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Research Article

On the Periods of 2-Step General Fibonacci Sequences in the Generalized Quaternion Groups

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We study 2-step general Fibonacci sequences in the generalized quaternion groups Q_{4n} . In cases where the sequences are proved to be simply periodic, we obtain the periods of 2-step general Fibonacci sequences.

1. Introduction

The study of the Fibonacci sequences in groups began with the earlier work of Wall [1] in 1960, where the ordinary Fibonacci sequences in cyclic groups were investigated. In the mideighties, Wilcox [2] extended the problem to the abelian groups. In 1990, Campbell et al. [3] expanded the theory to some classes of finite groups. In 1992, Knox proved that the periods of k-nacci (k-step Fibonacci) sequences in the dihedral groups are equal to 2k + 2, in [4]. In the progress of this study, the article [5] of Aydin and Smith proves that the lengths of the ordinary 2-step Fibonacci sequences are equal to the lengths of the ordinary 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and a prime exponent, in 1994.

Since 1994, the theory has been generalized and many authors had nice contributions in computations of recurrence sequences in groups and we may give here a brief of these attempts. In [6, 7] the definition of the Fibonacci sequence has been generalized to the ordinary 3-step Fibonacci sequences in finite nilpotent groups. Then in [8] it is proved that the period of 2-step general Fibonacci sequence is equal to the length of the fundamental period of the 2-step general recurrence constructed by two generating elements of a group of nilpotency class 2 and exponent p. In [9] Karaduman and Yavuz showed that the periods of the 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 5 and a prime exponent are $p \cdot k(p)$, for 2 , where <math>p is a prime and k(p) is the period of the ordinary

2-step Fibonacci sequence. The main role of [10, 11] in generalizing the theory was to study the 2-step general Fibonacci sequences in finite nilpotent groups of nilpotency class 4 and exponent p and to the 2-step Fibonacci sequences in finite nilpotent groups of nilpotency class p and exponent p, respectively.

One may consult [12, 13] to see the results of the Fibonacci sequences in the modular groups concerning the periodicity of 2-step Fibonacci sequences constructed by two generating elements.

Going on a detailed literature in this area of research, we have to mention certain essential computation on the Fibonacci lengths of new structures like the semidirect products, the direct products, and the automorphism groups of finite groups which have been studied in [14–19]. Finally, we refer to [20] where Karaduman and Aydin studied the periodicity property of 2-step general Fibonacci sequences in dihedral groups and the goal of this paper is to calculate the periods of 2-step general Fibonacci sequences in the generalized quaternion groups.

Let $G = \langle a_0, a_1, a_2, \dots, a_{j-1} \rangle$ be a finite group. A *k-nacci sequence* in group G is a sequence $\{x_n\}_{n=0}^{\infty}$ of group elements for which each element is defined by $x_0 = a_0$, $x_1 = a_1, \dots, x_{j-1} = a_{j-1}$,

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1}, & \text{for } j \le n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1}, & \text{for } n \ge k. \end{cases}$$
 (1.1)

This sequence of the group G is denoted by $F_k(G; a_0, a_1, a_2, \ldots, a_{j-1})$. We also call a 2-nacci sequence of group elements a *Fibonacci sequence* of a finite group. A finite group G is *k-nacci sequenceable* if there exists a *k*-nacci sequence of G such that every element of the group appears in the sequence. A sequence of group elements is *periodic* if after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the *period* of the sequence. For example, the sequence a,b,c,d,e,b,c,d,e,\ldots is periodic after the initial element a and has period 4. We denote the period of a k-nacci sequence $F_k(G; a_0, a_1, a_2, \ldots, a_{j-1})$ by $P_k(G; a_0, a_1, a_2, \ldots, a_{j-1})$. A sequence of group elements is called *simply periodic* with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a,b,c,d,e,f,a,b,c,d,e,f,\ldots$ is simply periodic with period 6. The following theorem is well known.

Theorem 1.1 (see [4]). Every k-nacci sequence in a finite group is simply periodic.

