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Research Article

Eigenvalue for Densely Defined $(S_+)_L$ Perturbations of Multivalued Maximal Monotone Operators in Reflexive Banach Spaces

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Let X be a real reflexive Banach space and let X^* be its dual. Let $\Omega \subset X$ be open and bounded such that $0 \in \Omega$. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone with $0 \in D(T)$ and $0 \in T(0)$. Using the topological degree theory developed by Kartsatos and Quarcoo we study the eigenvalue problem $Tx + \lambda Px \ni 0$, where the operator $P : X \supset D(P) \to X^*$ is a single-valued of class $(S_+)_L$. The existence of continuous branches of eigenvectors of infinite length then could be easily extended to the case where the operator $P : X \to 2^{X^*}$ is multivalued and is investigated.

1. Preliminaries

In what follows we assume that X is a real or complex Banach space and has been renormed such that it and its dual X^* are locally uniformly convex. The normalized duality mapping is defined by

$$J(x) = \left\{x^* \in X : \|x^*\| = \|x\|, \langle x, x^* \rangle = \|x\|^2\right\}. \quad (1)$$

The mapping $T : X \supset D(T) \to 2^{X^*}$ is said to be "monotone" if for every $x, y \in D(T)$, $u \in Tx$, and $v \in Ty$ we have

$$\langle u - v, x - y \rangle \geq 0. \quad (2)$$

A monotone operator T is "maximal monotone" if $G(T)$ is maximal in $X \times X^*$, when $X \times X^*$ is partially ordered by inclusion. In our setting, a monotone operator T is maximal if and only if $R(T + \lambda J) = X^*$ for every $\lambda > 0$. If $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone operator, the operator $T_t \equiv (T^{-1} + tJ^{-1})^{-1} : X \to X^*$ is called the Yosida approximant of T and the following, which can be found in [1, page 102] is true.

Lemma 1. *Let $T : X \supset D(T) \to 2^{X^*}$ be a maximal monotone operator with $0 \in D(T)$ and $0 \in T(0)$. Then*

(i) *T_t is a bounded maximal monotone mapping with $0 = T_t(0)$ for each $t > 0$;*

(ii) *$T_t x \rightharpoonup T^{\{0\}} x$ in X as $t \to 0$ for all $x \in D(T)$, where $T^{\{0\}} x$ denotes the element $y^* \in Tx$ of minimum norm;*

(iii) *$\|T_t x\| \to \infty$ as $t \to 0$ for all $x \notin \overline{D(T)}$.*

Also $T_t x \in TJ_t x$, where $J_t = I - tJ^{-1}T_t : X \to X$ and satisfies: if $x \in \overline{\operatorname{conv} D(T)}$, then $J_t x \to x$ in X as $t \to 0$, where $\operatorname{conv} M$ denote the convex hull of the set M.

The proof of the next lemma can be found in Kartsatos and Skrypnik [2],

Lemma 2. *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone such that $0 \in D(T)$ and $0 \in T(0)$. Then the mapping $(t, x) \to T_t x$ is continuous on the set $(0, \infty) \times X$.*

An operator $T : X \supset D(T) \to Y$, with Y another Banach space, is bounded if it maps bounded subsets of $D(T)$ onto bounded sets. It is compact if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact subsets of Y. It is demicontinuous if for every sequence $\{x_n\}$ such that $x_n \to x_0$ we have $Tx_n \rightharpoonup Tx_0$.

We say that the operator $T : X \supset D(T) \to 2^{X^*}$ satisfies condition (S_q) on a set $A \subset D(T)$ if for every sequence $\{x_n\} \subset A$ such that $x_n \rightharpoonup x_0 \in X$ and any $y_n^* \in Tx_n$, with $y_n^* \to$ (some) $y^* \in X^*$, we have $x_n \to x_0$. If $A = D(T)$, then we say that T satisfies (S_q).

