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Research Article

New Rough Set Approximation Spaces

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Rough set theory was introduced by Pawlak in 1982 to handle imprecision, vagueness, and uncertainty in data analysis. Our aim is to generalize rough set theory by introducing concepts of $\bigwedge_\beta$-lower and $\bigwedge_\beta$-upper approximations which depends on the concept of $\bigwedge_\beta$-sets. Also, we study some of their basic properties.

1. Introduction

Pawlak is credited with creating the "rough set theory" [1], a mathematical tool for dealing with vagueness or uncertainty. Since 1982, the theory and applications of rough set have impressively developed. There are many applications of rough set theory especially in data analysis, artificial intelligence, and cognitive sciences [2–4]. Some basic aspects of the research of rough sets and several applications have recently been presented by Pawlak and Skowron [5, 6]. Rough set theory [5–8] is an extension of set theory in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximation. Yao [9] classified broadly methods for the development of rough set theory into two classes, namely, the constructive and axiomatic (algebraic) approaches. In constructive methods, lower and upper approximations are constructed from the primitive notions, such as equivalence relations on a universe and neighborhood systems. In rough sets, the equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a nonempty intersection with the set. It is well known that a partition induces an equivalence relation on a set and vice versa. The properties of rough sets can thus be examined via either partition or equivalence classes. Rough sets are a suitable mathematical model of vague concepts. The main idea of rough sets corresponds to the lower and upper approximations. Pawlak's definitions for lower and upper approximations were originally introduced with reference to an equivalence relation. Many interesting properties of the lower and upper approximations have been derived by Pawlak and Skowron [5, 6] based on the equivalence relations. However, the equivalence relation appears to be a stringent condition that may limit the applicability of Pawlak's rough set model. Many extensions have been made in recent years by replacing equivalence relation or partition by notions such as binary relations [10–12], neighborhood systems, and Boolean algebras [12–16]. Abu-Donia [17] discussed three types of upper (lower) approximations depending on the right neighborhood by using general relation, also generalized this types by using a family of finite binary relations in two ways. Many proposals have been made for generalizing and interpreting rough sets [4, 18–25]. In 1983, Abd El-Monsef et al. [26] introduced the concept of β-open sets. In 1986, Maki [27] has introduced the concept of $\bigwedge$-sets in topological spaces as the sets that coincide with their kernel. The kernel of a set A is the intersection of all open supersets of A. In 2004, Noiri and Hatir [28] introduced the $\bigwedge_{sp}$-sets (or $\bigwedge_\beta$-sets) and investigated some of their properties. In 2008, Abu-Donia and Salama [29] introduced and investigated the concept of β-approximation space. The theory of rough sets can be generalized in several directions. Within the set-theoretic framework, generalizations of the element based definition can be obtained by using nonequivalence binary relations [9, 23, 30–32], generalizations of the granule based definition can be obtained by using coverings [12, 30, 33–35], and generalizations of subsystem based definition can be obtained by using other subsystems [36, 37]. In the standard rough set model, the same subsystem is used to define lower and upper approximation operators. When generalizing the subsystem based definition, one may use two subsystems, one for the lower approximation operator and the other for the upper approximation operator. Yao [24] defined a pair of generalized approximation operators by replacing the equivalence relations with the family of open sets for lower approximation operator "interior operators" and the equivalence relations with the family of closed sets for upper approximation operator "closure operators". In this paper we used a new subsystem called $\bigwedge_\beta$-sets to define new types of lower and upper approximation operators, called $\bigwedge_\beta$-lower approximation and $\bigwedge_\beta$-upper approximation. We study $\bigwedge_\beta$-rough sets, the comparison between this concept and rough sets is studied. Also, we give some counter examples.

2. Basic Concepts

A topological space [10] is a pair (X, τ) consisting of a set X and family τ of subset of X satisfying the following conditions:

(1) $\phi, X \in \tau$,
(2) τ is closed under arbitrary union,
(3) τ is closed under finite intersection.

The pair (X, τ) is called a topological space, the elements of X are called points of the space, the subsets of X belonging to τ are called open sets in the space, and the complement of the subsets of X belonging to τ are called closed set. The family τ of open subsets of X is also called a topology for X.

$\overline{A} = \bigcap\{F \subseteq X : A \subseteq F \text{ and } F \text{ is closed}\}$ is called τ-closure of a subset $A \subset X$.

Evidently, $\overline{A}$ is the smallest closed subset of X which contains A. Note that A is closed if and only if $A = \overline{A}$.

$A^\circ = \bigcup\{G \subseteq X : G \subseteq A \text{ and } G \text{ is open}\}$ is called the τ-interior of a subset $A \subseteq X$.

