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Research Article

The Centroid of a Lie Triple Algebra

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General results on the centroids of Lie triple algebras are developed. Centroids of the tensor product of a Lie triple algebra and a unitary commutative associative algebra are studied. Furthermore, the centroid of the tensor product of a simple Lie triple algebra and a polynomial ring is completely determined.

1. Introduction

In recent years, Lie triple algebras (i.e., Lie-Yamaguti algebras or general Lie triple systems) have attracted much attention in Lie theories. They contain Lie algebras and Lie triple systems as special cases ([1–6]). So, it is of vital importnce to study some properties of Lie triple algebras. The concept of Lie triple algebra has been introduced, originally, by Yamaguti as general Lie triple system by himself, Sagle, and others. Since Lie triple algebras are generalization of Lie algebras and Lie triple system, it is natural for us to imagine whether or not some results of Lie algebras and Lie triple system hold in Lie triple algebras. Now, as a generalization of Lie triple algebra, Hom-Lie-Yamaguti was introduced by Lister in [7].

Benkart and Neher studied centroid of Lie algebras in [8], and Melville investigated centroid of nilpotent Lie algebras in [9]. It turns out that result on the centroid of Lie algebras is a key ingredient in the classification of extended affine Lie algebras. The centroids of Lie triple systems were mentioned by Benito et al. [10]. Now, some results on centroids of Lie triple system and n-Lie algebras were developed in [11, 12].

In this paper we present new results concerning the centroids of Lie triple algebras and give some conclusion of the tensor product of a Lie triple algebra and a unitary commutative associative algebra. Furthermore, we completely determine the centroid of the tensor product of a simple Lie triple algebra and a polynomial ring. The organization of the rest of this paper is as follows. Section 1 is for basic notions and facts on Lie triple algebras. Section 2 is devoted to the structures and properties of the centroids of Lie triple algebras. Section 3 describes the structures of the centroids of tensor product of Lie triple algebras.

2. Preliminaries

Definition 1 (see [1]). A Lie triple algebra (also called a Lie-Yamaguti algebra or a general Lie triple system) $\mathfrak{g}$ is a vector space over an arbitrary field $\mathbf{F}$ with a bilinear map denoted by xy of $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$ and a ternary map denoted by $[x, y, z]$ of $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$ satisfying the following axioms:

(a) $xx = 0$,

(b) $[x, x, y] = 0$,

(c) $(xy)z+(yz)x+(zx)y+[x, y, z]+[y, z, x]+[z, x, y] = 0$,

(d) $[xy, z, w] + [yz, x, w] + [zx, y, w] = 0$,

(e) $[x, y, zw] = ([x, y, z])w + z([x, y, w])$,

(f) $([x, y, v])([z, w, v]) = [[x, y, z], w, v] + [z[x, y, w], v]$,

where $x, y, z, w, v \in \mathfrak{g}$.

Remark 2. Any Lie algebra is a Lie triple algebra relative to $xy = [x, y]$ and $D(x, y)z = [[x, y], z]$, for $x, y, z \in \mathfrak{g}$. If $D(x, y)z = 0$, for all $x, y, z \in \mathfrak{g}$, the axioms stated earlier reduce to that of Lie algebra, and if $xy = 0$, for all $x, y \in \mathfrak{g}$, the axioms stated earlier reduce to that of Lie triple system. In this sense, the Lie triple algebra is a more general concept than that of the Lie algebra and Lie triple system.

Definition 3 (see [1, 2]). A derivation of a Lie triple algebra $\mathfrak{g}$ is a linear transformation D of $\mathfrak{g}$ into $\mathfrak{g}$ satisfying the following conditions:

(1) $D(xy) = (Dx)y + x(Dy)$,

(2) $D([x, y, z]) = [Dx, y, z] + [x, Dy, z] + [x, y, Dz]$, for all $x, y, z \in \mathfrak{g}$.

Let $\mathrm{Der}(\mathfrak{g})$ be the set of all derivation of $\mathfrak{g}$; then $\mathrm{Der}(\mathfrak{g})$ is regarded as a subalgebra of the general Lie algebra $\mathrm{gl}(\mathfrak{g})$ and is called the derivation algebra of $\mathfrak{g}$.

