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Research Article

The Convergence Ball and Error Analysis of the Relaxed Secant Method

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A relaxed secant method is proposed. Radius estimate of the convergence ball of the relaxed secant method is attained for the nonlinear equation systems with Lipschitz continuous divided differences of first order. The error estimate is also established with matched convergence order. From the radius and error estimate, the relation between the radius and the speed of convergence is discussed with parameter. At last, some numerical examples are given.

1. Introduction

Many scientific problems can be concluded to the form of nonlinear systems. Finding the solutions of nonlinear systems is widely required in both mathematical physics and nonlinear dynamical systems. In this paper, we will establish the convergence ball and error analysis of the relaxed secant method of nonlinear systems. Consider

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on a convex subset Ω of a Banach space X with values in another Banach space Y. When F is nonlinear, iterative methods are generally adopted to solve the system:

$$x_{n+1} = \Psi(x_n), \quad x_0 \text{ is given.} \tag{2}$$

The most widely used iterative method is Newton's method which can be described as

$$x_{n+1} = x_n - F(x_n)^{-1} F(x_n), \quad x_0 \text{ is given.} \tag{3}$$

This method and Newton-like methods have been studied well by many authors (see [1–12]).

Newton's method requires that F is differentiable. Thus, when F is nondifferentiable, Newton method cannot be applied on it. We have to turn to other methods that do not need to evaluate derivatives. In their algorithms, instead of derivatives, divided differences are always used. The classical method of this type is the secant method.

Let $\Phi(X, Y)$ denote the space of the bounded linear maps from X to Y. If the following equality holds,

$$[X, Y; F](x - y) = F(x) - F(y), \tag{4}$$

then, we call the operator $[X, Y; F] \in \Phi(X, Y)$, at the points x and y $(x \neq y)$, a divided difference of order one of the nonlinear operator F.

By the above definition, secant method can be generalized to Banach spaces, it is described as the following scheme:

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \quad (n > 0) \ \ x_0, x_{-1} \in \Omega. \tag{5}$$

An interesting issue here is to estimate the radius of the convergence ball of an iterative method. Suppose x_* is a solution of the nonlinear system (1). Denote with $B(x_*, r) \subset X$ an open ball with center x_* and radius r. The open ball $B(x_*, r) \subset X$ is called a convergence ball of an iteration, if the sequence generated by the iterative method converges with any initial value in the ball. Under the assumption that the nonlinear operator F has Fréchet derivatives satisfying the Hölder condition,

$$\begin{aligned}&\left\|F'(x_*)^{-1}\left([x,y;F]-F'(z)\right)\right\|\\&\quad\leqslant K\left(\|x-z\|^p+\|y-z\|^p\right),\\&\qquad\forall x,y,z\in\Omega \text{ for some } K>0.\end{aligned}\tag{6}$$

Ren and Wu [13] have given the radius of the convergence ball which is $r_p=\sqrt[p]{(1+p)/K(1+2p)}$.

The convergence ball, the semilocal convergence of secant method, and secant-like method have been studied by many other authors (see [13–18]). In this paper, similar to the relaxed Newton's method in [7], we considered the relaxed secant method which can be written as the following form:

$$x_{n+1}=x_n-\lambda\left[x_{n-1},x_n;F\right]^{-1}F(x_n),\quad x_0,x_{-1}\in\Omega;\tag{7}$$

here, $\lambda\in(0,2)$ is called the relaxed parameter. When $\lambda=1$, it will be the normal secant method.

In this paper, we will study the convergence ball of (7) under the assumption that the nonlinear operator F has Fréchet derivatives satisfying the following Lipschitz condition:

$$\begin{aligned}&\left\|F'(x_*)^{-1}\left([x,y;F]-[v,w;F]\right)\right\|\\&\quad\leqslant K\left(\|x-v\|+\|y-w\|\right),\\&\qquad\forall x,y,v,w\in\Omega \text{ for some } K>0.\end{aligned}\tag{8}$$

Under the Lipschitz condition, the radius r_{λ_1} of the relaxed method is proved to be $\lambda/4K$ when $0<\lambda\leqslant 1$; and the radius r_{λ_2} of the relaxed method is proved to be $(2-\lambda)/4K\lambda$ when $1<\lambda<2$. The error estimate is also given.

