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Research Article

A New Family of Iterative Methods Based on an Exponential Model for Solving Nonlinear Equations

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We present two new families of iterative methods for obtaining simple roots of nonlinear equations. The first family is developed by fitting the model $m(x) = e^{px}(Ax^2 + Bx + C)$ to the function $f(x)$ and its derivative $f'(x)$, $f''(x)$ at a point x_n. In order to remove the second derivative of the first methods, we construct the second family of iterative methods by approximating the equation $f(x) = 0$ around the point $(x_n, f(x_n))$ by the quadratic equation. Analysis of convergence shows that the new methods have third-order or higher convergence. Numerical experiments show that new iterative methods are effective and comparable to those of the well-known existing methods.

1. Introduction

In this paper, we consider iterative methods to find a simple root α, that is, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a nonlinear equation:

$$f(x) = 0, \tag{1}$$

where $f : I \subset R \to R$ for an open interval I is a scalar function.

Many of the complex problems in science and engineering contains the function of nonlinear and transcendental nature in (1), so finding the simple roots of the nonlinear equation is one of the most important problems in numerical analysis. Numerical iterative methods are often used to obtain the approximate solution of such problems because it is not always possible to obtain its exact solution by usual algebraic process. We all know that Newton's method is an important and basic approach for solving nonlinear equations [1, 2], its formulation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

and this method converges quadratically. Earlier, many investigations [3–21] have been made to explain the root of nonlinear algebraic and transcendental equations.

The outline of the paper is as follows. In Section 2, we firstly describe a one-parameter family of third-order methods by fitting the model $m(x) = e^{px}(Ax^2 + Bx + C)$ to the function $f(x)$ and its derivative $f'(x)$, $f''(x)$ at a point x_n, and then we use a quadratic equation for approximating the equation $f(x) = 0$ to obtain a four-parameter family of second-derivative-free iterative methods. In Section 3, we obtain some different iterative methods by taking several parameters. In Section 4, different numerical tests confirm the theoretical results, and the new methods are comparable with other known methods and give better results in many cases. Finally, we infer some conclusions.

2. Development of Methods and Convergence Analysis

Consider the exponential model:

$$m(x) = e^{px}\left(Ax^2 + Bx + C\right), \tag{3}$$

where A, B, C, and p are parameters. We construct a new iteration scheme by fitting model (3) to the function $f(x)$ and its derivative $f'(x)$ and $f''(x)$ at a point x_n. Imposing the conditions as follows

$$m(x_n) = f(x_n),\quad m'(x_n) = f'(x_n),\quad m''(x_n) = f''(x_n) \tag{4}$$

at the point $(x_n, m(x_n))$ and then solving (3) and (4) for A, B, and C, we have

$$A = \frac{1}{2}\left[f''(x_n) + p^2 f(x_n) - 2pf'(x_n)\right] e^{-px_n}, \tag{5}$$

$$B = \left[f'(x_n) - pf(x_n) - \left(f''(x_n) + p^2 f(x_n) - 2pf'(x_n)\right) x_n\right] e^{-px_n}, \tag{6}$$

$$C = \left[f(x_n) - \frac{1}{2}\left[f''(x_n) + p^2 f(x_n) - 2pf'(x_n)\right] x_n^2 - \left[f'(x_n) - pf(x_n) - \left(f''(x_n) + p^2 f(x_n) - 2pf'(x_n)\right) x_n\right] x_n\right] e^{-px_n}. \tag{7}$$

From (3), (4), and (5), we take the values of A, B, and C into (3) and give

$$m(x) - f(x_n) e^{px-px_n} = \frac{1}{2}\left[f''(x_n) + p^2 f(x_n) - 2pf'(x_n)\right](x - x_n)^2 e^{px-px_n} + \left[f'(x_n) - pf(x_n)\right](x - x_n) e^{px-px_n}. \tag{8}$$

At the root estimate x_{n+1}, it follows that $m(x_{n+1}) = 0$. We consider reducing (8) as follows

$$\frac{1}{2}\left[f''(x_n) + p^2 f(x_n) - 2pf'(x_n)\right](x - x_n)^2 + \left[f'(x_n) - pf(x_n)\right](x - x_n) + f(x_n) = 0. \tag{9}$$

Thus from (9) we develop the iteration formula:

$$x_{n+1} = x_n - \frac{\left[f'(x_n) - pf(x_n)\right] - f'(x_n)\sqrt{1 - L_{f,p}(x_n)}}{f''(x_n) + p^2 f(x_n) - 2pf'(x_n)}, \tag{10}$$

where

$$L_{f,p}(x_n) = \frac{2f''(x_n) f(x_n) + p^2 f^2(x_n) - 2pf'(x_n) f(x_n)}{f'^2(x_n)}. \tag{11}$$

The square root is required in (10); however, this may cost expensively and even fail in the case $1-L_{f,p}(x_n) < 0$. In order to avoid the calculation of the square roots, we will derive some forms free from square roots by Taylor approximation [5].

It is easy to know that Taylor approximation of $\sqrt{1 - L_{f,p}(x_n)}$ is

$$\sqrt{1 - L_{f,p}(x_n)} = \sum_{k\geq 0} \binom{\frac{1}{2}}{k} \left(-L_{f,p}(x_n)\right)^k. \tag{12}$$

Using (12) in (10), we can obtain the following form:

$$x_{n+1} = x_n - \left(\left(\left[f'(x_n) - pf(x_n)\right] - f'(x_n)\sum_{k\geq 0}^{m} \binom{\frac{1}{2}}{k}\left(-L_{f,p}(x_n)\right)^k\right) \times \left(f''(x_n) + p^2 f(x_n) - 2pf'(x_n)\right)^{-1}\right). \tag{13}$$

We have the convergence analysis of the methods by (13).

Theorem 1. *Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \subset R \to R$ for an open interval I. If x_0 is sufficiently close to α, for $m \geq 1$, the methods defined by* (13) *are cubically convergent.*

Proof. Let $e_n = x_n-\alpha$, and we use the following Taylor expansions:

$$f(x_n) = f'(\alpha)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O\left(e_n^5\right)\right], \tag{14}$$

where $c_k = (1/k!)(f^{(k)}(\alpha)/f'(\alpha))$; furthermore, we have

$$f'(x_n) = f'(\alpha)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + O\left(e_n^5\right)\right], \tag{15}$$

$$f''(x_n) = f'(\alpha)\left[2c_2 + 6c_3 e_n + 12c_4 e_n^2 + 20c_5 e_n^3 + 30c_6 e_n^4 + O\left(e_n^5\right)\right]. \tag{16}$$

Since (15), we obtain

$$f'^2(x_n) = f'^2(\alpha)\left[1 + 4c_2 e_n + \left(6c_3 + 4c_2^2\right) e_n^2 + \left(8c_4 + 12c_2 c_3\right) e_n^3 + \left(10c_5 + 16c_2 c_4 + 9c_3^2\right) e_n^4 + O\left(e_n^5\right)\right]. \tag{17}$$