Definition 4. A Lie triple subalgebra I of $\mathfrak{g}$ is called an ideal if $\mathfrak{g}I \subseteq I$ and $[\mathfrak{g}, I, \mathfrak{g}] \subseteq I$.

Definition 5. A Lie triple algebra $\mathfrak{g}$ is perfect if $\mathfrak{g} = \mathfrak{g}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}, \mathfrak{g}]$.

Definition 6. Let I be a nonempty subset of $\mathfrak{g}$; we call $Z_{\mathfrak{g}}(I) = \{x \in \mathfrak{g} \mid xa = [x, a, y] = 0$, for all $a \in I, y \in \mathfrak{g}\}$ the centralizer of I in $\mathfrak{g}$. In particular, $Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid x\mathfrak{g} = [x, \mathfrak{g}, \mathfrak{g}] = 0\}$ is the center of $\mathfrak{g}$.

Definition 7. Suppose that I is an ideal of the Lie triple algebra $\mathfrak{g}$, on the quotient vector space

$$\frac{g}{I} = \{x = x + I \mid \forall x \in g\}. \quad (1)$$

Define the operator: $D(\overline{x}, \overline{y})(\overline{z}) = \overline{D(x, y)z}, \overline{x}\,\overline{y} = \overline{xy}$, where $x, y, z \in g$. Then g/I is also a Lie triple algebra, and it is called a quotient algebra of $\mathfrak{g}$ by I.

3. The Centroids of Lie Triple Algebras

Definition 8. Let $\mathfrak{g}$ be a Lie triple algebra over a field $\mathbf{F}$. The centroid of $\mathfrak{g}$ is the transform on $\mathfrak{g}$ given by $\Gamma(\mathfrak{g}) = \{\psi \in \mathrm{End}_{\mathbf{F}}(\mathfrak{g}) \mid \psi([x, y, z]) = [\psi(x), y, z], \psi(xy) = (\psi(x))y$, for all $x, y, z, \in \mathfrak{g}\}$.

By (a)–(f), we can conclude that if $\psi \in \Gamma(\mathfrak{g})$, then we have $\psi([x, y, z]) = [\psi(x), y, z] = [x, \psi(y), z] = [x, y, \psi(z)]$, $\psi(xy) = (\psi(x))y = x(\psi(y))$, for all $x, y, z, \in \mathfrak{g}$.

From the definition, it is clear that the scalars will always be in the centroid.

Proposition 9. *If $\mathfrak{g}$ is perfect, then the centroid $\Gamma(\mathfrak{g})$ is commutative.*

Proof. For all $x, y, z \in \mathfrak{g}, \phi, \psi \in \Gamma(\mathfrak{g})$, we have $\phi\psi([x, y, z]) = [\psi(x), \phi(y), z] = \psi\phi([x, y, z]), \phi\psi(xy) = \psi(x)\phi(y) = \psi\phi(xy)$. □

Proposition 10. *Let $\mathfrak{g}$ be a Lie triple algebra over a field $\mathbf{F}$ and B a subset of $\mathfrak{g}$. Then;*

(1) *$Z_{\mathfrak{g}}(B)$ is invariant under $\Gamma(\mathfrak{g})$,*

(2) *every perfect ideal of $\mathfrak{g}$ is invariant under $\Gamma(\mathfrak{g})$.*

Proof. (1) For any $\psi \in \Gamma(\mathfrak{g})$, $x \in Z_{\mathfrak{g}}(B)$, $y \in B$, $z \in \mathfrak{g}$, we have $[\psi(x), y, z] = [\psi(x), y, z] = \psi[x, y, z] = 0$, $(\psi(x))z = \psi(xz) = 0$, and $[z, y, \psi(x)] = \psi[z, y, x] = 0$, $z(\psi(x)) = \psi(zx) = 0$. Therefore, $\psi(x) \in \Gamma(\mathfrak{g})$, which implies that $Z_{\mathfrak{g}}(B)$ is invariant under $\Gamma(\mathfrak{g})$.

