{j=1}^{n} p_{2k}\beta_{2j}S_2^*G_j(I_j^*) \cdot \left[1 - \left[\frac{G_k(I_k^*)G_j(I_j)}{G_k(I_k)G_j(I_j^*)}\right]^{1/3} + \ln\left[\frac{G_k(I_k^*)G_j(I_j)}{G_k(I_k)G_j(I_j^*)}\right]^{1/3}\right] \\ & + 2\sum_{j=1}^{n} p_{1k}\beta_{1j}S_1^*G_j(I_j^*) \cdot \left[1 - \left[\frac{G_k(I_k^*)G_j(I_j)}{G_k(I_k)G_j(I_j^*)}\right]^{1/2} + \ln\left[\frac{G_k(I_k^*)G_j(I_j)}{G_k(I_k)G_j(I_j^*)}\right]^{1/2}\right] \\ & - \sum_{i}^{2}\sum_{j=1}^{n} p_{ik}\beta_{ij}S_i^*G_j(I_j^*)\ln\left[\frac{G_k(I_k^*)G_j(I_j)}{G_k(I_k)G_j(I_j^*)}\right]\Bigg\}. \end{aligned} \tag{23}$$

Using the fact that $1 - x + \ln x \leq 0$, where equality holds if only if $x = 1$, we obtain

$$\begin{aligned} B_2 & \leq -\sum_{k=1}^{n} v_k \sum_{i}^{2}\sum_{j=1}^{n} p_{ik}\beta_{ij}S_i^*G_j(I_j^*)\ln\left[\frac{G_k(I_k^*)G_j(I_j)}{G_k(I_k)G_j(I_j^*)}\right] \\ & = \sum_{k,j=1}^{n} v_k\overline{\beta}_{kj}\ln\frac{G_k(I_k)G_j(I_j^*)}{G_k(I_k^*)G_j(I_j)}. \end{aligned} \tag{24}$$

In the following, we will show that

$$\overline{H}_n := \sum_{k,j=1}^{n} v_k\overline{\beta}_{kj}\ln\frac{G_k(I_k)G_j(I_j^*)}{G_k(I_k^*)G_j(I_j)} \equiv 0. \tag{25}$$

We first give the proof of (25) for $n = 2$, which would give a reader the basic yet clear ideas without being hidden by the complexity of terms caused by larger values of n. When $n = 2$, we have $\overline{H}_2 = \sum_{k,j=1}^{2} v_k\overline{\beta}_{kj}\ln(G_k(I_k)G_j(I_j^*)/G_k(I_k^*)G_j(I_j))$. Formula (13) gives $v_1 = \overline{\beta}_{21}$ and $v_2 = \overline{\beta}_{12}$ in this case. Expanding $\overline{H}_2$ yields

$$\begin{aligned} \overline{H}_2 & = \overline{\beta}_{21}\overline{\beta}_{11}\ln\frac{G_1(I_1)G_1(I_1^*)}{G_1(I_1^*)G_1(I_1)} + \overline{\beta}_{12}\overline{\beta}_{22}\ln\frac{G_2(I_2)G_2(I_2^*)}{G_2(I_2^*)G_2(I_2)} \\ & \quad + \overline{\beta}_{21}\overline{\beta}_{12}\ln\frac{G_1(I_1)G_2(I_2^*)}{G_1(I_1^*)G_2(I_2)} + \overline{\beta}_{12}\overline{\beta}_{21}\ln\frac{G_2(I_2)G_1(I_1^*)}{G_2(I_2^*)G_1(I_1)} \\ & = \overline{\beta}_{12}\overline{\beta}_{21}\left[\ln\frac{G_1(I_1)G_2(I_2^*)}{G_1(I_1^*)G_2(I_2)} + \ln\frac{G_2(I_2)G_1(I_1^*)}{G_2(I_2^*)G_1(I_1)}\right] \\ & = 0. \end{aligned} \tag{26}$$

For more general n, by a similar argument as in the proof of $\sum_{k,j=1}^{n} v_k\overline{\beta}_{kj}\ln(E_k^*E_j/E_kE_j^*) \equiv 0$ in [16], we obtain $\sum_{k,j=1}^{n} v_k\overline{\beta}_{kj}\ln(G_k(I_k)G_j(I_j^*)/G_k(I_k^*)G_j(I_j)) \equiv 0$.