2. Main Theorems

The generalized quaternion group Q_{4n} , $(n \ge 2)$ is a group with a presentation of the form

$$\langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle.$$
 (2.1)

It is easy to see that Q_{4n} is of order 4n, a has order 2n, b has order 4, and the relation $a^kb=ba^{-k}$ holds for all $k \in \mathbb{Z}$.

First we consider the following lemma which will be used frequently without further reference.

Lemma 2.1. For an integer r, where $1 \le r < 2n$, and for $s \in \{1,3\}$, the following relations hold in Q_{4n} :

- (i) $(a^rb^s)^t = b^{st}$, where t is any even integer;
- (ii) $(a^rb^s)^t = a^rb^{st}$, where t is any odd integer.

Proof. We may use an induction method on t for both parts, simultaneously.

Theorem 2.2. $P_k(Q_{4n}; a, b) = 2k + 2$.

Proof. Obviously, the order of ab is 4. If k = 2, then

$$x_0 = a,$$
 $x_1 = b,$ $x_2 = ab,$ $x_3 = a^{n-1},$ $x_4 = a^{n+2}b,$ $x_5 = ab,$ $x_6 = a,$ $x_7 = b,...,$ (2.2)

and hence the sequence has period 6. If k = 3, then

$$x_0 = a$$
, $x_1 = b$, $x_2 = ab$, $x_3 = (ab)^2 = b^2$, $x_4 = a^{-1}$,
 $x_5 = a^2b^3$, $x_6 = ab$, $x_7 = 1$, $x_8 = a$, $x_9 = b$,..., (2.3)

and so the period is 8. Now let $k \ge 4$. Then, the first k + 1 elements of $F_k(Q_{4n}; a, b)$ are

$$x_0 = a$$
, $x_1 = b$, $x_2 = ab$, $x_3 = (ab)^2$, $x_4 = (ab)^4$, ..., $x_k = (ab)^{2^{k-2}}$. (2.4)

Since ab is of order 4, this sequence reduces to

$$x_0 = a,$$
 $x_1 = b,$ $x_2 = ab,$ $x_3 = b^2,$ $x_j = 1,$ $(4 \le j \le k).$ (2.5)

Thus,

$$x_{k+1} = \prod_{i=1}^{k} x_i = b(ab)b^2 = a^{-1},$$

$$x_{k+2} = \prod_{i=2}^{k+1} x_i = (ab)b^2 a^{-1} = a^2 b^3,$$

$$x_{k+3} = \prod_{i=3}^{k+2} x_i = b^2 a^{-1} \left(a^2 b^3\right) = ab,$$

$$x_{k+4} = \prod_{i=4}^{k+3} x_i = a^{-1} \left(a^2 b^3\right) (ab) = 1.$$
(2.6)

It follows that $x_{k+j} = 1$ for j ($4 \le j \le k+1$). We also have

$$x_{2k+2} = \prod_{i=k+2}^{2k+1} x_i = (a^2b^3)(ab) = a,$$

$$x_{2k+3} = \prod_{i=k+3}^{2k+2} x_i = (ab)a = b.$$
(2.7)

It shows that $P_k(Q_{4n}; a, b) = 2k + 2$.

Let $G = \langle a,b \rangle$ be a finite nonabelian 2-generated group. A 2-step general Fibonacci sequence in the group G is defined by $x_0 = a$, $x_1 = b$, $x_i = x_{i-2}^m x_{i-1}^l$, for $i \ge 2$, and the integers m and l. Now we study this sequence for group Q_{4n} .

Theorem 2.3. Let m and l be integers. If $m \equiv 0 \pmod{2n}$ or $m \equiv 0 \pmod{4}$, then 2-step general Fibonacci sequence in Q_{4n} is not simply periodic.

Proof. First we consider the case $m \equiv 0 \pmod{2n}$. Then the sequence is

$$x_0 = a,$$
 $x_1 = b,$ $x_2 = a^m b^l = b^l,$ $x_3 = b^m b^{l^2} = b^{m+l^2}, \dots$ (2.8)

Obviously, the cycle does not begin again with a and b, and hence the sequence is not simply periodic.

Now let $m \equiv 0 \pmod{4}$. Four cases occur.