Since (14), we get

$$f^2(x_n) = f'^2(\alpha)\left[e_n^2 + 2c_2 e_n^3 + \left(2c_3 + c_2^2\right) e_n^4 + O\left(e_n^5\right)\right]. \tag{18}$$

From (14) and (15), we also easily have

$$f(x_n) f'(x_n) = f'^2(\alpha)\left[e_n + 3c_2 e_n^2 + \left(2c_2^2 + 4c_3\right) e_n^3 + \left(5c_4 + 5c_2 c_3\right) e_n^4 + O\left(e_n^5\right)\right]. \quad (19)$$

We obtain the following expression by taking into account (14) and (16):

$$\begin{aligned} f(x_n) f''(x_n) &= f'^2(\alpha)\left[2c_2 e_n + \left(2c_2^2 + 6c_3\right) e_n^2 + \left(8c_2 c_3 + 12c_4\right) e_n^3 \right. \\ &\quad \left. + \left(14c_2 c_4 + 6c_3^2 + 20c_5\right) e_n^4 + O\left(e_n^5\right)\right]. \end{aligned} \quad (20)$$

From (18), (19), and (20), we obtain

$$\begin{aligned} &2f''(x_n) f(x_n) + p^2 f^2(x_n) - 2pf'(x_n) f(x_n) \\ &= f'^2(\alpha)\left[\left(4c_2 - 2p\right) e_n + \left(4c_2^2 + 12c_3 + p^2 - 6pc_2\right) e_n^2 \right. \\ &\quad + \left(16c_2 c_3 + 24c_4 + 2p^2 c_2 - 4pc_2^2 - 8pc_3\right) e_n^3 \\ &\quad + \left(28c_2 c_4 + 12c_3^2 + 40c_5 + 2p^2 c_3 + p^2 c_2^2 \right. \\ &\quad \left.\left. - 10pc_4 - 10pc_2 c_3\right) e_n^4 + O\left(e_n^5\right)\right]. \end{aligned} \quad (21)$$

From (17) and (21), we have

$$\begin{aligned} L_{f,p}(x_n) = f'(\alpha)&\left[\left(4c_2 - 2p\right) e_n \right. \\ &+ \left(12c_3 - 12c_2^2 + p^2 + 2pc_2\right) e_n^2 \\ &+ \left(24c_4 - 56c_2 c_3 - 2p^2 c_2 - 4pc_2^2 \right. \\ &\left.\left. + 4pc_3 + 32c_2^3\right) e_n^3 + O\left(e_n^4\right)\right]. \end{aligned} \quad (22)$$

From (14) and (15), we have

$$\begin{aligned} &f'(x_n) - pf(x_n) \\ &= f'(\alpha)\left[1 + \left(2c_2 - p\right) e_n + \left(3c_3 - pc_2\right) e_n^2 \right. \\ &\quad \left. + \left(4c_4 - pc_3\right) e_n^3 + \left(5c_5 - pc_4\right) e_n^4 + O\left(e_n^5\right)\right]. \end{aligned} \quad (23)$$

Using (14), (15), and (16), we have

$$\begin{aligned} &f''(x_n) + p^2 f(x_n) - 2pf'(x_n) \\ &= f'(\alpha)\left[\left(2c_2 - 2p\right) + \left(6c_3 + p^2 - 4pc_2\right) e_n \right. \\ &\quad + \left(12c_4 + p^2 c_2 - 6pc_3\right) e_n^2 + \left(20c_5 + p^2 c_3 - 8pc_4\right) e_n^3 \\ &\quad \left. + \left(30c_6 + p^2 c_4 - 10pc_5\right) e_n^4 + O\left(e_n^5\right)\right]. \end{aligned} \quad (24)$$

We know that

$$\begin{aligned} &\sum_{k\geq 0}^{m} \binom{\frac{1}{2}}{k} \left(-L_{f,p}(x_n)\right)^k \\ &= 1 - \frac{1}{2} L_{f,p}(x_n) - \frac{1}{8} L_{f,p}(x_n)^2 - \frac{1}{16} L_{f,p}(x_n)^3 + \cdots \\ &= 1 - \left(2c_2 - p\right) e_n + \left(pc_2 + 4c_2^2 - 6c_3 - p^2\right) e_n^2 \\ &\quad - \left(2p^2 c_2 - 16c_2 c_3 - 4pc_3 + 12c_4 + 8c_2^3 - p^3\right) e_n^3 \\ &\quad + O\left(e_n^4\right). \end{aligned} \quad (25)$$

From (15) and (25), we obtain

$$\begin{aligned} &f'(x_n) \sum_{k\geq 0}^{m} \binom{\frac{1}{2}}{k} \left(-L_{f,p}(x_n)\right)^k \\ &= f'(\alpha)\left[1 + pe_n + \left(3pc_2 - p^2 - 3c_3\right) e_n^2 \right. \\ &\quad + \left(7pc_3 - 4p^2 c_2 - 2c_2 c_3 + p^3 - 8c_4 + 2pc_2^2\right) e_n^3 \\ &\quad \left. + O\left(e_n^4\right)\right]. \end{aligned} \quad (26)$$

From (23) and (26), we have

$$\begin{aligned} &\left[f'(x_n) - pf(x_n)\right] - f'(x_n) \sum_{k\geq 0}^{m} \binom{\frac{1}{2}}{k} \left(-L_{f,p}(x_n)\right)^k \\ &= 2\left(c_2 - p\right) e_n - \left(4pc_2 - 6c_3 - p^2\right) e_n^2 \\ &\quad - \left(8pc_3 - 12c_4 - 4p^2 c_2 - 2c_2 c_3 + p^3 + 2pc_2^2\right) e_n^3 \\ &\quad + O\left(e_n^4\right). \end{aligned} \quad (27)$$

Using (13), (24), and (27), we have

$$\begin{aligned} x_{n+1} = x_n - &\left(\left(\left[f'(x_n) - pf(x_n)\right] \right.\right. \\ &\left. - f'(x_n) \sum_{k\geq 0}^{m} \binom{\frac{1}{2}}{k} \left(-L_{f,p}(x_n)\right)^k \right) \\ &\left. \times \left(f''(x_n) + p^2 f(x_n) - 2pf'(x_n)\right)^{-1} \right) \\ = x_n - e_n - &\left(\frac{1}{2}p^2 + c_3 - pc_2\right) e_n^3 + O\left(e_n^4\right). \end{aligned} \quad (28)$$

From $e_{n+1} = x_{n+1} - \alpha$, we have

$$e_{n+1} = \left(pc_2 - \frac{1}{2}p^2 - c_3\right) e_n^3 + O\left(e_n^4\right), \quad (29)$$

which completes the proof. □

The family methods given by (13) are novel third-order methods, but the methods depend on the second derivatives in computing process, and therefore their practical applications are restricted in some cases. In recent years, several methods with free second derivatives have been developed; see [4–15] and references therein.