From (21), (22), and (24), we see that if $\dot{V} = 0$, then

$$S_i = S_i^*, \quad i = 1, 2. \tag{27}$$

If (27) holds, it follows from (1) that

$$\begin{aligned} 0 & = \varphi(S_1^*) - \sum_{j=1}^{n}\beta_{1j}S_1^*G_j(I_j) - aS_1^*, \\ 0 & = aS_1^* - \sum_{j=1}^{n}\beta_{2j}S_2^*G_j(I_j) - d_2S_2^*. \end{aligned} \tag{28}$$

Then, we obtain that

$$\dot{I}_k = p_{1k}(\varphi(S_1^*) - aS_1^*) + p_{2k}(aS_1^* - d_2S_2^*) - m_kI_k, \quad k = 1, 2, \ldots, n. \tag{29}$$

This implies that

$$\begin{aligned} \lim_{t\to+\infty} I_k & = \frac{p_{1k}(\varphi(S_1^*) - aS_1^*) + p_{2k}(aS_1^* - d_2S_2^*)}{m_k} \\ & = I_k^*. \end{aligned} \tag{30}$$

By the characteristics of V, we obtain the idea that the largest invariant subset of the set where $\dot{V} = 0$ is the singleton $\{P^*\}$. By LaSalle's Invariance Principle, P^* is globally asymptotically stable for $R_0 > 1$. This completes the proof. □

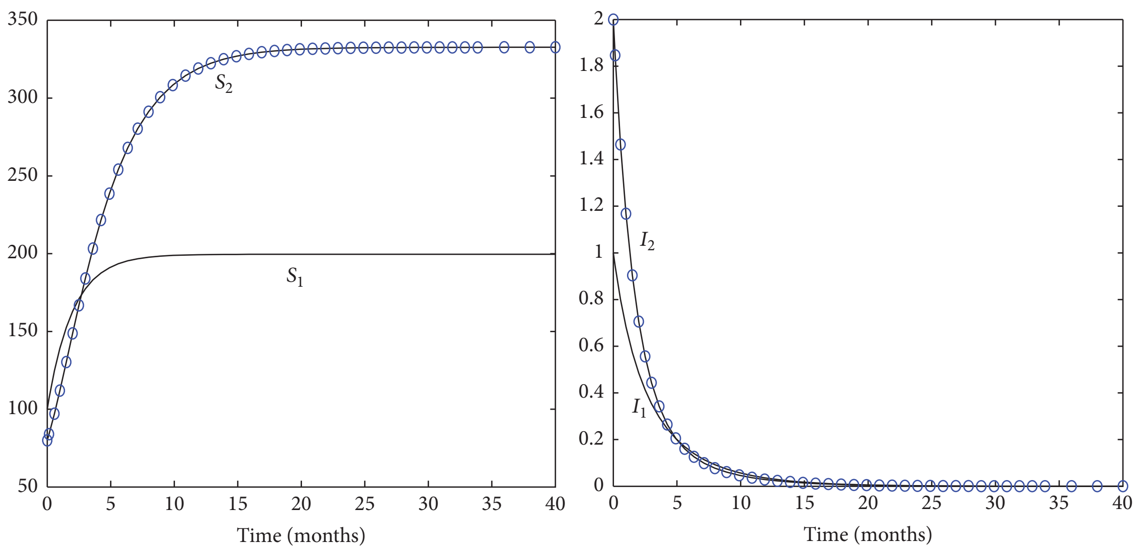

FIGURE 1: Dynamical behavior of system (31) with parameter values in (32) and Case 1. $R_0 \approx 0.55$. The initial conditions are $S_1(0) = 100$, $S_2(0) = 80$, $I_1(0) = 1$, and $I_2(0) = 2$. We see that the disease dies out in both subgroups.

4. Numerical Examples

In the section, numerical simulations are presented to support and complement the theoretical findings. We consider the following model:

$$\begin{aligned}
\dot{S_1} &= \varphi(S_1) - \sum_{j=1}^{2} \beta_{1j} S_1 G_j(I_j) - aS_1, \\
\dot{S_2} &= aS_1 - \sum_{j=1}^{2} \beta_{2j} S_2 G_j(I_j) - d_2 S_2, \\
\dot{I_1} &= p_{11} \sum_{j=1}^{2} \beta_{1j} S_1 G_j(I_j) + p_{21} \sum_{j=1}^{2} \beta_{2j} S_2 G_j(I_j) - m_1 I_1, \\
\dot{I_2} &= p_{12} \sum_{j=1}^{2} \beta_{1j} S_1 G_j(I_j) + p_{22} \sum_{j=1}^{2} \beta_{2j} S_2 G_j(I_j) - m_2 I_2,
\end{aligned} \tag{31}$$

where $G_j(I_j) = I_j/(1 + \alpha_j I_j)$. Clearly, (A1) and (A2) hold. We fix the parameters as follows:

$$\begin{aligned}
b &= 100, \\
d_1 &= 0.001, \\
d_2 &= 0.3, \\
a &= 0.5, \\
\alpha_1 &= \alpha_2 = 0.1, \\
m_1 &= 0.5, \\
m_2 &= 0.6, \\
p_{11} &= p_{22} = 0.6, \\
p_{12} &= p_{21} = 0.4.
\end{aligned} \tag{32}$$

Then, we have $P_0 \approx (199.6008, 332.6680, 0, 0)$.