2. Convergence Ball

Theorem 1. *Suppose $F(x_*)=0$, where the nonlinear operator F is Fréchet differentiable on Ω, $F'(x_*)^{-1}$ exists, the Lipschitz condition (8) holds, and $0<\lambda<2$. Denote*

$$\begin{aligned}r_{\lambda_1}&=\frac{\lambda}{4K},\\r_{\lambda_2}&=\frac{(2-\lambda)}{4K\lambda}.\end{aligned}\tag{9}$$

When $0<\lambda\leqslant 1$, starting from any two initial points x_0,x_{-1} in ball $B(x_*,r_{\lambda_1})$, the sequence $\{x_n\}$ generated by the relaxed secant method (7) converges to the solution x_*. When $1<\lambda<2$, the sequence $\{x_n\}$ generated by the relaxed secant method (7) converges to the solution x_*, with any two initial points x_0,x_{-1} in ball $B(x_*,r_{\lambda_2})$. x_* is the unique solution in ball $B(x_*,1/K)$, that is bigger than ball $B(x_*,r_{\lambda_1})$ and ball $B(x_*,r_{\lambda_2})$. Moreover, we have the following error estimate:

$$\begin{aligned}\|x_n-x_*\|&\leqslant\left(\frac{2-\lambda}{1-2K\theta}-1\right)^n,\quad \text{if } 0<\lambda\leqslant 1,\\\|x_n-x_*\|&\leqslant\left(\frac{(4\lambda-2)K\theta+\lambda-1}{1-2K\theta}\right)^n,\quad \text{if } 1<\lambda<2,\end{aligned}\tag{10}$$

where $\theta=\max\{\|x_0-x_*\|,\|x_{-1}-x_*\|\}$.

Proof. We will prove the above theorem by induction. Firstly, when $0<\lambda<2$, by Lipschitz condition, it is easy to get

$$\begin{aligned}&\left\|I-F'(x_*)^{-1}[x_{-1},x_0;F]\right\|\\&\quad=\left\|F'(x_*)^{-1}\left(F'(x_*)-[x_{-1},x_0;F]\right)\right\|\\&\quad\leqslant K\left(\|x_{-1}-x_*\|+\|x_0-x_*\|\right)<1.\end{aligned}\tag{11}$$

By Banach lemma, we can know $[x_{-1},x_0;F]$ is invertible. Since x_1 is well defined and

$$\begin{aligned}&\left\|\left(F'(x_*)^{-1}[x_{-1},x_0;F]\right)^{-1}\right\|\\&\quad\leqslant\frac{1}{1-K\left(\|x_{-1}-x_*\|+\|x_0-x_*\|\right)},\end{aligned}\tag{12}$$

we can conduct

$$\begin{aligned}&\left\|F'(x_*)^{-1}[x_{-1},x_0;F]\right\|\\&\quad=\left\|F'(x_*)^{-1}\left([x_{-1},x_0;F]-F'(x_*)+F'(x_*)\right)\right\|\\&\quad\leqslant K\left(\|x_{-1}-x_*\|+\|x_0-x_*\|\right)+1.\end{aligned}\tag{13}$$

Then, we can give the estimate of $\|x_1-x_*\|$ when $0<\lambda\leqslant 1$. From $F(x_*)=0$, we have