Evidently, A° is the union of all open subsets of X which containing in A. Note that A is open if and only if $A = A^\circ$. And $b(A) = \overline{A} - A^\circ$ is called the τ-boundary of a subset $A \subseteq X$.

Let A be a subset of a topological spaces (X, τ). Let $\overline{A}$, A° and $b(A)$ be closure, interior, and boundary of A, respectively. A is exact if $b(A) = \phi$, otherwise A is rough. It is clear A is exact if and only if $\overline{A} = A^\circ$.

Definition 1 (see [26]). A subset A of a topological space (X, τ) is called β-open if $A \subseteq \overline{(\overline{A})^\circ}$.

The complement of β-open set is called β-closed set. We denote the family of all β-open (resp., β-closed) sets by $\beta O(X)$ (resp., $\beta C(X)$).

Remark 2. For any topological space (X, τ), we have $\tau \subseteq \beta O(X)$

Definition 3 (see [28]). Let A be a subset of a topological space (X, τ). A subset $\bigwedge_\beta(A)$ is defined as follows: $\bigwedge_\beta(A) = \cap\{G : A \subseteq G, G \in \beta O(X)\}$.

The complement of $\bigwedge_\beta(A)$-set is called $\bigvee_\beta(A)$-set.

Noiri and Hater [28] stated some properties of $\bigwedge_\beta(A)$ in the following lemma.

Lemma 4. *For subsets A, B, and A_α ($\alpha \in \Delta$) of a topological space (X, τ), the following hold.*

(1) $A \subseteq \bigwedge_\beta(A)$.
(2) *If* $A \subset B$, *then* $\bigwedge_\beta(A) \subseteq \bigwedge_\beta(B)$.
(3) $\bigwedge_\beta(\bigwedge_\beta(A)) = \bigwedge_\beta(A)$.
(4) *If* $A \in \beta O(X)$, $A = \bigwedge_\beta(A)$.
(5) $\bigwedge_\beta(\bigcup\{A_\alpha : \alpha \in \Delta\}) = \bigcup\{\bigwedge_\beta(A_\alpha) : \alpha \in \Delta\}$.
(6) $\bigwedge_\beta(\bigcap\{A_\alpha : \alpha \in \Delta\}) \subseteq \bigcap\{\bigwedge_\beta(A_\alpha) : \alpha \in \Delta\}$.

Definition 5. A subset A of a topological space (X, τ) is called $\bigwedge_\beta$-set if $A = \bigwedge_\beta(A)$.

Lemma 6. *For subsets A and $A_\alpha, \alpha \in \Delta$ of a topological space (X, τ), the following hold.*

(1) $\bigwedge_\beta(A)$ *is* $\bigwedge_\beta$*-set.*
(2) *If A is β-open, then A is $\bigwedge_\beta$-set.*
(3) *If A_a is $\bigwedge_\beta$-set for each $\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} A_a$ is $\bigwedge_\beta$-set.*
(4) *If A_a is $\bigwedge_\beta$-set for each $\alpha \in \Delta$, then $\bigcap_{\alpha\in\Delta} A_a$ is $\bigwedge_\beta$-set.*

Definition 7. Let (X, τ) be a topological space the subset $A \subseteq X$ is called:

(1) α-open [38] if $\subseteq \overline{A^\circ}^\circ$,
(2) Preopen [39] if $\subseteq (\overline{A})^\circ$.

Remark 8. the class of all $\bigwedge_\beta$-sets is stronger than open (resp., α-open, preopen, and β-open) sets as shown in the following diagram:

$$\text{Open} \longrightarrow \alpha\text{-open} \longrightarrow \text{Preopen} \longrightarrow \beta\text{-open} \longrightarrow \bigwedge\nolimits_\beta\text{-set}. \tag{1}$$

Motivation for rough set theory has come from the need to represent subsets of a universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space $K = (X, R)$, where X is a set called the universe and R is an equivalence relation [2]. The equivalence classes of R are also known as the granules, elementary sets, or blocks; we will use $R_x \subseteq X$ to denote the equivalence class containing $x \in X$. In the approximation space, we consider two operators

$$\begin{aligned} \overline{R}(A) &= \{x \in X : R_x \cap A \neq \phi\}, \\ \underline{R}(A) &= \{x \in X : R_x \subseteq A\}, \end{aligned} \tag{2}$$

called the lower approximation and upper approximation of $A \subseteq X$, respectively. Also let $\mathrm{POS}_R(A) = \underline{R}(A)$ denote the positive region of A, $\mathrm{NEG}_R(A) = X - \overline{R}(A)$ denote the negative region of A and $\mathrm{BN}_R(A) = \overline{R}(A) - \underline{R}(A)$ denote the borderline region of X.