Definition 3. Let X be a separable reflexive Banach space and let L be a dense subspace of X. A mapping $P : D(P) \subset X \to X^*$ is said to be of class $(S_+)_L$ if, for any sequence of finite dimensional subspaces F_n of L with $\overline{U_{n=1}^{\infty} F_n} = X$, $h \in X^*$, $\{x_n\}_{n=1}^{\infty} \subset D(P)$ with $x_n \rightharpoonup x_0$ and

$$\limsup_{n\to\infty} \langle Px_n - h, x_n \rangle \le 0, \qquad \lim_{n\to\infty} \langle Px_n - h, v \rangle = 0 \tag{3}$$

for all $v \in U_{n=1}^{\infty} F_n$, we have $x_n \to x_0$, $x \in D(P)$ and $Px_0 = h$.

If $h = 0$, then we call P a mapping of class $(S_+)_{0,L}$.

Definition 4. Let X be a separable reflexive Banach space and let L be a dense subspace of X. A multivalued mapping $P : D(P) \subset X \to 2^{X^*}$ is said to be of class $(S_+)_L$ if it satisfies the following conditions:

(i) Px is bounded closed and convex for each $x \in D(P)$,

(ii) P is weakly upper semicontinuous in each finite dimensional space, that is, for each finite dimensional space F of L, $F \cap D(P) \neq \emptyset$, $P : F \cap D(P) \to 2^{X^*}$ is upper semicontinuous in the weak topology,

(iii) if for any sequence of finite dimensional subspaces F_n of L with $L \subset \overline{U_{n=1}^{\infty} F_n}$, $h \in X^*$, $\{x_n\}_{n=1}^{\infty} \subset D(P) \cap L$ and $x_n \rightharpoonup x_0$ such that

$$\limsup_{n\to\infty} \langle p_n - h, x_n \rangle \le 0, \qquad \lim_{n\to\infty} \langle p_n - h, v \rangle = 0 \tag{4}$$

for all $v \in U_{n=1}^{\infty} F_n$ and some $p_n \in Px_n$, then we have $x_n \to x_0$, $x \in D(P)$ and $h \in Px_0$.

If $h = 0$, then we call P a mapping of class $(S_+)_{0,L}$.

We will need the following two conditions:

$(P1)$ there exist a subspace L of X such that $L \subset D(P)$, $\overline{L} = X$ and that the operator P satisfies condition $(S_+)_L$;

$(P2)$ there exist a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing such that

$$\langle p, x \rangle \ge -\phi(\|x\|), \quad p \in Px, \; x \in D(P). \tag{5}$$

The following lemma can be found in Zeidler [3, page 915].

Lemma 5. *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone. Thus the following are true:*

(i) *$\{x_n\} \subset D(T) x_n \to x_0$ and $Tx_n \ni y_n \rightharpoonup y_0$ imply $x_0 \in D(T)$, and $y_0 \in Tx_0$;*

(ii) *$\{x_n\} \subset D(T) x_n \rightharpoonup x_0$ and $Tx_n \ni y_n \to y_0$ imply $x_0 \in D(T)$, and $y_0 \in Tx_0$.*

We will need the following lemma from Adhikari and Kartsatos [4].

Lemma 6. *Assume that the operators $T : X \supset D(T) \to 2^{X^*}$ and $T_0 : X \supset D(T_0) \to X^*$ are maximal monotone, with $0 \in D(T) \cap D(T_0)$ and $0 \in T(0) \cap T_0(0)$. Assume further that $T + T_0$ is maximal monotone. Assume that there is a positive sequence $\{t_n\}$ such that $t_n \downarrow 0$, a sequence $\{x_n\} \subset D(T_0)$ and a sequence $w_n \in T_0 x_n$ such that $x_n \rightharpoonup x_0$ and $T_{t_n} x_n + w_n \rightharpoonup y_0^* \in X^*$. Then the following are true:*

(i) *the inequality*

$$\lim_{n\to\infty} \left\langle T_{t_n} x_n + w_n, x_n - x_0 \right\rangle < 0 \tag{6}$$

is impossible;

(ii) *if*

$$\lim_{n\to\infty} \left\langle T_{t_n} x_n + w_n, x_n - x_0 \right\rangle = 0, \tag{7}$$

then $x_0 \in D(T + T_0)$ and $y^* \in (T + T_0) x_0$.