$$\begin{aligned}\|x_1-x_*\|&=\left\|x_0-x_*-\lambda[x_{-1},x_0;F]^{-1}F(x_0)\right\|\\&=\left\|\left(F'(x_*)^{-1}[x_{-1},x_0;F]\right)^{-1}F'(x_*)^{-1}\right.\\&\quad\left.\cdot\left([x_{-1},x_0;F](x_0-x_*)-\lambda\left(F(x_0)-F(x_*)\right)\right)\right\|\\&\leqslant\left\|\left(F'(x_*)^{-1}[x_{-1},x_0;F]\right)^{-1}\right\|\left\|F'(x_*)^{-1}\right.\\&\quad\cdot\left([x_{-1},x_0;F](x_0-x_*)\right)\\&\quad\left.-\lambda\int_0^1F'(tx_0+(1-t)x_*)\,dt\,(x_0-x_*)\right\|\\&\leqslant\left\|\left(F'(x_*)^{-1}[x_{-1},x_0;F]\right)^{-1}\right\|\|x_0-x_*\|\\&\quad\cdot\left(\lambda\left\|F'(x_*)^{-1}\right.\right.\\&\quad\left.\cdot\left([x_{-1},x_0;F]-\int_0^1F'(tx_0+(1-t)x_*)\,dt\right)\right\|+(1\\&\quad\left.-\lambda)\left\|F'(x_*)^{-1}[x_{-1},x_0;F]\right\|\right).\end{aligned}\tag{14}$$

Using Lipschitz condition with (12) and (13), we have

$$\begin{aligned}\|x_1 - x_*\| &\leqslant \frac{r_{\lambda_1}}{1-2Kr_{\lambda_1}}\Big(\int_0^1 \lambda K(\|x_{-1} - tx_0 - (1-t)x_*\| \\ &+ \|x_0 - tx_0 - (1-t)x_*\|)\,dt + (1-\lambda)(K(\|x_{-1} - x_*\| \\ &+ \|x_0 + x_*\|) + 1)\Big) \\ &= \frac{r_{\lambda_1}}{1-2Kr_{\lambda_1}}\Big(\int_0^1 \lambda K(\|t(x_{-1} - x_0) + (1-t)(x_{-1} - x_*)\| \\ &+ (1-t)\|x_0 - x_*\|)\,dt + (1-\lambda)(K(\|x_{-1} - x_*\| \\ &+ \|x_0 + x_*\|) + 1)\Big).\end{aligned} \tag{15}$$

Obviously, we have

$$\|x_{-1} - x_0\| \leqslant \|x_0 - x_*\| + \|x_{-1} - x_*\|. \tag{16}$$

From $x_{-1}, x_0 \in B(x_*, r_{\lambda_1})$, together with (15), (16), and $r_{\lambda_1} = \lambda/4K$, we have

$$\begin{aligned}&\|x_1 - x_*\| \\ &\quad < \frac{r_{\lambda_1}}{1-2Kr_{\lambda_1}}\left(2K\lambda r_{\lambda_1} + (1-\lambda)\left(2Kr_{\lambda_1} + 1\right)\right) \\ &\quad = r_{\lambda_1}.\end{aligned} \tag{17}$$

This means $x_1 \in B(x_*, r_{\lambda_1})$.

Similar to the procession above, when $1 < \lambda < 2$, we can get that

$$\begin{aligned}\|x_1 - x_*\| &= \left\|x_0 - x_* - \lambda[x_{-1}, x_0; F]^{-1} F(x_0)\right\| \\ &\leqslant \left\|\left(F'(x_*)^{-1}[x_{-1}, x_0; F]\right)^{-1}\right\| \|x_0 - x_*\| \\ &\cdot \Bigg(\lambda \Bigg\|F'(x_*) \\ &\cdot \left([x_{-1}, x_0; F] - \int_0^1 F'(tx_0 + (1-t)x_*)\,dt\right)\Bigg\| + (\lambda \\ &- 1)\left\|F'(x_*)^{-1}[x_{-1}, x_0; F]\right\|\Bigg).\end{aligned} \tag{18}$$

By (13) and (18) and Lipschitz condition we can get

$$\begin{aligned}\|x_1 - x_*\| &\leq \frac{r_{\lambda_2}}{1-2Kr_{\lambda_2}}\Big(\int_0^1 \lambda K(\|x_{-1} - tx_0 - (1-t)x_*\| \\ &+ \|x_0 - tx_0 - (1-t)x_*\|)\,dt + (\lambda-1)(K(\|x_{-1} - x_*\| \\ &+ \|x_0 - x_*\|) + 1)\Big) \\ &= \frac{r_{\lambda_2}}{1-2Kr_{\lambda_2}}\Big(\int_0^1 \lambda K(\|t(x_{-1} - x_0) + (1-t)(x_{-1} - x_*)\| \\ &+ (1-t)\|x_0 - x_*\|)\,dt + (\lambda-1)(K(\|x_{-1} - x_*\| \\ &+ \|x_0 - x_*\|) + 1)\Big) \leqslant \frac{r_{\lambda_2}}{1-2K\lambda_2}\left(2K\lambda r_{\lambda_2} + (\lambda-1)\left(2Kr_{\lambda_2}\right.\right. \\ &\left.\left. + 1\right)\right).\end{aligned} \tag{19}$$