Let X be a finite nonempty universe, $A \subseteq X$, the degree of completeness can also be characterized by the accuracy measure as follows:

$$\alpha_R(A) = \frac{|\underline{R}(A)|}{|\overline{R}(A)|}, \qquad A \neq \phi, \tag{3}$$

where $|\cdot|$ represents the cardinality of set. Accuracy measures try to express the degree of completeness of knowledge. $\alpha_R(A)$ is able to capture how large the boundary region of the data sets is; however, we cannot easily capture the structure of the knowledge. A fundamental advantage of rough set theory is the ability to handle a category that cannot be sharply defined given a knowledge base. Characteristics of the potential data sets can be measured through the rough sets framework. We can measure inexactness and express topological characterization of imprecision with the following.

(1) If $\underline{R}(A) \neq \phi$ and $\overline{R}(A) \neq X$, then A is roughly R-definable.
(2) If $\underline{R}(A) = \phi$ and $\overline{R}(A) \neq X$, then A is internally R-undefinable.
(3) If $\underline{R}(A) \neq \phi$ and $\overline{R}(A) = X$, then A is externally R-undefinable.
(4) If $\underline{R}(A) = \phi$ and $\overline{R}(A) = X$, then A is totally R-undefinable.

We denote the set of all roughly R-definable (resp., internally R-undefinable, externally R-undefinable, and totally R-undefinable) sets by $\mathrm{RD}(X)$ (resp., $\mathrm{IUD}(X)$, $\mathrm{EUD}(X)$, and $\mathrm{TUD}(X)$).

With $\alpha_R(A)$ and classifications above we can characterize rough sets by the size of the boundary region and structure. Rough sets are treated as a special case of relative sets and integrated with the notion of Belnap's logic [22].

Remark 9. We denote the relation which used to get a subbase for a topology τ on X and a class of β-open sets ($\beta O(X)$) by R_β. Also, we denote β-approximation space by (X, R_β).

Definition 10. Let (X, R_β) be a β-approximation space β-lower (resp., β-upper) approximation of any nonempty subset A of X is defined as:

$$\underline{R}_\beta(A) = \bigcup\{G \in \beta O(X) : G \subseteq A\},$$
$$\overline{R}_\beta(A) = \bigcap\{F \in \beta C(X) : F \supseteq A\}.$$

We can get the the β-approximation operator as follows.

(1) Get the right neighborhoods xR from the given relation R as $xR = \{y : xRy\}$.
(2) Using right neighborhoods xR as a sub-base to get the topology τ.
(3) Using the open sets in the topology τ to get the family of β-open sets "from Definition 1."
(4) Using the set of all β-open sets to get β-approximation operators (from Definition 10).

Definition 11. Let (X, R_β) be a β-approximation space and $A \subseteq X$. Then there are memberships $\underline{\in}$, $\overline{\in}$, $\underline{\in}_\beta$, and $\overline{\in}_\beta$, say, strong, weak, β-strong, and β-weak memberships respectively which defined by

(1) $x \,\underline{\in}\, A$ iff $x \in \underline{R}(A)$,
(2) $x \,\overline{\in}\, A$ iff $x \in \overline{R}(A)$,
(3) $x \,\underline{\in}_\beta\, A$ iff $x \in \underline{R}_\beta(A)$,
(4) $x \,\overline{\in}_\beta\, A$ iff $x \in \overline{R}_\beta(A)$.

Remark 12. According to Definition 11, β-lower and β-upper approximations of a set $A \subseteq X$ can be written as

$$\underline{R}_\beta(A) = \{x \in A : x \,\underline{\in}_\beta\, A\},$$
$$\overline{R}_\beta(A) = \{x \in A : x \,\overline{\in}_\beta\, A\}.$$

Definition 13. Let (X, R_β) be a β-approximation space and $A \subseteq X$. the β-accuracy measure of A defined as follows:

$$\alpha_{R_\beta}(A) = \frac{|\underline{R}_\beta(A)|}{|\overline{R}_\beta(A)|}, \qquad A \neq \phi. \tag{4}$$

Definition 14. Let (X, R_β) be a β-approximation space, the set $A \subseteq X$ is called

(1) roughly R_β-definable, if $\underline{R}_\beta(A) \neq \phi$ and $\overline{R}_\beta(A) \neq X$,
(2) internally R_β-undefinable, if $\underline{R}_\beta(A) = \phi$ and $\overline{R}_\beta(A) \neq X$,
(3) externally R_β-undefinable, if $\underline{R}_\beta(A) \neq \phi$ and $\overline{R}_\beta(A) = X$,
(4) totally R_β-undefinable, if $\underline{R}_\beta(A) = \phi$ and $\overline{R}_\beta(A) = X$.

We denote the set of all roughly R_β-definable (resp., internally R_β-undefinable, externally R_β-undefinable, and totally R_β-undefinable) sets by $\beta\mathrm{RD}(X)$ (resp., $\beta\mathrm{IUD}(X)$, $\beta\mathrm{EUD}(X)$ and $\beta\mathrm{TUD}(X)$).