The proof of the following lemma is given in [5] proof of Theorem 7 but we will repeat it here for completeness and future reference.

Lemma 7. *Let $T : X \supset D(T) \to 2^{X^*}$ be a quasibounded maximal monotone operator such that $0 \in T(0)$. Let $\{t_n\} \subset (0, \infty)$ and $\{t_n\} \subset X$ be such that*

$$\|u_n\| \le S, \quad \left\langle T_{t_n} u_n, u_n \right\rangle \le S_1, \tag{8}$$

where S, S_1 are positive constants. Then there exists a number $K > 0$ such that $\|T_{t_n} u_n\| \le K$ for all $n = 1, 2, \ldots$.

Proof. Let

$$w_n = T_{t_n} u_n = \left(T^{-1} + t_n J^{-1}\right)^{-1} u_n. \tag{9}$$

We have that

$$w_n \in T J_{t_n} u_n = T x_n, \quad t_n w_n = J\left(u_n - x_n\right), \tag{10}$$

where $x_n = J_{t_n} u_n$. Thus,

$$\begin{aligned} \langle w_n, x_n \rangle &= \left\langle w_n, u_n - t_n J^{-1} w_n \right\rangle \\ &= \langle w_n, u_n \rangle - t_n \left\langle w_n, J^{-1} w_n \right\rangle \\ &= \langle w_n, u_n \rangle - t_n \|w_n\|^2 \\ &\le \left\langle T_{t_n} u_n, u_n \right\rangle \\ &\le S_1. \end{aligned} \tag{11}$$

From (11) we obtain

$$t_n \|w_n\|^2 = \langle w_n, u_n \rangle - \langle w_n, x_n \rangle. \tag{12}$$

Now since $0 \in T(0)$ and $w_n \in Tx_n$, we have that $\langle w_n, x_n \rangle \ge 0$, which implies $t_n \|w_n\|^2 \le S_1$. We claim that $\{w_n\}$ is bounded, if not we may assume that $\|w_n\| \to \infty$ and $\|w_n\| \le \|w_n\|^2$ for all n. Thus $t_n\|w_n\| \le S_1$ and

$$t_n \|w_n\| = \|J(u_n - x_n)\| = \|u_n - x_n\| \tag{13}$$

which implies that $\{x_n\}$ is bounded. Now since T is strongly quasibounded, the boundedness of $\{x_n\}$ and $\{\langle w_n, x_n\rangle\}$ imply the boundedness of $\{w_n\}$, that is, a contradiction. It follows that $\{T_{t_n} u_n\}$ is bounded and the proof is complete. □

We denote by J_ψ the duality mapping with gauge function ψ. The function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing such that $\psi(0) = 0$ and $\psi(r) \to \infty$ as $r \to \infty$. This mapping J_ψ is continuous, bounded, surjective, strictly and maximal monotone, and satisfies condition (S_+). Also,

$$\langle J_\psi x, x\rangle = \psi(\|x\|)\|x\|, \quad \|J_\psi x\| = \psi(\|x\|), \quad x \in X \tag{14}$$

for these facts we refer to Petryshyn [6, pages 32-33 and 132].

2. The Eigenvalue Result

In this section we are using the topological degree developed by Kartsatos and Quarcoo in [7]. The following result will also improve Theorem 4 of Kartsatos and Skrypnik in [8] since we are no longer assuming that the perturbation is quasibounded.

Theorem 8. *Let Ω be open and bounded with $0 \in \Omega$. Assume that the operator $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone with $0 \in D(T)$ and $0 \in T(0)$. Assume that the operator $P : X \supset D(P) \to X^*$, with $L \subset D(P)$, satisfies condition $(S_+)_L$ and (P2). Let ϵ, ϵ_0, and Λ be positive numbers. Assume that*

($\mathscr{P}$) *there exists $\lambda \in (0, \Lambda]$ such that*

$$Tx + \lambda Px + \epsilon J_\psi x \ni 0 \tag{15}$$

has no solution in $D(T + P) \cap \Omega$.