In order to avoid the calculation of the second derivatives, we consider approximating the equation $f(x) = 0$ around the point $(x_n, f(x_n))$ by the quadratic equation in x and y in the following form [8]:

$$a_1 x^2 + a_2 y^2 + a_3 xy + a_4 x + a_5 y + a_6 = 0, \tag{30}$$

where $a_i \in R$, $i = 1, 2, \ldots, 6$, are parameters. We impose the tangency conditions

$$\begin{aligned} y(x_n) &= f(x_n), \\ y'(x_n) &= f'(x_n), \\ y(w_n) &= f(w_n), \end{aligned} \tag{31}$$

where x_n is nth iterate and

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{32}$$

From (30) and (31), we have

$$\begin{aligned} f''(x_n) &\approx y''(x_n) \\ &= L_{a_i}(x_n, w_n) \\ &= \left(\left[2a_1 + 2a_2 f'^2(x_n) + 2a_3 f'(x_n)\right] f'^2(x_n) f(w_n)\right) \\ &\quad \times \left(a_1 f^2(x_n) + a_2 f'^2(x_n)\left[f(x_n) - f(w_n)\right]^2 \right. \\ &\quad \left. + a_3 f(x_n) f'(x_n)\left[f(x_n) - f(w_n)\right]\right)^{-1}, \end{aligned} \tag{33}$$

where $i = 1, 2, 3$.

From (33) we can approximate

$$L_{a_i,f,p}(x_n, w_n) = \frac{2L_{a_i}(x_n, w_n) f(x_n) + p^2 f^2(x_n) - 2pf'(x_n) f(x_n)}{f'^2(x_n)}. \tag{34}$$

Using $L_{a_i,f,p}(x_n, w_n)$ instead of $L_{f,p}(x_n)$ (11), we obtain a new four-parameter family of methods free from second derivative:

$$\begin{aligned} x_{n+1} = x_n - \Bigg(\Bigg(&\left[f'(x_n) - pf(x_n)\right] \\ &- f'(x_n) \sum_{k\geq 0}^{m} \binom{\frac{1}{2}}{k} \left(-L_{a_i,f,p}(x_n, w_n)\right)^k\Bigg) \\ &\times \left(L_{a_i}(x_n, w_n) + p^2 f(x_n) - 2pf'(x_n)\right)^{-1}\Bigg), \end{aligned} \tag{35}$$

where a_i $(i = 1, 2, 3)$, $p \in R$, $m \geq 0$.

We also have the convergence analysis of the methods by (35).

Theorem 2. *Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \subset R \to R$ for an open interval I. If x_0 is sufficiently close to α, for $m \geq 1$, a_i $(i = 1, 2, 3)$, $p \in R$, the methods defined by (35) are at least cubically convergent; as particular cases, if $m \geq 2$, $a_2 = a_3 = p = 0$, $a_1 \in R$ and the methods have convergence order four.*

Proof. Let $e_n = x_n - \alpha$, and we use the following Taylor expansions:

$$f(x_n) = f'(\alpha)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O\left(e_n^5\right)\right], \tag{36}$$

where $c_k = (1/k!)(f^{(k)}(\alpha)/f'(\alpha))$; furthermore, we have

$$\begin{aligned} f^2(x_n) &= f'^2(\alpha)\left[e_n^2 + 2c_2 e_n^3 + \left(2c_3 + c_2^2\right) e_n^4 + O\left(e_n^5\right)\right], \end{aligned} \tag{37}$$

$$\begin{aligned} f'(x_n) &= f'(\alpha)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + O\left(e_n^5\right)\right]. \end{aligned} \tag{38}$$

Dividing (36) by (38)

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} &= e_n - c_2 e_n^2 + 2\left(c_2^2 - c_3\right) e_n^3 \\ &\quad + \left(7c_2 c_3 - 4c_2^3 - 3c_4\right) e_n^4 + O\left(e_n^5\right). \end{aligned} \tag{39}$$

From (39), we get

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= \alpha + c_2 e_n^2 - 2\left(c_2^2 - c_3\right) e_n^3 \\ &\quad - \left(7c_2 c_3 - 4c_2^3 - 3c_4\right) e_n^4 + O\left(e_n^5\right). \end{aligned} \tag{40}$$

Expanding $f(w_n)$ in Taylor's Series about α and using (40), we get

$$\begin{aligned} f(w_n) &= f'(\alpha)\left[w_n - \alpha + c_2(w_n-\alpha)^2 + c_3(w_n-\alpha)^3 + c_4(w_n-\alpha)^4 + \cdots\right] \\ &= f'(\alpha)\left[c_2e_n^2 + 2\left(c_3 - c_2^2\right)e_n^3 + \left(5c_2^3 + 3c_4 - 7c_2c_3\right)e_n^4 + O\left(e_n^5\right)\right]. \end{aligned} \tag{41}$$

From (36) and (41), we have

$$f(x_n)f(w_n) = f'^2(\alpha)\left[c_2e_n^3 + \left(2c_3 - c_2^2\right)e_n^4 + O\left(e_n^5\right)\right]. \tag{42}$$

From (38), we obtain

$$\begin{aligned} f'^2(x_n) = f'^2(\alpha)\Big[&1 + 4c_2e_n + \left(6c_3 + 4c_2^2\right)e_n^2 + \left(8c_4 + 12c_2c_3\right)e_n^3 \\ &+ \left(10c_5 + 16c_2c_4 + 9c_3^2\right)e_n^4 + O\left(e_n^5\right)\Big]. \end{aligned} \tag{43}$$

From (38), (41), and (43) we also easily obtain

$$\begin{aligned} &\left[2a_1 + 2a_2f'^2(x_n) + 2a_3f'(x_n)\right]f'^2(x_n)f(w_n) \\ &= f'^3(\alpha)\Big[2\left(a_2f'^2(\alpha) + a_3f'(\alpha) + a_1\right)c_2e_n^2 \\ &\quad + 4\left(a_2f'^2(\alpha)c_3 + 3a_2f'^2(\alpha)c_2^2 + a_3f'(\alpha)c_3 + 2f'(\alpha)a_3c_2^2 + a_1c_3 + a_1c_2^2\right)e_n^3 \\ &\quad + 2\left(3a_2f'^2(\alpha)c_4 + 21f'^2(\alpha)a_2c_2c_3 + 5f'(\alpha)a_3c_2^3 + 3a_3f'(\alpha)c_4 + 14f'(\alpha)a_3c_2c_3 + 7a_1c_3c_2\right)e_n^4 + O\left(e_n^5\right)\Big]. \end{aligned} \tag{44}$$