(2) Let J be any perfect ideal of $\mathfrak{g}$; then $J = JJ = [J, J, J]$. For any $y \in J$, there exist $a, b, c, d, e \in J$, such that $y = ab = [c, d, e]$, and then we have $\psi(y) = \psi(ab) = (\psi(a))b \in J\mathfrak{g} \subseteq J, \psi(y) = \psi([c, d, e]) = [c, \psi(d), e] \in [J, \mathfrak{g}, \mathfrak{g}] \subseteq J$. Hence J is invariant under $\Gamma(\mathfrak{g})$. □

Definition 11. Let $\varphi(\mathfrak{g}) \subseteq Z(\mathfrak{g})$ and $\varphi(\mathfrak{g}\mathfrak{g}) = \varphi([\mathfrak{g}, \mathfrak{g}, \mathfrak{g}]) = 0$; then φ is called a central derivation.

The set of all central derivation of $\mathfrak{g}$ is denoted by $C(\mathfrak{g})$. Clearly, $C(\mathfrak{g}) \subseteq \Gamma(\mathfrak{g})$ and $C(\mathfrak{g})$ is an ideal of $\Gamma(\mathfrak{g})$. A more precise relationship is summarized as follows.

Next, we will develop some general results on centroids of Lie triple algebra.

Proposition 12. *If $\mathfrak{g}$ has no nonzero ideals I and J with $[\mathfrak{g}, I, J] = 0$, $IJ = 0$, then $\Gamma(\mathfrak{g})$ is an integral domain.*

Proof. Clearly, $\mathrm{id} \in \Gamma(\mathfrak{g})$. If there exist $\psi, \varphi \in \Gamma(\mathfrak{g}), \psi \neq 0, \phi \neq 0$ such that $\psi\varphi = 0$, then there exist $x, y \in \mathfrak{g}$ such that $\psi(x) \neq 0$ and $\phi(y) \neq 0$. Then, $[\mathfrak{g}, \psi(x), \phi(y)] = \psi\phi([\mathfrak{g}, x, y]) = 0, (\psi(x))(\phi(y)) = \psi\phi(xy) = 0$. Therefore, $\psi(x)$ and $\phi(y)$ can span two nonzero ideals I, J of $\mathfrak{g}$ such that $[\mathfrak{g}, I, J] = IJ = 0$, which is a contradiction. Hence, $\Gamma(\mathfrak{g})$ has no zero divisor; it is an integral domain. □

Theorem 13. *If $\mathfrak{g}$ is a simple Lie triple algebra over an algebraically closed field $\mathbf{F}$, then $\Gamma(\mathfrak{g}) = \mathbf{F}\mathrm{id}$.*

Proof. Let $\phi \in \Gamma(\mathfrak{g}) \subseteq \mathrm{End}_{\mathbf{F}}(\mathfrak{g})$. Since $\mathbf{F}$ is algebraically closed, ϕ has an eigenvalue λ. We denote the corresponding eigenspace to be $E_\lambda(\phi) = \{v \in \mathfrak{g} \mid \phi(v) = \lambda(v)\}$, so $E_\lambda(\phi) \neq 0$, for any $v \in E_\lambda(\phi), x, y \in \mathfrak{g}$, and we have $\phi([v, x, y]) = [\phi(v), x, y] = \lambda[v, x, y], \phi((vx)) = \phi(v)x = (\lambda v)x = \lambda(vx)$, so $[v, x, y] \in E_\lambda(\phi), vx \in E_\lambda(\phi)$. It follows that $E_\lambda(\phi)$ is an ideal of $\mathfrak{g}$. But, $\mathfrak{g}$ is simple, so $E_\lambda(\phi) = \mathfrak{g}$; that is, $E_\lambda(\phi) = \lambda\mathrm{id}_{\mathfrak{g}}$. This proves the theorem. □

When $\Gamma(\mathfrak{g}) = \mathbf{F}\mathrm{id}$, the Lie triple algebra $\mathfrak{g}$ is said to be central. Furthermore, if $\mathfrak{g}$ is simple, $\mathfrak{g}$ is said to be central simple. Every simple Lie triple algebra is central simple over its centroid.