Case 1 ($l \equiv 0 \pmod{4}$). Then

$$x_0 = a,$$
 $x_1 = b,$ $x_2 = a^m,$ $x_3 = a^{ml},$ $x_4 = a^{m^2 + ml^2},$ $x_5 = a^{2m^2l + ml^3},$ $x_6 = a^{m^3 + 3m^2l^2 + ml^4},$ (2.9)

Since the cycle does not begin again with *a* and *b*, the sequence is not simply periodic.

Case 2 ($l \equiv 1 \pmod{4}$). Then

$$x_0 = a,$$
 $x_1 = b,$ $x_2 = a^m b,$ $x_3 = a^m b,$ $x_4 = a^m b, \dots$ (2.10)

Clearly, the sequence is not simply periodic.

Case 3 ($l \equiv 2 \pmod{4}$). Note that b^2 is a central element of Q_{4n} . Thus

$$x_0 = a$$
, $x_1 = b$, $x_2 = a^m b^2$, $x_3 = a^{ml}$, $x_4 = a^{m^2 + ml^2}$, $x_5 = a^{2m^2 l + ml^3}$,.... (2.11)

Similar to Case 1, the sequence is not simply periodic.

Case 4 ($l \equiv 3 \pmod{4}$). So

$$x_0 = a$$
, $x_1 = b$, $x_2 = a^m b^3$, $x_3 = a^m b$, $x_4 = a^m b^3$, $x_5 = a^m b$,.... (2.12)

Thus the sequence is not simply periodic.

Theorem 2.4. *Let n be an even integer.*

- (i) If $m \equiv 1 \pmod{2n}$ and $l \equiv 0 \pmod{2n}$, then 2-step general Fibonacci sequence in Q_{4n} is simply periodic with period 2.
- (ii) If $m \equiv -1 \pmod{2n}$ and $l \equiv -1 \pmod{2n}$, then 2-step general Fibonacci sequence in Q_{4n} is simply periodic with period 3.
- (iii) If $m \equiv 1 \pmod{2n}$ and $l \equiv 1 \pmod{2n}$, then 2-step general Fibonacci sequence in Q_{4n} is simply periodic with period 6.

Proof. (i) Since *n* is even, $m \equiv 1 \pmod{4}$ and $l \equiv 0 \pmod{4}$. Thus

$$x_0 = a,$$
 $x_1 = b,$ $x_2 = a,$ $x_3 = b,...,$ (2.13)

and the period is 2.

(ii) Since n is even, $m \equiv 3 \pmod 4$ and $l \equiv 3 \pmod 4$. Note that m and l are odd integers. So the sequence reduces to

$$x_0 = a$$
, $x_1 = b$, $x_2 = a^{-1}b^3$, $x_3 = a$, $x_4 = b$,..., (2.14)

and the period is 3.

(iii) Because n is even, $m \equiv 1 \pmod{4}$ and $l \equiv 1 \pmod{4}$. Furthermore, m and l are odd integers. Then the sequence reduces to

$$x_0 = a,$$
 $x_1 = b,$ $x_2 = ab,$ $x_3 = a^{n-1},$ $x_4 = a^2b^3,$ $x_5 = ab,$ $x_6 = a,$ $x_7 = b,...,$ (2.15)

and the period is 6.

Theorem 2.5. *Let n be an odd integer.*

- (i) If $m \equiv 1 \pmod{2n}$, $m \equiv 1 \pmod{4}$, $l \equiv 0 \pmod{4}$ and (l,n) = 1, then 2-step general Fibonacci sequence in Q_{4n} is simply periodic with period 2n.
- (ii) If $m \equiv 1 \pmod{2n}$, $m \equiv 3 \pmod{4}$, $l \equiv 0 \pmod{4}$ and (l,n) = 1, then 2-step general Fibonacci sequence in Q_{4n} is simply periodic with period 4n.

Proof. (i) By induction on $k \ge 0$, we can show that $x_{2k} = a$ and $x_{2k+1} = a^{-kl}b$. In particular, the period must be even. Now we have

$$x_{2k+1} = b \iff 2n \mid kl \iff n \mid k \text{ (because } (l, n) = 1),$$
 (2.16)

and so the period is 2n.