Substituting (36), (37), (38), (41), and (43) in the denominator of $L_{a_i}(x_n, w_n)$, we obtain

$$\begin{aligned} &a_1f^2(x_n) + a_2f'^2(x_n)\left[f(x_n) - f(w_n)\right]^2 + a_3f(x_n)f'(x_n)\left[f(x_n) - f(w_n)\right] \\ &= f'^2(\alpha)\Big[\left(a_1 + a_2f'^2(\alpha) + a_3f'(\alpha)\right)e_n^2 + \left(2a_1c_2 + 4a_2c_2f'^2(\alpha) + 3a_3c_2f'(\alpha)\right)e_n^3 \\ &\quad + \left(2a_1c_3 + a_1c_2^2 + 8a_2c_2^2f'^2(\alpha) + 4a_2c_3f'^2(\alpha) + 4a_3c_2^2f'(\alpha) + 3a_3c_3f'(\alpha)\right)e_n^4 + O\left(e_n^5\right)\Big]. \end{aligned} \tag{45}$$

Using (44) and (45), we have

$$\begin{aligned} &L_{a_i}(x_n, w_n) \\ &= f'(\alpha)\Big[2c_2 + 2\left(2a_2f'^2(\alpha)c_3 + 2a_2f'^2(\alpha)c_2^2 + 2a_3f'(\alpha)c_3 + f'(\alpha)a_3c_2^2 + 2a_1c_3\right)e_n / \left(a_2f'^2(\alpha) + a_3f'(\alpha) + a_1\right) \\ &\quad - \big(+16a_2^2f'^4(\alpha)c_2^3 - 14f'^3(\alpha)a_2c_2c_3a_3 - 6a_2f'^3(\alpha)c_4a_3 - 9f'^4(\alpha)a_2^2c_2c_3 \\ &\quad + 17f'^3(\alpha)a_3c_2^3a_2 - 5f'^2(\alpha)a_3^2c_2c_3 - 3a_2f'^2(\alpha)c_4a_1 + 2f'(\alpha)a_3c_2^3a_1 \\ &\quad - 3a_3f'(\alpha)c_4a_1 + a_1^2c_2^3 - 3a_2^2f'(\alpha)^4c_4 + 2f'(\alpha)^2a_3^2c_2^3 - 3a_3^2f'(\alpha)^2c_4 - a_1^2c_3c_2 \\ &\quad - 10f'^2(\alpha)a_2c_2c_3a_1 - 6f'(\alpha)a_3c_2c_3a_1 + 13a_1c_2^3a_2f'^2(\alpha)\big)e_n^2 \\ &\quad / \left(a_2f'^2(\alpha) + a_3f'(\alpha) + a_1\right)^2 + O\left(e_n^3\right)\Big]. \end{aligned} \tag{46}$$

From (36), (37), and (46), we have

$$\begin{aligned} &2L_{a_i}(x_n, w_n)f(x_n) + p^2f^2(x_n) - 2pf'(x_n)f(x_n) \\ &= f'^2(\alpha)\Big[2\left(2c_2 - p\right)e_n \\ &\quad + \big(12a_2f'^2(\alpha)c_2^2 + 8f'(\alpha)a_3c_2^2 + 4a_1c_2^2 + p^2a_2f'(\alpha)^2 + p^2a_2f'(\alpha) \\ &\quad + p^2a_1 + 8a_2f'^2(\alpha)c_3 + 8a_3f'(\alpha)c_3 + 8a_1c_3 - 6pc_2a_2f'^2(\alpha) - 6pc_2a_3f'(\alpha) \\ &\quad - 6pc_2a_1\big)e_n^2 / \left(a_2f'^2(\alpha) + a_3f'(\alpha) + a_1\right) \\ &\quad - 2\big(2pc_2^2c^2f'^2(\alpha) + 2pc_2^2a_2^2f'^4(\alpha) - 6f'^2(\alpha)a_3^2c_4 - 6f'^4(\alpha)a_2^2c_4 \\ &\quad + 28f'^4(\alpha)a_2^2c_2^3 + 2a_1^2c_2^3 - p^2c_2a_2^2f'^4(\alpha) - p^2c_2a_3^2f'^2(\alpha) - 2p^2c_2a_2f'^3(\alpha)a_3 \\ &\quad - 2p^2c_2a_2f'^2(\alpha)a_1 - 2p^2c_2a_3f'(\alpha)a_1 \end{aligned}$$

$$+2pc_2^2a_1^2+4pc_3a_1^2+2f'(\alpha)^2a_3^2c_2^3 -32f'^2(\alpha)a_2c_2c_3a_1-16f'^2(\alpha)c^2c_2c_3 -40f'^3(\alpha)a_2c_2c_3a_3-6f'^2(\alpha)a_2c_4a_1 +2f'(\alpha)a_3c_2^3a_1-4f'^2(\alpha)a_1c_2^3a_2 -12f'^3(\alpha)a_2c_4a_3-6f'(\alpha)a_3c_4a_1 -6f'(\alpha)^3a_3c_2^3a_2-24f'(\alpha)a_3c_2c_3a_1 -8a_1^2c_3c_2-24f'^4(\alpha)a_2^2c_2c_3 +4pc_2^2a_2f'(\alpha)^3a_3+4pc_2^2a_2f'^2(\alpha)a_1 +4pc_2^2a_3f'(\alpha)a_1+8pc_3a_2f'^3(\alpha)a_3 +8pc_3a_2f'^2(\alpha)a_1+8pc_3a_3f'(\alpha)a_1 -p^2c_2a_1^2+4pc_3a_3^2f'^2(\alpha)+4pc_3a_2^2f'(\alpha)^4\Big)e_n^2 \Big/\left(a_2f'^2(\alpha)+a_3f'(\alpha)+a_1\right)^2+O\left(e_n^3\right)\Big]. \quad (47)$$