Then

(i) *there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T + P) \cap \partial\Omega)$ such that*

$$Tx_0 + \lambda_0 Px_0 + \epsilon J_\psi x_0 \ni 0; \tag{16}$$

(ii) *if $0 \notin T(D(T) \cap \partial\Omega)$, T satisfies (S_q) on $\partial\Omega$, and property ($\mathscr{P}$) is satisfied for every $\epsilon \in (0, \epsilon_0]$, then there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T + P) \cap \partial\Omega)$ such that $Tx_0 + \lambda_0 Px_0 \ni 0$.*

Proof. (i) Assume that ($\mathscr{P}$) is true and that (16) is false. We consider the homotopy inclusion

$$H(t, x) \equiv Tx + t\Lambda Px + \epsilon J_\psi x \ni 0, \quad t \in [0, 1]. \tag{17}$$

It is clear that this inclusion has no solution $x \in D(H(t, \cdot)) \cap \partial\Omega$ for $t \in (0, 1]$, because $t\Lambda \in (0, \Lambda]$ and our assumption about (16). It is also true for $t = 0$ because $0 \in (T + \epsilon J_\psi)(0)$ and the operator $T + \epsilon J_\psi$ is strictly monotone, hence one-to-one.

We are going to show that $H(t, x)$ is an admissible homotopy for the degree in [4]. To this end, we set $T_t = T$, $P^t \equiv t\Lambda P + J_\psi$ and we recall the operators $T_{t,s} = T_s \equiv (T^{-1} + sJ^{-1})^{-1} : X \to X^*$, $s > 0$, and we set $J_{t,s} = J_s = I - sJ^{-1}T_{t,s} = I - sJ^{-1}T_s : X \to X$. We have $D(H(0, \cdot)) = D(T)$ and $D(H(t, \cdot)) = D(T + P)$, $t \in (0, 1]$. We also set $D^t = D(t\Lambda P) = D(tP)$. We have $D^0 = X$ and $D^t = D(P)$ for $t \in (0, 1]$. Let Ω be an open and bounded subset of X. We know that the equation

$$T^t x + P^t x \ni 0 \tag{18}$$

has no solution $x \in D(H(t, \cdot)) \cap \partial\Omega$ for any $t \in [0, 1]$. Now we consider the equation

$$T_s x + t\Lambda Px + \epsilon J_\psi = 0, \tag{19}$$

and we show that there exists $s_1 > 0$ such that

$$0 \notin (T_s + t\Lambda P + \epsilon J_\psi)(D^t \cap \partial\Omega), \quad (s, t) \in (0, s_1] \times [0, 1]. \tag{20}$$

Assume that this is not true, then there exist $\{s_n\} \subset (0, \infty)$ with $s_n \downarrow 0$, $\{t_n\} \subset [0, 1]$, with $t_n \to t_0$, $\{x_n\} \subset \partial\Omega$ with $x_n \rightharpoonup x_0$, and

$$T_{s_n} x_n + t_n \Lambda Px_n + \epsilon J_\psi x_n = 0. \tag{21}$$

Clearly we cannot have $t_n = 0$ for any n, since $(T_{s_n} + J_\psi)(0) = 0$ and the operator $T_{s_n} + J_\psi$ is strictly monotone, hence one-to-one. Thus $t_n > 0$, for all n. From (21) we have that

$$\begin{aligned} \langle T_{s_n} x_n, x_n\rangle &= -t_n\Lambda \langle Px_n, x_n\rangle - \epsilon \langle J_\psi x_n, x_n\rangle \\ &\le t_n \Lambda \phi(\|x_n\|) - \epsilon\psi(\|x_n\|) \\ &\le \Lambda\phi(S_1) + \epsilon\psi(S_1), \end{aligned} \tag{22}$$

where S_1 is the bound of $\{x_n\}$. Thus we have the boundedness of $\langle T_{s_n} x_n, x_n\rangle$. Using Lemma 7, we have the boundedness of $T_{s_n} x_n$. We may thus assume that $T_{s_n} x_n \rightharpoonup h_1^*$. From (21) we also have that the sequence $\{Px_n\}$ is bounded and we may assume that $Px_n \rightharpoonup h_0^*$.