For $x_{-1}, x_0 \in B(x_*, r_{\lambda_2})$,

$$\begin{aligned}&\|x_1 - x_*\| \\ &\quad \leqslant \frac{Kr_{\lambda_2}}{1-2K\lambda_2}\left(2K\lambda r_{\lambda_2} + (\lambda-1)\left(2Kr_{\lambda_2} + 1\right)\right) \\ &\quad = r_{\lambda_2}.\end{aligned} \tag{20}$$

This means that $x_1 \in B(x_*, r_{\lambda_2})$ when $1 < \lambda < 2$.

Now, suppose $\{x_k\}$ $(k = 1, 2, \ldots, n)$ is well defined, $x_k \in B(x_*, r_{\lambda_1})$, when $0 < \lambda \leqslant 1$; $\{x_k\}$ $(k = 1, 2, \ldots, n)$ is well defined, $x_k \in B(x_*, r_{\lambda_2})$, when $1 < \lambda < 2$. Similar to the argumentation about x_{-1} and x_0, when $0 < \lambda < 2$,

$$\begin{aligned}&\left\|I - F'(x_*)^{-1}[x_{n-1}, x_n; F]\right\| \\ &\quad \leqslant K(\|x_{n-1} - x_*\| + \|x_n - x_*\|) < 1.\end{aligned} \tag{21}$$

By the Banach lemma, it is obviously known that $[x_{n-1}, x_n; F]$ is invertible. Hence, x_{n+1} is well defined. We also get

$$\begin{aligned}&\left\|\left(F'(x_*)^{-1}[x_{n-1}, x_n; F]\right)^{-1}\right\| \\ &\quad \leq \frac{1}{1 - K(\|x_{n-1} - x_*\| + \|x_n - x_*\|)}.\end{aligned} \tag{22}$$

When $0 < \lambda \leqslant 1$,

$$\begin{aligned}\|x_{n+1} - x_*\| &\leqslant \frac{\|x_n - x_*\|}{1 - K(\|x_n - x_*\| + \|x_{n-1} - x_*\|)} \\ &\times \left(\lambda K \int_0^1 (\|t(x_{n-1} - x_n) + (1-t)(x_{n-1} - x_*)\|\right. \\ &+ (1-t)\|x_n - x_*\|)\,dt + (1-\lambda) \\ &\left.\cdot (K(\|x_{n-1} - x_*\| + \|x_n - x_*\|) + 1)\right).\end{aligned} \tag{23}$$

And when $1 < \lambda < 2$, we have

$$\begin{aligned}\|x_{n+1} - x_*\| &\leqslant \frac{\|x_n - x_*\|}{1 - K(\|x_n - x_*\| + \|x_{n-1} - x_*\|)} \\ &\times \left(\lambda K \int_0^1 (\|t(x_{n-1} - x_n) + (1-t)(x_{n-1} - x_*)\|\right. \\ &+ (1-t)\|x_n - x_*\|)\,dt + (\lambda-1) \\ &\left.\cdot (K(\|x_{n-1} - x_*\| + \|x_n - x_*\|) + 1)\right).\end{aligned} \tag{24}$$

By the assumptions that $x_{n-1}, x_n \in B(x_*, r_{\lambda_1})$ when $0 < \lambda \leqslant 1$ and $x_{n-1}, x_n \in B(x_*, r_{\lambda_2})$ when $1 < \lambda < 2$, similar to the discussions about x_1, it is known that $x_{n+1} \in B(x_*, r_{\lambda_1})$ when $0 < \lambda \leqslant 1$ and $x_{n+1} \in B(x_*, r_{\lambda_2})$ when $1 < \lambda < 2$.