Remark 15. For any β-approximation space (X, R_β) the following hold:

(1) $\beta\mathrm{RD}(X) \supseteq \mathrm{RD}(X)$,
(2) $\beta\mathrm{IUD}(X) \subseteq \mathrm{IUD}(X)$,
(3) $\beta\mathrm{EUD}(X) \subseteq \mathrm{EUD}(X)$,
(4) $\beta\mathrm{TUD}(X) \subseteq \mathrm{TUD}(X)$.

3. A New Type of Rough Classification Based on $\bigwedge_\beta$-Sets

In this section, we introduced and investigated the concept of $\bigwedge_\beta$-approximation space. Also, we introduce the concepts of $\bigwedge_\beta$-lower approximation and $\bigwedge_\beta$-upper approximation for any subset and study their properties.

Remark 16. We denote the relation which used to get a subbase for a topology τ on X and a class of $\bigwedge_\beta$-sets by $R_{\wedge_\beta}$. Also, we denote $\bigwedge_\beta$-approximation space by $(X, R_{\wedge_\beta})$.

Example 17. Let $X = \{a,b,c,d\}$ be a universe and a relation $R_{\wedge_\beta}$ defined by $R_{\wedge_\beta} = \{(a,a),(a,c),(a,d),(b,b),(b,d),(c,a),(c,b),(c,d),(d,a)\}$, thus $aR_{\wedge_\beta} = \{a,c,d\}$, $bR_{\wedge_\beta} = \{b,d\}$, $cR_{\wedge_\beta} = \{a,b,d\}$, and $dR_{\wedge_\beta} = \{a\}$. Then the topology associated with this relation is $\tau = \{X,\phi,\{a\},\{d\}\{a,d\},\{b,d\},\{a,b,d\},\{a,c,d\}\}$ and $\bigwedge_\beta$-sets $= \{X,\phi,\{a\},\{c\},\{d\},\{a,c\}\{a,d\},\{b,d\},\{c,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$. So $(X, R_{\wedge_\beta})$ is a $\bigwedge_\beta$-approximation space.

Definition 18. Let $(X, R_{\wedge_\beta})$ be a $\bigwedge_\beta$-approximation space. $\bigwedge_\beta$-lower approximation and $\bigwedge_\beta$-upper approximation of any nonempty subset A of X is defined as

$$\underline{\bigwedge}_\beta(A) = \bigcup\{G : G \text{ is } \bigwedge_\beta\text{-set}, G \subseteq A\},$$
$$\overline{\bigwedge}_\beta(A) = \bigcap\{F : F \text{ is } \bigvee_\beta\text{-set}, F \supseteq A\}.$$

The following proposition shows the properties of $\bigwedge_\beta$-lower approximation and $\bigwedge_\beta$-upper approximation of any nonempty subset.

Proposition 19. *Let $(X, R_{\wedge_\beta})$ be a $\bigwedge_\beta$-approximation space and $A, B \subseteq X$. Then:*

(1) $\underline{\bigwedge}_\beta(A) \subseteq A \subseteq \overline{\bigwedge}_\beta(A)$,

(2) $\underline{\bigwedge}_\beta(\phi) = \overline{\bigwedge}_\beta(\phi) = \phi$, $\underline{\bigwedge}_\beta(X) = \overline{\bigwedge}_\beta(X) = X$,

(3) *If* $A \subseteq B$ *then* $\underline{\bigwedge}_\beta(A) \subseteq \underline{\bigwedge}_\beta(B)$ *and* $\overline{\bigwedge}_\beta(A) \subseteq \overline{\bigwedge}_\beta(B)$,

(4) $\underline{\bigwedge}_\beta(X \setminus A) = X \setminus \overline{\bigwedge}_\beta(A)$,

(5) $\overline{\bigwedge}_\beta(X \setminus A) = X \setminus \underline{\bigwedge}_\beta(A)$,

(6) $\underline{\bigwedge}_\beta(\underline{\bigwedge}_\beta(A)) = \underline{\bigwedge}_\beta(A)$,

(7) $\overline{\bigwedge}_\beta(\overline{\bigwedge}_\beta(A)) = \overline{\bigwedge}_\beta(A)$,

(8) $\underline{\bigwedge}_\beta(\underline{\bigwedge}_\beta(A)) \subseteq \overline{\bigwedge}_\beta(A))$,

(9) $\underline{\bigwedge}_\beta(\overline{\bigwedge}_\beta(A)) \subseteq \overline{\bigwedge}_\beta(\overline{\bigwedge}_\beta(A))$,

(10) $\underline{\bigwedge}_\beta(A \cup B) \supseteq \underline{\bigwedge}_\beta(A) \cup \underline{\bigwedge}_\beta(B)$,

(11) $\overline{\bigwedge}_\beta(A \cup B) \supseteq \overline{\bigwedge}_\beta(A) \cup \overline{\bigwedge}_\beta(B)$,

(12) $\underline{\bigwedge}_\beta(A \cap B) \subseteq \underline{\bigwedge}_\beta(A) \cap \underline{\bigwedge}_\beta(A)$,

(13) $\overline{\bigwedge}_\beta(A \cap B) \subseteq \overline{\bigwedge}_\beta(A) \cap \overline{\bigwedge}_\beta(B)$.