Proposition 14. *Let $\mathfrak{g}$ be a Lie triple algebra over a field $\mathbf{F}$; then*

(1) *$\mathfrak{g}$ is indecomposable if and only if $\Gamma(\mathfrak{g})$ does not contain idempotents except 0 and id;*

(2) *if $\mathfrak{g}$ is perfect, then every $\phi \in \Gamma(\mathfrak{g})$ is symmetric $(f([a, b, c], d) = f(a[d, c, b]), f(ab, c) = f(a, bc))$ with respect to any invariant form on $\mathfrak{g}$.*

Proof. (1) If there exists $\phi \in \Gamma(\mathfrak{g})$ which is an idempotent and satisfies $\phi \neq 0, \mathrm{id}$, then $\phi^2(x) = \phi(x)$, for all $x \in \mathfrak{g}$. We assert that $\mathrm{Ker}\,\phi$ and $\mathrm{Im}\,\phi$ are ideals of $\mathfrak{g}$. In fact, for any $x \in \mathrm{Ker}\,\phi$ and $y, z \in \mathfrak{g}$, we have $\phi([x, y, z]) = [\phi(x), y, z] = 0$, $\phi(xy) = (\phi(x))y = 0y = 0$, which implies $[x, y, z] \in \mathrm{Ker}\,\phi$,

$xy \in \operatorname{Ker}\phi$. For any $x \in \operatorname{Im}\phi$, there exists $a \in \mathfrak{g}$ such that $x = \phi(a)$. Then, we have

$$[x, y, z] = [\phi(a), y, z] = \phi([a, y, z]) \in \operatorname{Im}\phi, \quad \forall y, z \in \mathfrak{g},$$
$$xy = \phi(a)\, y = \phi(ay) \in \operatorname{Im}\phi, \quad \forall y \in \mathfrak{g}. \tag{2}$$

This proves our assertion. Moreover, $\operatorname{Ker}\phi \cap \operatorname{Im}\phi = 0$. Indeed, if $x \in \operatorname{Ker}\phi \cap \operatorname{Im}\phi$, then there exists $y \in \mathfrak{g}$ such that $x = \phi(y)$ and $0 = \phi(x) = \phi^2(y) = \phi(y) = x$. We have a decomposition $x = \phi(x) + y$, for all $x \in \mathfrak{g}$, where $\phi(y) = 0$. So, we have $\mathfrak{g} = \operatorname{Ker}\phi \oplus \operatorname{Im}\phi$, which is a contradiction.

On the other hand, suppose $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then for any $x \in \mathfrak{g}$, we have $x = x_1 + x_2$, $x_i \in \mathfrak{g}_i$, $i = 1, 2$. We choose $\phi \in \Gamma(\mathfrak{g})$ such that $\phi(x_1) = x_1$ and $\phi(x_2) = 0$. Then, $\phi^2(x) = \phi(x_1) = x_1 = \phi(x)$. Hence, ϕ is an idempotent. By assumption, we have $\phi = 0$ or $\phi = \mathrm{id}$. If $\phi = 0$, then $x_1 = 0$, implying $\mathfrak{g} = \mathfrak{g}_2$. If $\phi = \mathrm{id}$, then $x_2 = 0$, implying $\mathfrak{g} = \mathfrak{g}_1$.

(2) Let f be an invariant $\mathbf{F}$-bilinear form on $\mathfrak{g}$. Then $f([a, b, c], d) = f(a[d, c, b])$, $f(ab, c) = f(a, bc)$, for all $a, b, c, d \in \mathfrak{g}$. Since $\mathfrak{g}$ is perfect, for $\phi \in \Gamma(\mathfrak{g})$, we have

$$\begin{aligned} f(\phi([a, b, c]), d) &= f([a, \phi(b), c], d) \\ &= f(a, [d, c, \phi(b)]) = f([a, \phi([d, c, b])]) \\ &= f(a, [\phi(d), c, b]) = f([a, b, c], \phi(d)), \\ f(\phi(ab), c) &= f(a(\phi(b)), c) \\ &= f(a, \phi(b) c) = f(a, b\phi(c)) \\ &= f(ab, \phi(c)). \end{aligned} \tag{3}$$

The result follows. □

Remark 15. Set $L(x)(y) = xy$, $L(x, y)(z) = [x, y, z]$, $R(x)(y) = yx$, $R(x, y)(z) = [z, x, y]$, for all $x, y, z, \in \mathfrak{g}$; then we call $L(x)$, and $L(x, y)$ the left multiplication operator of $\mathfrak{g}$; $R(x)$ and $R(x, x)$ right multiplication operator of $\mathfrak{g}$. Denote by $\operatorname{Mult}(\mathfrak{g}) = \operatorname{Mult}_{\mathbf{F}}(\mathfrak{g})$ the subalgebra of $\operatorname{End}_{\mathbf{F}}(\mathfrak{g})$ generated by the left and right multiplication operators of $\mathfrak{g}$. Then, $\Gamma(\mathfrak{g})$ is the centralizer of $\operatorname{Mult}(\mathfrak{g})$.