If $t_0 = 0$, then from

$$\epsilon\psi(\|x_n\|)\|x_n\| = \langle \epsilon J_\psi x_n, x_n\rangle \le -t_n\Lambda \langle Px_n, x_n\rangle \longrightarrow 0 \tag{23}$$

we obtain $x_n \to 0 \in \partial\Omega$, which is a contradiction to $0 \in \Omega$. Hence it follows that $t_0 > 0$. Since $\{J_\psi x_n\}$ is bounded, we may assume that $J_\psi x_n \rightharpoonup h_2^*$ and we have $t_0\Lambda h_0^* = -h_1^* - \epsilon h_2^*$.

From (21) it follows that

$$\langle T_{s_n} x_n + t_n\Lambda Px_n + \epsilon J_\psi x_n, x\rangle = 0, \quad \forall x \in U_{n=1}^\infty F_n. \tag{24}$$

Thus

$$\begin{gathered} \langle T_{s_n} x_n + t_n\Lambda Px_n + \epsilon J_\psi x_n, x_n\rangle = 0, \\ \lim_{n\to\infty} \langle T_{s_n} x_n + t_n\Lambda Px_n + \epsilon J_\psi x_n, v\rangle = 0, \quad \forall v \in U_{n=1}^\infty F_n. \end{gathered} \tag{25}$$

Now

$$\begin{aligned}\left\langle t_n \Lambda P x_n + h_1^* + \epsilon h_2^*, x_n \right\rangle = {} & \left\langle T_{s_n} x_n + t_n \Lambda P x_n + \epsilon J_\psi x_n, x_n \right\rangle \\ & - \left\langle T_{s_n} x_n + \epsilon J_\psi x_n, x_n - x_0 \right\rangle \\ & - \left\langle T_{s_n} x_n + \epsilon J_\psi x_n, x_0 \right\rangle \\ & + \left\langle h_1^* + \epsilon h_2^*, x_n \right\rangle,\end{aligned} \tag{26}$$

hence

$$\begin{aligned}& \limsup_{n\to\infty} \left\langle t_n \Lambda P x_n + h_1^* + \epsilon h_2^*, x_n \right\rangle \\ & \quad \le \limsup_{n\to\infty} \left\langle T_{s_n} x_n + \epsilon J_\psi x_n + t_n \Lambda P x_n, x_n \right\rangle \\ & \qquad - \liminf_{n\to\infty} \left\langle T_{s_n} x_n + \epsilon J_\psi x_n, x_n - x_0 \right\rangle \\ & \qquad - \left\langle h_1^* + \epsilon h_2^*, x_0 \right\rangle + \left\langle h_1^* + \epsilon h_2^*, x_0 \right\rangle \\ & \quad \le -\liminf_{n\to\infty} \left\langle T_{s_n} x_n + \epsilon J_\psi x_n, x_n - x_0 \right\rangle \\ & \quad \le 0,\end{aligned} \tag{27}$$

where the last inequality follows from Lemma 6. Now since P is of class $(S_+)_L$, it follows that $x_n \to x_0$, $x_0 \in D(P)$ and $t_0 \Lambda P x_0 = -h_1^* - \epsilon h_2^*$. Also we have that

$$\left\langle T_{s_n} x_n, x_n - x_0 \right\rangle = 0 \tag{28}$$

and using Lemma 6 we get that $x_0 \in D(T)$ and $T x_0 \ni h_1^* = -t_0 \Lambda P x_0 - \epsilon J_\psi x_0$. This is a contradiction since $x_n \to x_0$, we have $x_0 \in \partial\Omega$. We have shown that $H(t, x)$ is an admissible homotopy for our degree. We can now work as in Theorem 3 in [8] in order to show that $d(H(t, \cdot), \Omega, 0) = \text{const}$.

Thus

$$d(H(t, \cdot), \Omega, 0) = d\left(T_s + \epsilon J_\psi, \Omega, 0\right) = 1, \tag{29}$$

where the last equality follows from Theorem 3, (i) of [9].