Definition 20. Let $(X, R_{\wedge_\beta})$ be a $\bigwedge_\beta$-approximation space. The Universe X can be divided into 24 regions with respect to any $A \subseteq X$ as follows.

(1) The internal edg of A, $\underline{\mathrm{Edg}}(A) = A - \underline{R}(A)$.

(2) The β-internal edg of A, $\beta\underline{\mathrm{Edg}}(A) = A - \underline{R}_\beta(A)$.

(3) The $\bigwedge_\beta$-internal edg of A, $\bigwedge_\beta\underline{\mathrm{Edg}}(A) = A - \underline{\bigwedge}_\beta(A)$.

(4) The external edg of A, $\overline{\mathrm{Edg}}(A) = \overline{R}(A) - A$.

(5) The β-external edg of A, $\beta\overline{\mathrm{Edg}}(A) = \overline{R}_\beta(A) - A$.

(6) The $\bigwedge_\beta$-external edg of A, $\bigwedge_\beta\overline{\mathrm{Edg}}(A) = \overline{\bigwedge}_\beta(A) - A$.

(7) The boundary of A, $b(A) = \overline{R}(A) - \underline{R}(A)$.

(8) The β-boundary of A, $\beta b(A) = \overline{R}_\beta(A) - \underline{R}_\beta(A)$.

(9) The $\bigwedge_\beta$-boundary of A, $\bigwedge_\beta b(A) = \overline{\bigwedge}_\beta(A) - \underline{\bigwedge}_\beta(A)$.

(10) The exterior of A, $\mathrm{ext}(A) = X - \overline{R}(A)$.

(11) The β-exterior of A, $\beta\mathrm{ext}(A) = X - \overline{R}_\beta(A)$.

(12) The $\bigwedge_\beta$-exterior of A, $\bigwedge_\beta\ \mathrm{ext}(A) = X - \overline{\bigwedge}_\beta(A)$.

(13) $\overline{R}(A) - \underline{R}_\beta(A)$.

(14) $\overline{R}(A) - \underline{\bigwedge}_\beta(A)$.

(15) $\overline{R}(A) - \overline{\bigwedge}_\beta(A)$.

(16) $\overline{R}_\beta(A) - \underline{R}(A)$.

(17) $\overline{R}_\beta(A) - \underline{\bigwedge}_\beta(A)$

(18) $\overline{R}_\beta(A) - \overline{\bigwedge}_\beta(A)$.

(19) $\underline{R}_\beta(A) - \underline{R}_\beta(A)$.

(20) $\overline{\bigwedge}_\beta(A) - \underline{R}_\beta(A)$.

(21) $\overline{\bigwedge}_\beta(A) - \underline{R}(A)$.

(22) $\underline{\bigwedge}_\beta(A) - \underline{R}_\beta(A)$.

(23) $\underline{\bigwedge}_\beta(A) - \underline{R}(A)$.

(24) $\overline{R}(A) - \overline{R}_\beta(A)$.

Remark 21. As shown in Figure 1, the study of $\underline{\bigwedge}_\beta$-approximation spaces is a generalization for study of approximation spaces. Because of the elements of the regions $[\underline{R}_\beta(A)-\underline{R}(A)]$, $[\underline{\bigwedge}_\beta(A)-\underline{R}_\beta(A)]$, and $[\underline{\bigwedge}_\beta(A)-\underline{R}(A)]$ will be defined well in A, while this points was undefinable in Pawlak's approximation spaces. Also, the elements of the region $[\overline{R}(A) - \underline{\bigwedge}_\beta(A)]$, $[\overline{R}_\beta(A) - \overline{\bigwedge}_\beta(A)]$, and $[\overline{R}(A) - \overline{R}_\beta(A)]$ do not be belong to A, while these elements was not well defined in Pawlak's approximation spaces.

Figure 1 shows the above 24 regions.