Since (43) and (47), we get

$$L_{a_i,f,p}(x_n,w_n) = \Big[2(2c_2-p)e_n - \Big(4a_2f'^2(\alpha)c_2^2+8a_3f'(\alpha)c_2^2 +12a_1c_2^2-p^2a_2f'^2(\alpha)-p^2a_3f'(\alpha) -p^2a_1-8a_2f'^2(\alpha)c_3-8a_3f'(\alpha)c_3 -8a_1c_3-2pc_2a_2f'^2(\alpha)-2pc_2a_3f'(\alpha) -2pc_2a_1\Big)e_n^2\Big/\left(a_2f'^2(\alpha)+a_3f'(\alpha)+a_1\right) +2\Big(-2pc_2^2a_3^2f'^2(\alpha)-2pc_2^2a_2^2f'^4(\alpha) +6f'^2(\alpha)a_3^2c_4+6f'^4(\alpha)a_2^2c_4 -28f'^4(\alpha)a_2^2c_2^3+14a_1^2c_2^3 -p^2c_2a_2^2f'^4(\alpha)-p^2c_2a_3^2f'^2(\alpha) -2p^2c_2a_2f'^3(\alpha)a_3-2p^2c_2a_2f'^2(\alpha)a_1 -2p^2c_2a_3f'(\alpha)a_1-2pc_2^2a_1^2 +2pc_3a_1^2+6f'^2(\alpha)a_3^2c_2^3 -24f'^2(\alpha)a_2c_2c_3a_1-12f'^2(\alpha)a_3^2c_2c_3 -16f'^3(\alpha)a_2c_2c_3a_3+6f'^2(\alpha)a_2c_4a_1 +22f'(\alpha)a_3c_2^3a_1+20f'^2(\alpha)a_1c_2^3a_2 +12f'^3(\alpha)a_2c_4a_3+6f'(\alpha)a_3c_4a_1 +14f'^3(\alpha)a_3c_2^3a_2-32f'(\alpha)a_3c_2c_3a_1 -20a_1^2c_3c_2-4f'^4(\alpha)a_2^2c_2c_3 -4pc_2^2a_2f'^3(\alpha)a_3-4pc_2^2a_2f'^2(\alpha)a_1 -4pc_2^2a_3f'(\alpha)a_1+4pc_3a_2f'^3(\alpha)a_3 +4pc_3a_2f'^2(\alpha)a_1+4pc_3a_3f'(\alpha)a_1 -p^2c_2a_1^2+2pc_3a_3^2f'^2(\alpha) +2pc_3a_2^2f'^4(\alpha)\Big)e_n^3 \Big/\left(a_2f'^2(\alpha)+a_3f'(\alpha)+a_1\right)^2+O\left(e_n^4\right)\Big]. \quad (48)$$

Using (48), we write

$$\sum_{k\geq 0}^{m}\binom{\frac{1}{2}}{k}\left(-L_{a_i,f,p}(x_n,w_n)\right)^k = 1-\frac{1}{2}L_{a_i,f,p}(x_n,w_n)-\frac{1}{8}L_{a_i,f,p}(x_n,w_n)^2 -\frac{1}{16}L_{a_i,f,p}(x_n,w_n)^3+\cdots = 1-(2c_2-p)e_n +\frac{1}{2}\Big(4a_2f'^2(\alpha)c_2^2+8cf'(\alpha)c_2^2 +12a_1c_2^2-p^2a_2f'^2(\alpha)-p^2a_3f'(\alpha) -p^2a_1-8a_2f'^2(\alpha)c_3-8a_3f'(\alpha)c_3 -8a_1c_3-2pc_2a_2f'^2(\alpha)-2pc_2a_3f'(\alpha) -2pc_2a_1\Big)e_n^2\Big/\left(a_2f'^2(\alpha)+a_3f'(\alpha)+a_1\right) +O\left(e_n^3\right). \quad (49)$$

Using (36), (38), and (46), we obtain

$$L_{a_i}(x_n, w_n) + p^2 f(x_n) - pf'(x_n)$$
$$= f'(\alpha)\Big[2(c_2 - p)$$
$$+ \Big(4a_2 f'^2(\alpha) c_3$$
$$+ 4a_2 f'^2(\alpha) c_2^2 + 4a_3 f'(\alpha) c_3 + 2a_3 f'(\alpha) c_2^2$$
$$+ 4a_1 c_3 + p^2 a_2 f'^2(\alpha) + p^2 a_3 f'(\alpha) + p^2 a_1$$
$$-4pc_2 a_2 f'^2(\alpha) - 4pc_2 a_3 f'(\alpha) - 4pc_2 a_1\Big) e_n$$
$$/\Big(a_2 f'^2(\alpha) + a_3 f'(\alpha) + a_1\Big)$$
$$-\Big(6pc_3 a_1^2 - p^2 c_2 a_2^2 f'^4(\alpha) - p^2 c_2 a_1^2$$
$$+ 6pf'^4(\alpha) c_3 a_2^2 + 6pf'^2(\alpha) c_3 a_3^2$$
$$- 2p^2 c_2 a_3 f'(\alpha) a_1 - 2p^2 c_2 a_2 f'^2(\alpha) a_1$$
$$- 2p^2 c_2 a_2 f'^3(\alpha) a_3 - p^2 c_2 a_3^2 f'^2(\alpha)$$
$$+ 12 f'^3(\alpha) pc_3 a_2 a_3 + 12pf'^2(\alpha) c_3 a_2 a_1$$
$$+ 12pf'(\alpha) c_3 a_3 a_1 - 18a_2^2 f'^4(\alpha) c_2 c_3$$
$$+ 26a_1 c_2^3 a_2 f'^2(\alpha) - 20c_2 a_2 f'^2(\alpha) c_3 a_1$$
$$- 12c_2 a_3 f'(\alpha) c_3 a_1 - 12 f'^3(\alpha) a_2 c_4 a_3$$
$$- 6a_2^2 f'^4(\alpha) c_4 + 4a_3^2 f'^2(\alpha) c_2^3 - 6a_3^2 f'^2(\alpha) c_4$$
$$- 2a_1^2 c_3 c_2 + 2a_1^2 c_2^3 + 32a_2^2 f'^4(\alpha) c_2^3$$
$$- 28c_2 a_2 f'^3(\alpha) c_3 a_3 + 34 f'^3(\alpha) a_3 c_2^3 a_2$$
$$- 10a_3^2 f'^2(\alpha) c_2 c_3 - 6a_2 f'^2(\alpha) c_4 a_1$$
$$+4a_3 f'(\alpha) c_2^3 a_1 - 6a_3 f'(\alpha) c_4 a_1\Big) e_n^2$$
$$/\Big(a_2 f'^2(\alpha) + a_3 f'(\alpha) + a_1\Big)^2 + O\left(e_n^3\right)\Big]. \tag{50}$$