Consequently, the inclusion $H(t, x) \ni 0$ has a solution in Ω for each $t \in [0, 1]$. In particular, this says that $Tx + \lambda Px + \epsilon J_\psi x \ni 0$ has a solution in Ω for every $\lambda > 0$. This of course contradicts condition $(\mathscr{P})$ and finishes the proof of (i).

(ii) Let $\lambda_n \in (0, \Lambda]$, $x_n \in D(P) \cap \partial\Omega$ be such that, for some $u_n^* \in T x_n$,

$$u_n^* + \lambda_n P x_n + \left(\frac{1}{n}\right) J_\psi x_n = 0 \tag{30}$$

again, $\lambda_n = 0$ for any n is not possible. Since $\lambda_n > 0$, we have by property $(P2)$ that

$$\begin{aligned}\left\langle u_n^*, x_n \right\rangle & = -\lambda_n \left\langle P x_n, x_n \right\rangle - \frac{1}{n} \left\langle J_\psi x_n, x_n \right\rangle \\ & \le \lambda_n \phi\left(\|x_n\|\right) - \frac{1}{n} \psi\left(\|x_n\|\right) \|x_n\| \\ & \le \Lambda \phi\left(S_1\right) + \psi\left(S_1\right) S_1,\end{aligned} \tag{31}$$

where S_1 is the bound of $\{x_n\}$. This show that $\langle u_n^*, x_n \rangle$ is bounded and further we obtain the boundedness of $\{u_n^*\}$ by Lemma 2. We may thus assume that $\lambda_n \to \lambda_0$, $x_n \rightharpoonup x_0$, $u_n^* \rightharpoonup h_0^*$, then $\lambda_n P x_n \rightharpoonup -h_0^*$.

If $\lambda_0 = 0$, then (30) implies that

$$\lim_{n\to\infty} u_n^* = \lim_{n\to\infty} \left[-\lambda_n P x_n - \left(\frac{1}{n}\right) J_\psi x_n\right] = 0. \tag{32}$$

Since T satisfies (S_q), it follows that $x_n \to x_0 \in \partial\Omega$. Now, by the demiclosedness of T (see Lemma 6) we obtain that $x_0 \in D(T)$, $0 \in T x_0$ and this contradicts $0 \notin T(D(T) \cap \partial\Omega)$. Hence, $\lambda_0 > 0$. Repeating the proof of (i) starting from (21), we get again that

$$\limsup_{n\to\infty} \left\langle \lambda_n P x_n + h_0^*, x_n \right\rangle \le 0 \tag{33}$$

and since P is of class $(S_+)_L$, we have $x_n \to x_0$, $-h_0^* = \lambda_0 P x_0$. Using the demiclosedness of T, we obtain $x_0 \in D(T) \cap \partial\Omega$, $h_0 \in T x_0$ and $T x_0 + \lambda_0 P x_0 \ni 0$. The proof is now complete. □

3. Continuous Branches of Eigenvectors

In this section we are interested in showing that the results obtained in previous sections could be used in order to obtain the existence of continuous branches of eigenvectors. We need the following definition.

Definition 9. Let $T : X \supset D(T) \to 2^{X^*}$, $P : X \supset D(P) \to X^*$ be given and consider the problem

$$Tx + \lambda Px \ni 0. \tag{34}$$

An eigenvector x is solution of (34) for some eigenvalue λ with $x \in (D(T) \cap D(P))$. We say that the nonzero eigenvectors of the problem (34) form a continuous branch of infinite length if there exists $r_0 > 0$ such that, for every $r \ge r_0$, the sphere $\partial B_r(0)$ contains at least one nonzero eigenvector of (34).