Case 1. If $\beta_{11} = \beta_{21} = 0.001$, $\beta_{12} = \beta_{22} = 0.0001$, then we obtain $R_0 \approx 0.55$. By Theorem 1, the disease dies out in both subgroups. Numerical simulation illustrates this fact (see Figure 1).

Case 2. If $\beta_{11} = \beta_{21} = 0.01$, $\beta_{12} = \beta_{22} = 0.001$, then we have $R_0 \approx 5.53$, $P^* \approx (171.0530, 222.9491, 32.0764, 28.1767)$, $v_1 = \overline{\beta}_{21} = \sum_{i=1}^{2} p_{i2}\beta_{i1}S_i^* G_1(I_1^*) \approx 154.06$, $v_2 = \overline{\beta}_{12} = \sum_{i=1}^{2} p_{i1}\beta_{i2}S_i^* G_2(I_2^*) \approx 73.81$, and $\sum_{k=1}^{2} v_k(p_{1k} - p_{2k}) \approx 16.05$. By Theorem 2, the disease persists in both subgroups. Numerical simulation illustrates this fact (see Figure 2).

5. Conclusions

A differential infectivity epidemic model with stage structure has been used to describe the spreading of such a disease. We have focused on the theoretical analysis of the equilibriums. Using a graph-theoretic approach to the method of Lyapunov functions, we have proved the global stability of the endemic equilibrium. We have established uniform persistence and the sharp threshold. The work has potential extensions and improvements, which remains to be discussed in the future.

Competing Interests

The author declares that there are no competing interests.

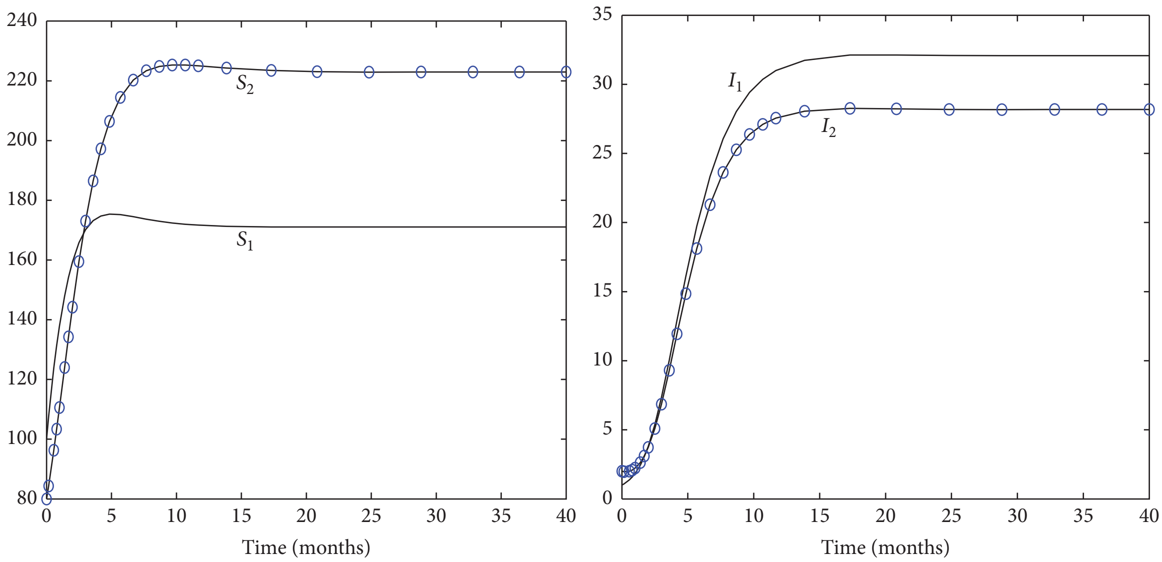

Figure 2: Dynamical behavior of system (31) with parameter values in (32) and Case 1. $R_0 \approx 5.53$. The initial conditions are $S_1(0) = 100$, $S_2(0) = 80$, $I_1(0) = 1$, and $I_2(0) = 2$. We see that the disease persists in both subgroups.

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