Taking into account (36), (38), (49), and (50), we finally obtain

$$x_{n+1}$$
$$= x_n - \Bigg(\Bigg(\Big[f'(x_n) - pf(x_n)\Big] - f'(x_n)$$
$$\times \sum_{k\geq 0}^{m} \binom{\frac{1}{2}}{k} \left(-L_{a_i,f,p}(x_n, w_n)\right)^k\Bigg)$$
$$\times \left(L_{a_i}(x_n, w_n) + p^2 f(x_n) - 2pf'(x_n)\right)^{-1}\Bigg)$$
$$= x_n - e_n + \frac{1}{2}\Big(-40 f'^3(\alpha) a_3 c_2^3 a_2 - 30a_1 c_2^3 a_2 f'^2(\alpha)$$
$$- 2a_3 f'(\alpha) c_2^3 a_1 - 2a_3^2 f'^2(\alpha) c_2^3$$
$$- 4a_2^2 f'^4(\alpha) c_2^3 + 2c_2^2 pa_1^2 + 6pf'(\alpha) c_2^2 a_3 a_1$$
$$+ 8pf'^2(\alpha) c_2^2 a_2 a_1 + 4pf'^2(\alpha) c_2^2 a_3^2$$
$$+ 10 f'^3(\alpha) pc_2^2 a_2 a_3 + 6pf'^4(\alpha) c_2^2 a_2^2$$
$$- 3p^2 c_2 a_3^2 f'^2(\alpha) - 6p^2 c_2 a_2 f'^2(\alpha) a_1$$
$$- 6p^2 c_2 a_3 f'(\alpha) a_1 - 3p^2 c_2 a_1^2$$
$$- 3p^2 c_2 a_2^2 f'^4(\alpha) - 6p^2 c_2 a_2 f'^3(\alpha) a_3$$
$$+ 2 f'^2(\alpha) p^3 a_1 a_2 + 2 f'^3(\alpha) p^3 a_2 a_3$$
$$+ p^3 a_1^2 + 2 f'(\alpha) p^3 a_1 a_3 + f'^4(\alpha) p^3 a_2^2$$
$$+ f'^2(\alpha) p^3 a_3^2\Big) e_n^3 / (c_2 - p)$$
$$/\Big(a_2 f'^2(\alpha) + a_3 f'(\alpha) + a_1\Big)^2 + O\left(e_n^4\right). \tag{51}$$

Taking into account the last expression (51) and $e_{n+1} = x_{n+1} - \alpha$, we have

$$e_{n+1} = \frac{1}{2}\Big(-40 f'^3(\alpha) a_3 c_2^3 a_2 - 30a_1 c_2^3 a_2 f'^2(\alpha)$$
$$- 2a_3 f'(\alpha) c_2^3 a_1 - 2a_3^2 f'^2(\alpha) c_2^3$$
$$- 4a_2^2 f'^4(\alpha) c_2^3 + 2c_2^2 pa_1^2 + 6pf'(\alpha) c_2^2 a_3 a_1$$
$$+ 8pf'^2(\alpha) c_2^2 a_2 a_1 + 4pf'^2(\alpha) c_2^2 a_3^2$$
$$+ 10 f'^3(\alpha) pc_2^2 a_2 a_3 + 6pf'^4(\alpha) c_2^2 a_2^2$$
$$- 3p^2 c_2 a_3^2 f'^2(\alpha) - 6p^2 c_2 a_2 f'^2(\alpha) a_1$$
$$- 6p^2 c_2 a_3 f'(\alpha) a_1 - 3p^2 c_2 a_1^2$$
$$- 3p^2 c_2 a_2^2 f'^4(\alpha) - 6p^2 c_2 a_2 f'^3(\alpha) a_3$$
$$+ 2 f'^2(\alpha) p^3 a_1 a_2 + 2 f'^3(\alpha) p^3 a_2 a_3$$
$$+ p^3 a_1^2 + 2 f'(\alpha) p^3 a_1 a_3 + f'^4(\alpha) p^3 a_2^2$$
$$+ f'^2(\alpha) p^3 a_3^2\Big) e_n^3 / (c_2 - p)$$
$$/\Big(a_2 f'^2(\alpha) + a_3 f'(\alpha) + a_1\Big)^2 + O\left(e_n^4\right). \tag{52}$$

This means that the methods defined by (35) are at least of order three for any a_i $(i = 1, 2, 3)$, $p \in R$. Furthermore, we consider that if $m \geq 2$, $a_2 = a_3 = p = 0$, then the methods defined by (35) are shown to converge the order four. □

3. Some Special Cases

From (33)–(35), we have

$$L_{a_1,a_2,a_3}(x_n, w_n) = \left(\left[2a_1 + 2a_2 f'^2(x_n) + 2a_3 f'(x_n)\right] f'^2(x_n) f(w_n)\right) \times \left(a_1 f^2(x_n) + a_2 f'^2(x_n)\left[f(x_n) - f(w_n)\right]^2 + a_3 f(x_n) f'(x_n)\left[f(x_n) - f(w_n)\right]\right)^{-1}, \tag{53}$$

where $a_1, a_2, a_3 \in R$. From (33) we can approximate

$$L_{a_1,a_2,a_3,f,p}(x_n, w_n) = \left(2L_{a_1,a_2,a_3}(x_n, w_n) f(x_n) + p^2 f^2(x_n) - 2pf'(x_n) f(x_n)\right) \times \left(f'^2(x_n)\right)^{-1}. \tag{54}$$

Let

$$K_{a_1,a_2,a_3,f,p}(x_n, w_n) = \left(\left[f'(x_n) - pf(x_n)\right] - f'(x_n)\sum_{k\geq 0}^{m}\binom{\frac{1}{2}}{k}\left(-L_{a_1,a_2,a_3,f,p}(x_n, w_n)\right)^k\right) \times \left(L_{a_1,a_2,a_3}(x_n, w_n) + p^2 f(x_n) - 2pf'(x_n)\right)^{-1}, \tag{55}$$

where $a_1, a_2, a_3, p \in R$, $m \geq 0$.

1^0: If $a_1 = a_2 = a_3 = p = 1$, and $m = 3$, from (55), we obtain a third-order method (LM1):

$$x_{n+1} = x_n - K_{1,1,1,f,1}(x_n, w_n). \tag{56}$$

2^0: If $a_1 = a_2 = 0$, $a_3 = p = 1$, and $m = 3$, from (55) we also obtain a third-order method (LM2):

$$x_{n+1} = x_n - K_{0,0,1,f,1}(x_n, w_n). \tag{57}$$

3^0: If $a_1 = 1$, $a_2 = 0$, $a_3 = p = 1$, and $m = 3$, from (55) we obtain a new third-order method (LM3):

$$x_{n+1} = x_n - K_{1,0,1,f,1}(x_n, w_n). \tag{58}$$

4^0: If $a_1 = -1$, $a_2 = a_3 = 0$, $p = 1$, and $m = 3$, from (55) we obtain a third-order method (LM4):

$$x_{n+1} = x_n - K_{-1,0,0,f,1}(x_n, w_n). \tag{59}$$

5^0: If $a_1 = a_2 = 0$, $a_3 = -1$, $p = 1$, and $m = 3$, from (55) we obtain a new third-order method (LM5):

$$x_{n+1} = x_n - K_{0,0,-1,f,1}(x_n, w_n). \tag{60}$$

4. Numerical Examples

In this section, some numerical examples commonly used in the literature are presented in Table 1 to check the effectiveness of the new methods. The following methods were compared: Newton method (NM), the method of Weerakoon and Fernando [10] (WF), the method of Potra and Pták (PP) [11], Chebyshev's method (CHM) [12, 13], Halley's method (HM) [12], and our new methods (56) (LM1), (57) (LM2), (58) (LM3), (59) (LM4), and (60) (LM5). Displayed in Table 1 are the number of iterations (IT), the number of function evaluations (NFE) counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative, the value $f(x_{n+1})$, the computing time (TIME, the unit of time is one second), and the distance of two consecutive approximations $\delta = |x_{n+1} - x_n|$. All computations were done using Matlab 7.1 environment with a ADM athlon (tm) II X2 250-3.01 GHz based PC. We accept an approximate solution rather than the exact root, depending on the precision ϵ of the computer. We use the following stopping criteria for computer programs: $|f(x_{n+1})| < \epsilon$, we used the fixed stopping criterion $\epsilon = 10^{-15}$.