Theorem 22. *For any topological space (X, τ) generated by a binary relation R on X, we have, $\underline{R}(A) \subseteq \underline{R}_\beta(A) \subseteq \underline{\bigwedge}_\beta(A) \subseteq A \subseteq \overline{\bigwedge}_\beta(A) \subseteq \overline{R}_\beta(A) \subseteq \overline{R}(A)$.*

Proof. $\underline{R}(A) = \bigcup\{G \in \tau : G \subseteq A\} \subseteq \bigcup\{G \in \beta O(X) : G \subseteq A\} = \underline{R}_\beta(A) \subseteq \bigcup\{G \in \bigwedge_\beta\text{-set} : G \subseteq A\} = \underline{\bigwedge}_\beta(A) \subseteq A$, that is, $\underline{R}(A) \subseteq \underline{R}_\beta(A) \subseteq \underline{\bigwedge}_\beta(A) \subseteq A$.

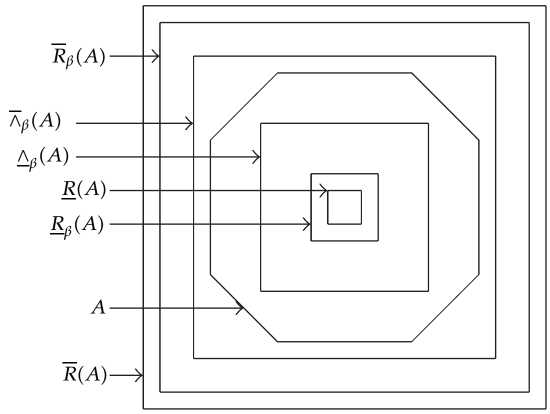

Figure 1

Also, $\overline{R}(A) = \bigcap\{F \in \tau^c : F \supseteq A\} \supseteq \bigcap\{F \in \beta C(X) : F \supseteq A\} = \overline{R}_\beta(A) \supseteq \bigcap\{F \in \bigvee_\beta\text{-set} : F \supseteq A\} = \overline{\bigwedge}_\beta(A) \supseteq A$, that is, $\overline{R}(A) \supseteq \overline{R}_\beta(A) \supseteq \overline{\bigwedge}_\beta(A) \supseteq A$.

Consequently, $\underline{R}(A) \subseteq \underline{R}_\beta(A) \subseteq \underline{\bigwedge}_\beta(A) \subseteq A \subseteq \overline{\bigwedge}_\beta(A) \subseteq \overline{R}_\beta(A) \subseteq \overline{R}(A)$. □

Definition 23. Let $(X, R_{\wedge_\beta})$ be a $\bigwedge_\beta$-approximation space and $A \subseteq X$. Then there are memberships $\underline{\in}_{\wedge_\beta}$ and $\overline{\in}_{\wedge_\beta}$, say, $\bigwedge_\beta$-strong and $\bigwedge_\beta$-weak memberships respectively which defined by

(1) $x \underline{\in}_{\wedge_\beta} A$ iff $x \in \underline{\bigwedge}_\beta(A)$,

(2) $x \overline{\in}_{\wedge_\beta} A$ iff $x \in \overline{\bigwedge}_\beta(A)$.

Remark 24. According to Definition 28, $\bigwedge_\beta$-lower and $\bigwedge_\beta$-upper approximations of a set $A \subseteq X$ can be written as

(1) $\underline{\bigwedge}_\beta(A) = \{x \in A : x \underline{\in}_{\wedge_\beta} A\}$,

(2) $\overline{\bigwedge}_\beta(A) = \{x \in A : x \overline{\in}_{\wedge_\beta} A\}$.

Remark 25. Let $(X, R_{\wedge_\beta})$ be a $\bigwedge_\beta$-approximation space and $A \subseteq X$. Then

(1) $x \overline{\in} A \Rightarrow x \underline{\in}_\beta A \Rightarrow x \underline{\in}_{\wedge_\beta} A$,

(2) $x \overline{\in}_{\wedge_\beta} A \Rightarrow x \overline{\in}_\beta A \Rightarrow x \overline{\in} A$.

The converse of Remark 25 may not be true in general as seen in the following example.

Example 26. In Example 17. Let $A = \{b, c\}$, we have $c \underline{\in}_{\wedge_\beta} A$ but $c \underline{\notin}_\beta A$. Let $A = \{c, d\}$, $c \underline{\in}_\beta A$ but $c \underline{\notin} A$. Let $A = \{d\}$ we have $c \overline{\in} A$ but $c \overline{\notin}_\beta A$. Let $A = \{a, d\}$, $c \overline{\in}_\beta A$ but $c \overline{\notin}_{\wedge_\beta} A$.