Theorem 10. *Assume that the operator $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone with $0 \in D(T)$ and $0 \in T(0)$. Assume that the operator $P : X \supset D(P) \to X^*$ with $L \subset D(P)$ is of class $(S_+)_L$ and satisfies $(P2)$. Let Λ be a positive number. Assume that $Tx \ni 0$ implies $x = 0$, T satisfies (S_q) and*

$(\mathscr{P}1)$ there exists $\alpha > 0$ and $\lambda \in (0, \Lambda]$ such that

$$|Tx + \lambda Px| \ge \alpha, \quad x \in D(T + P). \tag{35}$$

Then the nonzero eigenvectors of problem (34) form a continuous branch of infinite length with corresponding eigenvalues $\lambda \in (0, \Lambda]$.

Proof. Let $r_0 > 0$ be given. Let $\epsilon_0 > 0$ be so small that $\epsilon_0 r_0 < \alpha$. Then

$$\begin{aligned}|Tx + \lambda Px + \epsilon Jx| & \ge |Tx + \lambda Px| - |\epsilon Jx| \\ & \ge \alpha - \epsilon \|x\| \\ & \ge \alpha - \epsilon_0 r_0,\end{aligned} \tag{36}$$

$x \in D(T+P)$, which implies that the inclusion

$$Tx + \lambda Px + \epsilon Jx \ni 0 \tag{37}$$

has no solution $x \in D(T+P) \cap B_r(0)$ for any $\epsilon \in (0, \epsilon_0]$. Since $0 \notin T(\partial B_{r_0}(0))$ and T is of type (S_q), Theorem 8 implies the existence of a solution $x_{\lambda_0} \in D(T+P) \cap \partial B_{r_0}(0)$, for some $\lambda_0 \in (0, \Lambda]$. The same argument can be repeated for any $r > r_0$. This complete the proof. □

Remark 11. This result is also true when $P : X \supset D(P) \to 2^{X^*}$ is assumed to be multivalued. We have the following theorem without proof.

Theorem 12. *Assume that the operator $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone with $0 \in D(T)$ and $0 \in T(0)$. Assume that the operator $P : X \supset D(P) \to 2^{X^*}$ with $L \subset D(P)$ is of class $(S_+)_L$ and satisfies (P2). Let Λ be a positive number. Assume that $Tx \ni 0$ implies $x = 0$, T satisfies (S_q), and*

($\mathscr{P}$1) there exists $\alpha > 0$ and $\lambda \in (0, \Lambda]$ such that

$$|Tx + \lambda Px| \geq \alpha, \quad x \in D(T+P). \tag{38}$$

Then the nonzero eigenvectors of problem (34) form a continuous branch of infinite length with corresponding eigenvalues $\lambda \in (0, \Lambda]$.

In the following result, we assume that the operator T is defined and bounded on all of X. In this case we demonstrate the fact that the assumption $|Tx + Px| \geq 0$ may be replaced by the assumption $\|Px\| \geq 0$ on $D(P)$.

Theorem 13. *Assume that the operator $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone and bounded with $0 \in D(T)$ and $0 \in T(0)$. Assume that the operator $P : X \supset D(P) \to X^*$, with $L \subset D(P)$ satisfies condition $(S_+)_L$ and (P2). Assume that $Tx \ni 0$ implies $x = 0$, T satisfies (S_q) and there exist $\alpha > 0$ such that*

$$\|Px\| \geq \alpha, \quad x \in D(P). \tag{39}$$

Then the nonzero eigenvectors of the problem (34) form a continuous branch of infinite length.

Proof. We show that the problem (34) has eigenvectors on the set $\partial B_r(0)$ for every $r > 0$. To this end, let fix $r > 0$, $\epsilon > 0$ and show the existence of $\tilde{\lambda} > 0$ such that

$$d\left(T + \tilde{\lambda} P + \epsilon J, B_r(0), 0\right) = 0. \tag{40}$$

If this is not true, then there exist a sequence of $\{\lambda_n\} \subset (0, \infty)$ such that $\lambda_n \to \infty$ and one of the following is true:

(i) the degree $d(T + \lambda_n P + \epsilon J, B_r(0), 0)$ is not well-defined;

(ii) $d(T + \lambda_n P + \epsilon J, B_r(0), 0) \neq 0$.