We used the following test functions and display the computed approximate zero x^* [16]:

$$\begin{aligned}
f_1(x) &= x^3 + 4x^2 - 10, \quad x^* = 1.3652300134140969,\\
f_2(x) &= x^2 - e^x - 3x + 2, \quad x^* = 0.25753028543986076,\\
f_3(x) &= \sin(x) e^x + \ln\left(1 + x^2\right), \quad x^* = 0,\\
f_4(x) &= (x-1)^3 - 1, \quad x^* = 2,\\
f_5(x) &= \cos x - x, \quad x^* = 0.73908513321516067,\\
f_6(x) &= \sin^2 x - x^2 + 1, \quad x^* = 1.4044916482153411,\\
f_7(x) &= e^{x^2+7x-30} - 1, \quad x^* = 3.
\end{aligned} \tag{61}$$

5. Conclusions

In this paper, we presented two new families of iterative methods for solving nonlinear equations. One is developed by fitting the model $m(x) = e^{px}(Ax^2 + Bx + C)$ to the function $f(x)$ and its derivative $f'(x)$, $f''(x)$ at a point x_n. The other family of iterative methods was constructed by approximating the equation $f(x) = 0$ around the point $(x_n, f(x_n))$ with the quadratic equation to avoid the calculation of the second derivatives. Analysis of convergence shows that the new methods have third-order or higher convergence: if $m \geq 2$, $a_2 = a_3 = p = 0$, then the methods defined by (35) are shown to converge the order four. We observed from numerical examples that the proposed methods are

Table 1: Comparison of various third-order methods and Newton's method.

	IT	NFE	$f(x_{n+1})$	Time	δ
$f_1 : x_0 = 1$					
NM	5	10	0	0.062577	$2.126987475037367e-011$
WF	3	9	0	0.018731	$2.284722713019605e-006$
PP	4	12	0	0.044520	$1.558753126573720e-013$
CHM	4	12	0	0.040102	$1.643130076445232e-014$
HM	3	9	0	0.019876	$3.698649917449615e-007$
LM1	4	12	0	0.029137	$1.043609643147647e-014$
LM2	3	9	0	0.022242	$1.788683664960544e-006$
LM3	3	9	0	0.023550	$1.295199573814188e-006$
LM4	3	9	0	0.019227	$3.091731315407742e-010$
LM5	3	9	0	0.020131	$1.788683664960544e-006$
$f_1 : x_0 = 2$					
NM	5	10	0	0.057231	$5.020497351182485e-010$
WF	4	12	0	0.039384	$4.440892098500626e-016$
PP	4	12	0	0.044239	$7.949196856316121e-014$
CHM	4	12	0	0.042782	$2.065014825802791e-014$
HM	3	9	0	0.017733	$3.107350415199051e-006$
LM1	4	12	0	0.032638	$2.706235235905297e-011$
LM2	4	12	0	0.037830	$5.551115123125783e-015$
LM3	4	12	0	0.034087	$3.108624468950438e-015$
LM4	4	12	0	0.038399	$2.220446049250313e-016$
LM5	4	12	0	0.035809	$5.551115123125783e-015$
$f_2 : x_0 = 0$					
NM	4	8	0	0.035702	$2.665312415217613e-012$
WF	3	9	0	0.019699	$7.801814749797131e-012$
PP	3	9	0	0.021596	$1.219191414492116e-012$
CHM	3	9	0	0.023312	$8.906764215055318e-013$
HM	3	9	0	0.017431	$7.374600929921371e-012$
LM1	3	9	0	0.016351	$9.004856806882344e-007$
LM2	3	9	0	0.017242	$8.067152316715287e-007$
LM3	3	9	0	0.018978	$8.334935165388302e-007$
LM4	3	9	0	0.016178	$7.334822825222354e-007$
LM5	3	9	0	0.016795	$8.067152316715287e-007$
$f_2 : x_0 = 0.5$					
NM	4	8	0	0.044392	$1.791899961745003e-013$
WF	3	9	0	0.023132	$6.424749621203318e-012$
PP	3	9	0	0.024293	$4.607425552194400e-014$
CHM	3	9	0	0.026689	$3.087480271446452e-011$
HM	3	9	0	0.022199	$4.208039472430869e-011$
LM1	3	9	0	0.024120	$2.850281847210923e-007$
LM2	3	9	$4.440892098500626e-016$	0.021035	$2.744941123289379e-007$
LM3	3	9	0	0.021310	$2.778229745148408e-007$
LM4	3	9	0	0.020886	$2.655207446689012e-007$
LM5	3	9	$4.440892098500626e-016$	0.021592	$2.744941123289379e-007$
$f_3 : x_0 = 1$					
NM	7	14	$3.537126081266182e-024$	0.085178	$1.085848323840232e-012$
WF	4	12	$2.621304391538411e-016$	0.030341	$4.330310691887267e-006$
PP	5	15	$8.196910187379942e-033$	0.063971	$8.806888499109001e-012$
CHM	5	15	$6.352230116524407e-022$	0.053064	$2.520356663650445e-011$
HM	5	15	$7.257520328033309e-029$	0.055315	$8.459855063117184e-015$
LM1	5	15	$5.355132874954819e-017$	0.041370	$4.825251676864564e-007$
LM2	4	12	$6.348042447121620e-018$	0.026722	$5.276208195236892e-013$
LM3	3	9	$1.559423551656343e-017$	0.019476	$5.545966966175517e-011$
LM4	4	12	$6.485252546346663e-018$	0.030513	$1.317913979899399e-009$
LM5	4	12	$6.348042447121620e-018$	0.031575	$5.276208195236892e-013$

Table 1: Continued.