Table 1

The set $A \subseteq X$	$\alpha_R(A)$	$\alpha_{R_\beta}(A)$	$\alpha_{\wedge_\beta}(A)$
$\{a\}$	1/2	1	1
$\{b\}$	0	0	0
$\{c\}$	0	0	1
$\{d\}$	1/3	1/2	1/2
$\{a, b\}$	1/3	1/2	1/2
$\{a, c\}$	1/2	1	1
$\{a, d\}$	1/2	1/2	2/3
$\{b, c\}$	0	0	1/2
$\{b, d\}$	2/3	1	1
$\{c, d\}$	1/3	2/3	2/3
$\{a, b, c\}$	1/3	2/3	2/3
$\{a, b, d\}$	3/4	3/4	1
$\{a, c, d\}$	3/4	3/4	3/4
$\{b, c, d\}$	2/3	1	1

Let X be a finite nonempty universe, $A \subseteq X$, we can characterize the degree of completeness by a new tool named $\bigwedge_\beta$-accuracy measure defined as follows.

$$\alpha_{\wedge_\beta}(A) = \frac{\left|\underline{\bigwedge}_\beta(A)\right|}{\left|\overline{\bigwedge}_\beta(A)\right|}, \quad A \neq \phi. \tag{5}$$

Example 27. In Example 17, we can deduce the following table showing the degree of accuracy measure $\alpha_R(A)$, β-accuracy measure $\alpha_{R_\beta}(A)$ and $\bigwedge_\beta$-accuracy measure $\alpha_{\wedge_\beta}(A)$ for some sets, see Table 1.

We see that the degree of exactness of the set $A = \{a, c\}$ by using accuracy measure equal to 50%, by using $\bigwedge_\beta$-accuracy measure equal to 100%. Also, the set $A = \{a, b, d\}$ by using β-accuracy measure equal to 75% and by using $\bigwedge_\beta$-accuracy measure equal to 100%. Consequently $\bigwedge_\beta$-accuracy measure is better than accuracy and β-accuracy measures in this case.

We investigate $\bigwedge_\beta$-rough equality and $\bigwedge_\beta$-rough inclusion based on rough equality and inclusion which introduced by Novotný and Pawlak in [7, 40].

Definition 28. Let $(X, R_{\wedge_\beta})$ be a $\bigwedge_\beta$-approximation space, $A, B \subseteq X$. Then we say that A and B are

(i) $\bigwedge_\beta$-roughly bottom equal $(A \eqsim_{\wedge_\beta} B)$ if $\underline{\bigwedge}_\beta(A) = \underline{\bigwedge}_\beta(B)$,

(ii) $\bigwedge_\beta$-roughly top equal $(A \simeq_{\wedge_\beta} B)$ if $\overline{\bigwedge}_\beta(A) = \overline{\bigwedge}_\beta(B)$,

(iii) $\bigwedge_\beta$-roughly equal $(A \approx_{\wedge_\beta} B)$ if $(A \eqsim_{\wedge_\beta} B)$ and $(A \simeq_{\wedge_\beta} B)$.

Example 29. In Example 17, we have the sets $\{a, c\}, \{a, b, c\}$ are $\bigwedge_\beta$-roughly bottom equal and $\{c, d\}$, $\{b, c, d\}$ are $\bigwedge_\beta$-roughly top equal.

Definition 30. Let $(X, R_{\wedge_\beta})$ be a $\bigwedge_\beta$-approximation space, $A, B \subseteq X$. Then we say that

(i) A is $\bigwedge_\beta$-roughly bottom included in $B(A \underset{\sim \wedge_\beta}{\subset} B)$ if $\underline{\bigwedge}_\beta(A) \subseteq \underline{\bigwedge}_\beta(B)$,

(ii) A is $\bigwedge_\beta$-roughly top included in $B(A \tilde{\subset}_{\wedge_\beta} B)$ if $\overline{\bigwedge}_\beta(A) \subseteq \overline{\bigwedge}_\beta(B)$.

(iii) A is $\bigwedge_\beta$-roughly included in $B(A \underset{\sim \wedge_\beta}{\tilde{\subset}} B)$ if $(A \underset{\sim \wedge_\beta}{\subset} B)$ and $(A \tilde{\subset}_{\wedge_\beta} B)$.

Example 31. In Example 17, we have $\{b, c\}$ is $\bigwedge_\beta$- roughly bottom included in $\{a, b, c\}$. Also, $\{b, c\}$ is $\bigwedge_\beta$- roughly top included in $\{a, b, c\}$. Also, $\{b, c\}$ is $\bigwedge_\beta$-roughly included in $\{a, b, c\}$.

4. $\bigwedge_\beta$-Rough Sets

We introduced a new concept of $\bigwedge_\beta$-rough set.

Definition 32. For any $\bigwedge_\beta$-approximation space $(X, R_{\wedge_\beta})$, a subset A of X is called:

(1) $\bigwedge_\beta$-definable ($\bigwedge_\beta$-exact) if $\overline{\bigwedge}_\beta(A) = \underline{\bigwedge}_\beta(A)$,

(2) $\bigwedge_\beta$-rough if $\overline{\bigwedge}_\beta(A) \neq \underline{\bigwedge}_\beta(A)$.