(ii) By induction on $k \ge 0$, it may be shown that

$$x_{2k} = a,$$
 $x_{2k+1} = \begin{cases} a^{-kl}b & \text{if } k \text{ is even,} \\ a^{-kl}b^3 & \text{if } k \text{ is odd.} \end{cases}$ (2.17)

Therefore, the period must be even. Now we have

$$x_{2k+1} = b \iff 2n \mid kl \iff 2n \mid k$$
 (because $(l, n) = 1$ and n is odd), (2.18)

and thus the period is 4n.

Theorem 2.6. Let n and l be odd integers, $m \equiv -1 \pmod{2n}$ and (l+1,n) = 1.

- (i) If $m \equiv 1 \pmod{4}$, then 2-step general Fibonacci sequence in Q_{4n} is simply periodic with period 6n.
- (ii) If $m \equiv 3 \pmod{4}$ and $l \equiv 1 \pmod{4}$, then 2-step general Fibonacci sequence in Q_{4n} is simply periodic with period 6n.
- (iii) If $m \equiv 3 \pmod{4}$ and $l \equiv 3 \pmod{4}$, then 2-step general Fibonacci sequence in Q_{4n} is simply periodic with period 3n.

Proof. (i) Two cases occur.

Case 1 ($l \equiv 1 \pmod{4}$). By induction on $k \ge 0$, we can prove that the following relations hold:

$$x_{6k} = a,$$

$$x_{6k+1} = a^{-2k(l+1)}b,$$

$$x_{6k+2} = a^{-(2k(l+1)+1)}b,$$

$$x_{6k+3} = a^{n+1},$$

$$x_{6k+4} = a^{-(n+(2k+1)(l+1))}b,$$

$$x_{6k+5} = a^{-((2k+1)(l+1)+1)}b.$$
(2.19)

Consequently, the period must be a multiple of 6. Now we have

$$x_{6k+1} = b \iff 2n \mid 2k(l+1) \iff n \mid k \quad \text{(because } (l+1,n) = 1),$$
 (2.20)

and hence the period is 6*n*.

Case 2 ($l \equiv 3 \pmod{4}$). The proof is similar to Case 1 except that

$$x_{6k+2} = a^{-(n+2k(l+1)+1)}b,$$

$$x_{6k+4} = a^{-(2k+1)(l+1)}b.$$
(2.21)

(ii) It is easily shown that the following relations hold for every $k \ge 0$:

$$x_{3k} = a$$
, $x_{6k} = a$, $x_{6k+1} = a^{-2k(l+1)}b$, $x_{6k+2} = a^{-(2k(l+1)+1)}b$, $x_{6k+3} = a$, $x_{6k+4} = a^{-(n+(2k+1)(l+1))}b$, $x_{6k+5} = a^{-(n+1+(2k+1)(l+1))}b$. (2.22)

Particularly, the period must be a multiple of 3. Since n + (2k + 1)(l + 1) is odd, then it is not a multiple of 2n. Thus $x_{6k+4} \neq b$. Further,

$$x_{6k+1} = b \iff 2n \mid 2k(l+1) \iff n \mid k \quad \text{(because } (l+1,n) = 1),$$
 (2.23)

and hence the period is 6*n*.

(iii) By induction on $k \ge 0$, we can prove the following relations:

$$x_{3k} = a,$$

 $x_{3k+1} = a^{-k(l+1)}b,$ (2.24)
 $x_{3k+2} = a^{-(n+1+k(l+1))}b.$

Therefore, the period must be a multiple of 3. Now we have

$$x_{3k+1} = b \iff 2n \mid k(l+1) \iff n \mid k \quad \text{(because } (l+1,n) = 1),$$
 (2.25)

and so the period is 3n.

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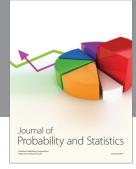
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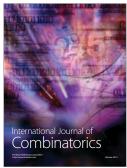








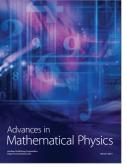


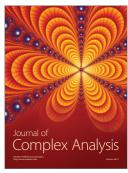




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