	IT	NFE	$f(x_{n+1})$	Time	δ
$f_3 : x_0 = 0.5$					
NM	6	12	5.905159674954809e − 020	0.071738	1.402992074360412e − 010
WF	4	12	1.764824578467612e − 017	0.039948	4.200981459101664e − 009
PP	4	12	1.694834561079519e − 019	0.056046	2.767209186089879e − 007
CHM	4	12	8.798671206634291e − 017	0.036458	3.059650585008672e − 007
HM	4	12	3.044907255908736e − 017	0.072238	5.518067735074518e − 009
LM1	4	12	1.293991275513200e − 017	0.033751	1.322689611777333e − 013
LM2	4	12	2.828488650436813e − 016	0.032997	4.626219030783813e − 006
LM3	4	12	6.789745028898642e − 018	0.036425	1.912594303370619e − 014
LM4	4	12	1.506225012219799e − 017	0.034137	2.365541379047147e − 011
LM5	4	12	2.828488650436813e − 016	0.035087	4.626219030783813e − 006
$f_4 : x_0 = 2.5$					
NM	6	12	0	0.071353	1.154631945610163e − 014
WF	4	12	0	0.034140	7.314593375440381e − 012
PP	4	12	0	0.042598	4.221685223626537e − 010
CHM	4	12	0	0.036365	9.853584614916144e − 011
HM	4	12	0	0.038080	4.662936703425658e − 014
LM1	3	9	0	0.018372	9.336336148635382e − 010
LM2	4	12	0	0.035562	1.610311883837312e − 010
LM3	4	12	0	0.036985	2.312174895990893e − 010
LM4	4	12	0	0.035463	5.127454016928823e − 012
LM5	4	12	0	0.033362	1.610311883837312e − 010
$f_4 : x_0 = 3.5$					
NM	7	14	0	2.151288	2.877564853065451e − 011
WF	5	15	0	0.054803	6.550315845288424e − 013
PP	5	15	0	0.063380	4.512221707386743e − 010
CHM	5	15	0	0.061753	4.188738245147761e − 011
HM	4	12	0	0.046608	4.485352507632712e − 006
LM1	5	15	6.661338147750939e − 016	0.050909	2.781577124189028e − 008
LM2	5	15	0	0.048688	9.426681657487279e − 011
LM3	10	30	0	0.111640	1.081885248055414e − 009
LM4	4	12	0	0.040109	2.154232348061669e − 011
LM5	5	15	0	0.043482	9.426681657487279e − 011
$f_5 : x_0 = 0$					
NM	5	10	0	0.050123	1.701233598438989e − 010
WF	3	9	0	0.023397	7.792236328407753e − 007
PP	4	12	0	0.038373	1.500558566291943e − 010
CHM	4	12	0	0.037498	5.327979279989847e − 009
HM	4	12	0	0.035701	1.121325254871408e − 014
LM1	4	12	3.330669073875470e − 016	0.028362	4.678117765610779e − 006
LM2	4	12	0	0.028666	7.181037887660224e − 007
LM3	6	18	0	0.053924	6.676881270095691e − 013
LM4	4	12	0	0.028247	2.163133006050089e − 010
LM5	4	12	0	0.029150	7.181037887660224e − 007
$f_5 : x_0 = 1$					
NM	4	8	0	0.039328	1.701233598438989e − 010
WF	2	6	4.440892098500626e − 016	0.003092	2.674277017133964e − 005
PP	3	9	0	0.020461	9.809075773858922e − 011
CHM	3	9	0	0.019378	1.600380383770528e − 009
HM	3	9	0	0.018186	6.624212289807474e − 010
LM1	3	9	0	0.018199	7.500783549829748e − 008
LM2	3	9	1.110223024625157e − 016	0.017732	5.330569952111119e − 008
LM3	3	9	0	0.013523	8.804162354714151e − 008
LM4	3	9	0	0.013782	3.531516445942629e − 008
LM5	3	9	1.110223024625157e − 016	0.016581	5.330569952111119e − 008

Table 1: Continued.

	IT	NFE	$f(x_{n+1})$	Time	δ
$f_6 : x_0 = 1$					
NM	6	12	$3.330669073875470e-016$	0.065520	$3.059774655866931e-013$
WF	4	12	$4.440892098500626e-016$	0.030929	$1.793023507445923e-010$
PP	16	48	$4.440892098500626e-016$	0.266014	$1.531728257564424e-007$
CHM	5	15	$4.440892098500626e-016$	0.061006	$6.883094094689568e-010$
HM	4	12	$4.440892098500626e-016$	0.043655	$2.686739719592879e-013$
LM1	4	12	$3.330669073875470e-016$	0.030877	$1.659126835917846e-008$
LM2	4	12	$3.330669073875470e-016$	0.031156	$9.103828801926284e-014$
LM3	4	12	$3.330669073875470e-016$	0.033076	$1.261090962767497e-006$
LM4	5	15	$4.440892098500626e-016$	0.041102	$8.215650382226158e-014$
LM5	4	12	$3.330669073875470e-016$	0.030923	$9.103828801926284e-014$
$f_6 : x_0 = 2.5$					
NM	6	12	$3.330669073875470e-016$	0.064979	$1.404654170755748e-012$
WF	4	12	$3.330669073875470e-016$	0.038404	$4.229505634611996e-012$
PP	4	12	$3.330669073875470e-016$	0.042164	$1.030850205196998e-008$
CHM	4	12	$3.330669073875470e-016$	0.043437	$1.475204565171140e-007$
HM	4	12	$4.440892098500626e-016$	0.043896	$9.462626682221753e-009$
LM1	5	15	$3.330669073875470e-016$	0.053132	$6.201483770951199e-012$
LM2	4	12	$3.330669073875470e-016$	0.036191	$4.450250368215336e-008$
LM3	6	18	$3.330669073875470e-016$	0.064485	$8.881784197001252e-016$
LM4	4	12	$3.330669073875470e-016$	0.035393	$5.568381311604753e-009$
LM5	4	12	$3.330669073875470e-016$	0.036946	$4.450250368215336e-008$
$f_7 : x_0 = 3.25$					
NM	8	16	0	0.114189	$9.720393379097914e-010$
WF	6	18	0	0.065645	$1.691979889528739e-013$
PP	6	18	0	0.073469	$1.131490456884876e-010$
CHM	6	18	0	0.082677	$2.398081733190338e-014$
HM	5	15	0	0.055630	$3.082423205569285e-012$
LM1	4	12	0	0.036191	$2.613820271335499e-011$
LM2	5	15	0	0.049599	$4.440892098500626e-016$
LM3	4	12	0	0.038172	$1.546353226355990e-006$
LM4	6	18	0	0.051577	$4.440892098500626e-016$
LM5	5	15	0	0.048340	$4.440892098500626e-016$
$f_7 : x_0 = 3.45$					
NM	11	22	0	0.147308	$4.008793297316515e-011$
WF	8	24	0	0.106276	$7.105427357601002e-015$
PP	8	24	0	0.116999	$2.160227552394645e-010$
CHM	7	21	0	0.100725	$1.268533917908599e-006$
HM	6	18	0	0.062876	$1.694565332499565e-008$
LM1	6	18	0	0.060920	$4.440892098500626e-016$
LM2	6	18	0	0.060951	$1.753264200488047e-011$
LM3	6	18	0	0.060627	$1.518252190635394e-011$
LM4	7	21	0	0.087873	$9.144596191390519e-011$
LM5	6	18	0	0.060607	$1.753264200488047e-011$

efficient and demonstrate equal or better performance as compared with other well-known methods.

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