Example 33. Let $(X, R_{\wedge_\beta})$ be a $\bigwedge_\beta$-approximation space as in Example 17. We have the set $\{c\}$ is $\bigwedge_\beta$-exact while $\{c, d\}$ is $\bigwedge_\beta$-rough set.

Proposition 34. *Let $(X, R_{\wedge_\beta})$ be a $\bigwedge_\beta$-approximation space. Then*

(1) *every exact set in X is β-exact,*

(2) *every β-exact set in X is $\bigwedge_\beta$-exact,*

(3) *every $\bigwedge_\beta$-rough set in X is β-rough,*

(4) *every β-rough set in X is rough.*

Proof. Obvious.

The converse of all parts of Proposition 34 may not be true in general as seen in the following example. □

Example 35. Let $(X, R_{\wedge_\beta})$ be an $\bigwedge_\beta$-approximation space as in Example 17. Then the set $\{b, d\}$ is β-exact but not exact, the set $\{c\}$ is $\bigwedge_\beta$-exact but not β-exact, the set $\{c\}$ is β-rough but not $\bigwedge_\beta$-rough and the set $\{a, c\}$ is rough but not β-rough.

Definition 36. Let $(X, R_{\wedge_\beta})$ be a $\bigwedge_\beta$-approximation space, the set $A \subseteq X$ is called:

(1) roughly $\bigwedge_\beta$-definable, if $\underline{\bigwedge}_\beta(A) \neq \phi$ and $\overline{\bigwedge}_\beta(A) \neq X$,

(2) internally $\bigwedge_\beta$-undefinable, if $\underline{\bigwedge}_\beta(A) = \phi$ and $\overline{\bigwedge}_\beta(A) \neq X$,

(3) externally $\bigwedge_\beta$-undefinable, if $\underline{\bigwedge}_\beta(A) \neq \phi$ and $\overline{\bigwedge}_\beta(A) = X$,

(4) totally $\bigwedge_\beta$-undefinable, if $\underline{\bigwedge}_\beta(A) = \phi$ and $\overline{\bigwedge}_\beta(A) = X$.

We denote the set of all roughly $\bigwedge_\beta$-definable (resp., internally $\bigwedge_\beta$-undefinable, externally $\bigwedge_\beta$-undefinable and totally $\bigwedge_\beta$-undefinable) sets by $\bigwedge_\beta\mathrm{RD}(X)$ (resp., $\bigwedge_\beta\mathrm{IUD}(X)$, $\bigwedge_\beta\mathrm{EUD}(X)$ and $\bigwedge_\beta\mathrm{TUD}(X)$).

Remark 37. For any $\bigwedge_\beta$-approximation space $(X, R_{\wedge_\beta})$. The following are hold:

(1) $\bigwedge_\beta\mathrm{RD}(X) \supseteq \beta\mathrm{RD}(X) \supseteq \mathrm{RD}(X)$,

(2) $\bigwedge_\beta\mathrm{IUD}(X) \subseteq \beta\mathrm{IUD}(X) \subseteq \mathrm{IUD}(X)$,

(3) $\bigwedge_\beta\mathrm{EUD}(X) \subseteq \beta\mathrm{EUD}(X) \subseteq \mathrm{EUD}(X)$,

(4) $\bigwedge_\beta\mathrm{TUD}(X) \subseteq \beta\mathrm{TUD}(X) \subseteq \mathrm{TUD}(X)$.

Example 38. In Example 17, we have the set $\{a, d\} \in \bigwedge_\beta\mathrm{RD}(X)$ but $\{a, d\} \notin \beta\mathrm{RD}(X)$. The set $\{c\} \in \beta\mathrm{IUD}(X)$ but $\{c\} \notin \bigwedge_\beta\mathrm{IUD}(X)$. The set $\{a, b, d\} \in \beta\mathrm{EUD}(X)$ but $\{a, b, d\} \in \bigwedge_\beta\mathrm{EUD}(X)$.

5. Conclusion

In this paper, we used the class of $\bigwedge_\beta$-sets to introduce a new type of approximations named $\bigwedge_\beta$-approximation operator. Also, using $\bigwedge_\beta$-approximation we can obtain 24 dissimilar granules of the universe of discourse. Our approach is the largest granulation based on β-open sets in topological spaces. This made the accuracy measures higher than the use of any type of near open sets such as, α-open, β-open, and preopen sets. Some important properties of the classical. Pawlak's rough sets are generalized. Also, we defined the concept of rough membership function using $\bigwedge_\beta$-sets. It is a generalization of classical rough membership function of Pawlak rough sets. The generalized rough membership function can be used to analyze which decision should be made according to a conditional attribute in decision information system. The rough set approach to approximation of sets leads to useful forms of granular computing that are part of computational intelligence.

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