In the case (i) there exist eigenvectors $x_n \in \partial B_r(0)$ such that

$$Tx_n + \lambda_n Px_n + \epsilon Jx_n \ni 0. \tag{41}$$

In the case (ii) there exist eigenvectors $x_n \in B_r(0)$ such that (41) holds. Thus in either case, there exist a sequence $\{x_n\} \subset \overline{B_r(0)}$ such that (41) is true. But this leads to a contradiction because $\|\lambda_n Px_n + \epsilon Jx_n\| \geq \alpha\lambda_n - \epsilon r \to \infty$, while the set Tx_n lie in a bounded set. Thus, (40) is true for some $\tilde{\lambda} > 0$. We consider the homotopy

$$H(t, x) \equiv Tx + t\tilde{\lambda} Px + \epsilon Jx, \quad t \in [0, 1], \; x \in D(H(t, \cdot)), \tag{42}$$

either there exist $t_0 \in [0,1]$ and $x_{t_0} \in \partial B_r(0)$ such that

$$Tx_{t_0} + t_0 \tilde{\lambda} Px_{t_0} + \epsilon Jx_{t_0} \ni 0, \tag{43}$$

or

$$d(H(t, \cdot)) = d(H(1, \cdot)) = 0 = d(H(0, \cdot)) = 1, \quad t \in [0, 1] \tag{44}$$

that is a contradiction. The last equality in (44) follows from Theorem 3, (i) in [2]. It follows that (43) is true. Naturally, we must have $t_0 \neq 0$ in (43) because otherwise $0 \in (T + \epsilon J)(\partial B_r(0))$. This cannot happen because we already have $0 \in (T + \epsilon J)(0)$ and $T + \epsilon J$ is one-to-one. From (43) we obtain sequences $\lambda_n \in (0, \infty)$, $\{x_n\} \subset \partial B_r(0)$ such that

$$Tx_n + \lambda_n Px_n + \left(\frac{1}{n}\right) Jx_n \ni 0. \tag{45}$$

We may assume that $x_n \rightharpoonup x_0$. Again, $\{\lambda_n\}$ cannot contain a subsequence $\{\lambda_{n_k}\}$ such that $\lambda_{n_k} \to \infty$ as $k \to \infty$ because the sequence $Tx_{n_k} + (1/n_k)Jx_{n_k}$ lies in a bounded set and $\lambda_{n_k}\|Px_{n_k}\| \to \infty$ as $k \to \infty$. Also we have that the sequence $\{Px_n\}$ is bounded. Thus we may assume that $Px_n \rightharpoonup p^* \in X^*$. Since the sequence $\{\lambda_n\}$ is bounded, we may assume that $\lambda_n \to \lambda_0$. Again, $\lambda_0 \neq 0$ otherwise $\lambda_n Px_n + (1/n)Jx_n \to 0$ and the (S_q) property of T would imply that $x_n \to x_0 \in \partial\Omega$. Since T is demiclosed, Lemma 6 would imply $Tx_0 \ni 0$. This is a contradiction to our assumption that $Tx \ni 0$ implies $x = 0$. It follows that $\lambda_0 > 0$ and we may also assume that $\lambda_n > 0$ for all n.

Now proceeding as in the proof of Theorem 8, it is easy to see that

$$\begin{gathered}\limsup_{n\to\infty} \langle \lambda_n Px_n + \lambda_0 p^*, x_n \rangle \leq 0, \\ \lim_{n\to\infty} \langle \lambda_n Px_n + \lambda_0 p^*, v \rangle = 0, \quad \forall v \in U_{n=1}^{\infty} F_n.\end{gathered} \tag{46}$$

Now since P is of class $(S_+)_L$, we conclude that $x_n \to x_0$, $x_0 \in D(P)$ and $Px_0 = p^*$. Thus since also $y_n^* \rightharpoonup -\lambda_0 Px_0$, $x_n \to x_0$ and the demiclosedness of T, we obtain $x_0 \in D(P) \cap \partial B_r(0)$ and $Tx_0 + \lambda_0 Px_0 \ni 0$. Since $r > 0$ is arbitrary, the nonzero eigenvectors of problem (34) form a continuous branch of infinite length. □

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