Therefore, starting from any two initial points x_{-1}, x_0, x_n, the sequence $\{x_n\}$, generated by the relaxed secant method, is well defined when $0 < \lambda \leqslant 1$, $x_n \in B(x_*, r_{\lambda_1})$, and when $1 < \lambda < 2$, $x_n \in B(x_*, r_{\lambda_2})$. It means that the following holds:

$$\|x_n - x_*\| < r_{\lambda_1}, \quad (0 < \lambda \leqslant 1,\ n \geq -1), \tag{25}$$

$$\|x_n - x_*\| < r_{\lambda_2}, \quad (1 < \lambda < 2,\ n \geq -1). \tag{26}$$

Denote

$$\theta_n = \|x_n - x_*\|, \tag{27}$$

$$\theta = \max\{\theta_0, \theta_{-1}\}. \tag{28}$$

When $0 < \lambda \leqslant 1$, from (14) we can get

$$\begin{aligned} \|x_{n+1} - x_*\| \leqslant{} & \frac{\|x_n - x_*\|}{1 - K(\|x_n - x_*\| + \|x_{n-1} - x_*\|)} \\ & \times (\lambda K(\|x_n - x_*\| + \|x_{n-1} - x_*\|) \\ & + (1-\lambda)(K(\|x_n - x_*\| + \|x_{n-1} - x_*\|) + 1)). \end{aligned} \tag{29}$$

Then, by (27), we have

$$\begin{aligned} \theta_{n+1} \leqslant{} & \frac{\theta_n}{1 - K(\theta_n + \theta_{n-1})} (\lambda K(\theta_n + \theta_{n-1}) \\ & + (1-\lambda)(K(\theta_n + \theta_{n-1}) + 1)). \end{aligned} \tag{30}$$

By (23), we know $\theta_n < r_{\lambda_1}$ for all n. Then by (29) and (30), we can induct

$$\theta_{n+1} < \theta_n < \theta_{n-1} < \cdots < \theta_1 < \theta. \tag{31}$$

Then we can see

$$\begin{aligned} \theta_{n+1} & \leqslant \frac{\theta_n}{1 - K(\theta_n + \theta_{n-1})} (K(\theta_n + \theta_{n-1}) + 1 - \lambda) \\ & = \theta_n \frac{K(\theta_n + \theta_{n-1}) - 1 + 2 - \lambda}{1 - K(\theta_n + \theta_{n-1})} \\ & = \theta_n \left(\frac{2-\lambda}{1 - K(\theta_n + \theta_{n-1})} - 1 \right) \\ & < \theta_n \left(\frac{2-\lambda}{1 - 2K\theta} - 1 \right) < \theta \left(\frac{2-\lambda}{1 - 2K\theta} - 1 \right)^{n+1}. \end{aligned} \tag{32}$$

Obviously, $0 < (2-\lambda)/(1-2K\theta) - 1 < 1$. The sequence $\{x_n\}$ converges to the exact solution x_* from (32).

When $1 < \lambda < 2$, from (24),

$$\begin{aligned} \|x_{n+1} - x_*\| \leqslant{} & \frac{\|x_n - x_*\|}{1 - K(\|x_n - x_*\| + \|x_{n-1} - x_*\|)} \\ & \times \left(\lambda K \int_0^1 (\|t(x_{n-1} - x_n) + (1-t)(x_{n-1} - x_*)\| \right. \\ & + (1-t)\|x_n - x_*\|)\,dt + (\lambda - 1) \\ & \left. \cdot (K(\|x_{n-1} - x_*\| + \|x_n - x_*\|) + 1) \right). \end{aligned} \tag{33}$$

Then, by (27), we have

$$\begin{aligned} \theta_{n+1} \leqslant{} & \frac{\theta_n}{1 - K(\theta_n + \theta_{n-1})} (\lambda K(\theta_n + \theta_{n-1}) \\ & + (\lambda - 1)(K(\theta_n + \theta_{n-1}) + 1)) \\ = {} & \frac{\theta_n}{1 - K(\theta_n + \theta_{n-1})} ((2\lambda - 1) K(\theta_n + \theta_{n-1}) + \lambda \\ & - 1). \end{aligned} \tag{34}$$

By (24) and (26), we know $\theta_n < r_{\lambda_2}$ for all n. Then by (26) and (34), when $1 < \lambda < 2$, we can induct

$$\theta_{n+1} < \theta_n < \theta_{n-1} < \cdots < \theta_1 < \theta. \tag{35}$$

So we have

$$\begin{aligned} \theta_{n+1} & \leqslant \theta_n \left(\frac{2(2\lambda - 1)K\theta - 1}{1 - 2K\theta} \right) \\ & < \theta \left(\frac{(4\lambda - 2)K\theta + \lambda - 1}{1 - 2K\theta} \right)^{n+1}. \end{aligned} \tag{36}$$

It is easy to proof that $0 < ((4\lambda - 2)K\theta + \lambda - 1)/(1 - 2K\theta) < 1$. So the sequence $\{x_n\}$ converges to the solution x_*.

Now we show the uniqueness. Assume that there exists another solution $y_* \in B(x_*, 1/K)$. Consider the operator $A = [x_*, y_*; F]$. Because $A[y_* - x_*] = F(y_*) - F(x_*)$, we can get $y_* = x_*$ if the operator A is invertible. From (4), we get

$$\begin{aligned} \left\| I - F'(x_*)^{-1} A \right\| & = \left\| F'(x_*)^{-1} (F'(x_*) - A) \right\| \\ & \leqslant K \|y_* - x_*\| < 1. \end{aligned} \tag{37}$$

So, we can tell that operator A is invertible by Banach lemma. From the definition of r_λ and (9), it is easy to verify that ball $B(x_*, 1/K)$ is bigger than ball $B(x_*, r_{\lambda_1})$ and ball $B(x_*, r_{\lambda_2})$. Proof completes. □

Remark 2. When $\lambda = 1$, the radius of the convergence ball is $1/4K$. We denote $r_1 = 1/4K$. From (9), we know when $0 < \lambda < 1$, $r_{\lambda_1} < r_1$, and when $1 < \lambda < 2$, $r_{\lambda_2} < r_1$. So we have the biggest convergence ball when $\lambda = 1$.

3. Numerical Examples

In this section, we applied the convergence ball result given in Section 2 to solve some numerical problems.

Table 1: Relaxed secant method with different λ.

λ	n	x_n	$\|x_n - x_*\|$	CPU time
0.9	1	1.0315	0.0315	0.000877
	2	1.0057	0.0057	
	3	1.0006	6.4804×10^{-4}	
	4	1.0001	6.6458×10^{-5}	
	5	1.0000	6.6652×10^{-6}	
	6	1.0000	6.6672×10^{-7}	
1	1	1.0128	0.0128	0.000864
	2	1.0012	0.0012	
	3	1.0000	7.3141×10^{-6}	
	4	1.0000	4.2172×10^{-7}	
1.1	1	0.9940	0.006	0.000095
	2	1.0000	1.6176×10^{-6}	
	3	1.0000	1.6708×10^{-7}	

Example 1. Let us consider

$$F(x) = x^2 - 1, \quad x \in [0, 2]. \tag{38}$$

Then $F'(x) = 2x$. $F(x) = 0$ has a root $x_* = 1$ and $F'(x_*) = 2$. It is easy to obtain

$$\begin{aligned}&\left\|F'(x_*)^{-1}([x, y; F] - [u, v; F])\right\| \\ &\quad \leqslant \frac{1}{2}(\|x - u\| + \|y - v\|).\end{aligned} \tag{39}$$

Set $\lambda_1 = 0.9$, $\lambda_2 = 1$, $\lambda_3 = 1.1$. Then the radius of the convergence balls is $r_1 = 9/20$, $r_2 = 1/2$, $r_3 = 9/22$. Choose the initial points $x_{-1} = 1.15$, $x_0 = 1.2$ and they are in the convergence ball of the relaxed secant method. From Table 1, we can see the sequence $\{x_n\}$ converges to x_* with different λ.

As we know, when $\lambda = 1$, the relaxed secant method reduces to normal secant method. From Table 1, we can see that relaxed secant method in the case of $\lambda = 1.1$ outperforms the normal secant method in the sense of iteration number and CPU time.

Example 2. Let us consider the following numerical problem which has been studied in [3, 17, 18]:

$$\begin{aligned}F(x) &= e^x - 1, \\ D &= [-1, 1].\end{aligned} \tag{40}$$

Then $F'(x) = e^x$, $x_* = 0$, and $F'(x_*) = 1$.

Similar to the process in [17], we know $|e^x - e^y| \leqslant e|x - y|$. Then,

$$\left|\int_0^1 \left(e^{tx+(1-t)y} - e^{tu+(1-t)v}\right) dt\right| \leqslant \frac{e}{2}(|x - u| + |y - v|). \tag{41}$$

For $[x, y; F] = \int_0^1 e^{tx+(1-t)y} dt$ and $\|F'(x_*)^{-1}([x, y; F] - [u, v; F])\| \leqslant \|[x, y; F] - [u, v; F]\|$, we can get

$$\begin{aligned}&\left\|F'(x_*)^{-1}([x, y; F] - [u, v; F])\right\| \\ &\quad \leqslant \frac{e}{2}(\|x - u\| + \|y - v\|).\end{aligned} \tag{42}$$

Table 2: Relaxed secant method with different λ.

λ	n	x_n	$\|x_n - x_*\|$	CPU time
0.999	1	0.0038	0.0038	0.000094
	2	1.8245×10^{-4}	1.8245×10^{-4}	
	3	1.6290×10^{-7}	1.6290×10^{-7}	
1	1	0.0039	0.0039	0.000094
	2	1.9078×10^{-4}	1.9078×10^{-4}	
	3	3.7011×10^{-4}	3.7011×10^{-4}	
1.01	1	0.0048	0.0048	0.000108
	2	2.8301×10^{-4}	2.8301×10^{-4}	
	3	3.4788×10^{-6}	3.4788×10^{-6}	
	4	3.4931×10^{-8}	3.4931×10^{-8}	

So $K = e/2$ in this problem. Set $\lambda_1 = 0.999$, $\lambda_2 = 1$, $\lambda_3 = 1.01$. Then, the radius of the convergence balls is $r_1 = 999/2000e$, $r_2 = 1/2e$, $r_3 = 99/202e$. Set the initial points $x_{-1} = 0.08$, $x_0 = 0.1$, and they are in the convergence ball of the relaxed secant method. From Table 2, we can see the sequence $\{x_n\}$ converges to the solution x_*.

From Table 2, we can know that the relaxed scant method ($\lambda = 0.999$) performs the same as the normal secant method in the sense of the iteration number and CPU time, while the solution gotten by the relaxed secant method is closer to the exact solution than that by the normal secant method.

Example 3. Let us consider the nonlinear system:

$$\begin{aligned}2x_1 - \frac{1}{9}x_1^2 - x_2 &= 0, \\ -x_1 + 2x_2 - \frac{1}{9}x_2^2 &= 0.\end{aligned} \tag{43}$$

It comes from the following nonlinear boundary value problem of second order:

$$\begin{aligned}x'' + x^2 &= 0, \\ x(0) = x(1) &= 0,\end{aligned} \tag{44}$$

which has been studied by many authors [5, 13, 16].

Now, define the operator $F : R^2 \to R^2$ such that $F = (F_1, F_2)$. We take $F_1(x_1, x_2) = 2x_1 - (1/9)x_1^2 - x_2 = 0$, $F_2(x_1, x_2) = -x_1 + 2x_2 - (1/9)x_2^2 = 0$, $x = (x_1, x_2) \in R^2$. Then, notice $0 < \lambda \leqslant 1$; it is easy to know F is Fréchet differentiable in R^2 and we get

$$F'(x) = \begin{pmatrix} 2 - \frac{2}{9}x_1 & -1 \\ -1 & 2 - \frac{2}{9}x_2 \end{pmatrix}. \tag{45}$$

Let $x = (x_1, x_2) \in R^2$ and $\|x\| = \|x\|_\infty = \max_{1\leqslant i\leqslant 2}|x_i|$. The corresponding norm on $A \in R^2 \times R^2$ is

$$\|A\| = \max_{1\leqslant i\leqslant 2}\sum_{j=1}^{2}\left|a_{ij}\right|. \tag{46}$$

Table 3: Relaxed secant method with different λ.

λ	n	x_n	$\|x_n - x_*\|$	CPU time
0.9999	1	(8.9732, 8.9732)	0.0277	0.000288
	2	(9.0016, 9.0016)	0.0016	
	3	(9.0000, 9.0000)	5.2307×10^{-6}	
	4	(9.0000, 9.0000)	4.2965×10^{-10}	
1	1	(8.9722, 8.9722)	0.0278	0.001643
	2	(9.0016, 9.0016)	0.0016	
	3	(9.0000, 9.0000)	5.0744×10^{-6}	
	4	(9.0000, 9.0000)	9.2414×10^{-10}	
0.8	1	(8.9455, 8.9455)	0.545	0.001608
	2	(9.0000, 9.0000)	4.0974×10^{-5}	
	3	(9.0000, 9.0000)	2.0838×10^{-6}	
	4	(9.0000, 9.0000)	1.1794×10^{-7}	

It can be verified easily that $x_* = (9,9)$ is a solution of (24) and from (26) we get

$$F'(x) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{47}$$

$F'(x)$ is invertible. Similar to [13], we can deduce that Lipschitz continuous condition is satisfied for $K = 1/9$. Set $\lambda_1 = 0.9999$, $\lambda_2 = 1$, $\lambda_3 = 1.06$. Then the radius of the convergence ball is $r_1 = 2.499975$, $r_2 = 9/4$, $r_3 = 326/212$. Set the two initial points $x_{-1} = (9.5, 9.5)$, $x_0 = (8.5, 8.5)$ and they are in the convergence ball. For results, see Table 3.

Table 3 shows the sequence $\{x_n\}$ generated by the relaxed secant method. From this table, it is known that the sequence $\{x_n\}$ converges, and also the error estimation holds. Moreover, relaxed secant method has more choices than secant method, and optimal parameter λ makes the presented method outperforms the normal secant method.

Example 4. Consider the nonlinear conservative system given in [15]:

$$\frac{d^2 x(t)}{dt^2} = -e^{x(t)}, \quad x(0) = x(1) = 0. \tag{48}$$

Applying the centered finite difference scheme, we can get the nonlinear system:

$$F(x) = Mx + h^2 \phi(x), \tag{49}$$

where $h = 1/(N+1)$ is the step-size and N is a prescribed positive integer. x, $\phi(x)$ are vectors with forms of

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \quad \phi(x) = \begin{pmatrix} e^{x_1} \\ e^{x_2} \\ \vdots \\ e^{x_m} \end{pmatrix}, \tag{50}$$

Table 4: Numerical results for nonlinear conservative systems.

λ	IT	$\|x_{n+1} - x_n\|$	CPU time
0.99	75	8.7445×10^{-7}	0.002581
1	94	1.9319×10^{-7}	0.002931
0.001	63	9.3315×10^{-7}	0.001471

Table 5: Approximated solution.

i	x_i^*
1	0.026205377
2	0.049844664
3	0.070856372
4	0.089184975
5	0.104780806
6	0.117601571
7	0.127610864
8	0.134780833
9	0.139090798
10	0.140529159
11	0.139090906
12	0.134780768
13	0.127610728
14	0.117601588
15	0.104780816
16	0.089184869
17	0.070856314
18	0.049844674
19	0.026205359

and the matrix M has the form

$$A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix}. \tag{51}$$

Take the same parameters used in [15], $N = 19$, $h = 1/20$, and the initial points $x_{-1}(t) = (5/2)t(1-t)$ and $x_0(t) = (1/2)t(1-t)$, $t \in [0,1]$. Then, we can solve this problem by our relaxed secant method, and we compare it with normal secant method. For the results, see Table 4.

From the results, we can know that, in this example, the relaxed secant method performs better. And we list the approximation solution which is gotten by the relaxed secant method in the situation $\lambda = 0.99$ in Table 5.

Competing Interests

The authors declare that they have no competing interests.

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