Theorem 16. *Let $\pi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be an epimorphism of Lie triple algebra; for any $f \in \operatorname{End}_{\mathbf{F}}(\mathfrak{g}_1; \operatorname{Ker}\pi) := \{g \in \operatorname{End}_{\mathbf{F}}(\mathfrak{g}_2) \mid g(\operatorname{Ker}\pi) \subseteq \operatorname{Ker}\pi\}$, there exists a unique $\overline{f} \in \operatorname{End}_{\mathbf{F}}(\mathfrak{g}_2)$ satisfying $\pi \circ f = \overline{f} \circ \pi$. Moreover, the following results hold.*

(1) *The map $\pi_{\mathrm{End}} : \operatorname{End}_{\mathbf{F}}(\mathfrak{g}_1; \operatorname{Ker}\pi) \to \operatorname{End}_{\mathbf{F}}(\mathfrak{g}_2)$, $f \mapsto \overline{f}$ is a homomorphism with the following properties: $\pi_{\mathrm{End}}(\operatorname{Mult}(\mathfrak{g}_1)) = \operatorname{Mult}(\mathfrak{g}_2)$, $\pi_{\mathrm{End}}(\Gamma(\mathfrak{g}_1) \cap \operatorname{End}_{\mathbf{F}}(\mathfrak{g}_1; \operatorname{Ker}\pi)) \subseteq \Gamma(\mathfrak{g}_2)$. There is a homomorphism $\pi_\Gamma : \Gamma(\mathfrak{g}_1) \cap \operatorname{End}_{\mathbf{F}}(\mathfrak{g}_1; \operatorname{Ker}\pi) \to \Gamma(\mathfrak{g}_2)$, $f \mapsto \overline{f}$.*

If $\operatorname{Ker}\pi = Z(\mathfrak{g}_1)$, then every $\phi \in \Gamma(\mathfrak{g}_1)$ leaves $\operatorname{Ker}\pi$ invariant; that is, π_Γ is defined on all of $\Gamma(\mathfrak{g}_1)$.

(2) *Suppose $\mathfrak{g}_1$ is perfect and $\operatorname{Ker}\pi \subseteq Z(\mathfrak{g}_1)$; then $\pi_\Gamma : \Gamma(\mathfrak{g}_1) \cap \operatorname{End}_{\mathbf{F}}(\mathfrak{g}_1; \operatorname{Ker}\pi) \to \Gamma(\mathfrak{g}_2)$, $f \mapsto \overline{f}$ is injective.*

(3) *If $\mathfrak{g}_1$ is perfect, $Z(\mathfrak{g}_2) = 0$, and $\operatorname{Ker}\pi \subseteq Z(\mathfrak{g}_1)$, then $\pi_\Gamma : \Gamma(\mathfrak{g}_1) \to \Gamma(\mathfrak{g}_2)$ is a monomorphism.*

Proof. (1) It is easy to see that π_{End} is a homomorphism. Since $\operatorname{Ker}\pi$ is an ideal of $\mathfrak{g}_1$ and all left and right multiplication operators of $\mathfrak{g}_1$ leave $\operatorname{Ker}\pi$ invariant, $\operatorname{Mult}(\mathfrak{g}_1) \subseteq \operatorname{End}_{\mathbf{F}}(\mathfrak{g}_1; \operatorname{Ker}\pi)$. Furthermore, for the left multiplication operator $L(x, y)$, $L(x)$ on $\mathfrak{g}_1$, we have $\pi \circ L(x, y) = L(\pi(x), \pi(y)) \circ \pi$, $\pi \circ L(x) = L(\pi(x)) \circ \pi$, so $\pi_{\mathrm{End}}(L(x, y)) = L(\pi(x), \pi(y))$, $\pi_{\mathrm{End}}(L(x)) = L(\pi(x))$. For the right multiplication, we have the analogous formula $\pi_{\mathrm{End}}(R(x, y)) = R(\pi(x), \pi(y))$, $\pi_{\mathrm{End}}(R(x)) = R(\pi(x))$. Moreover, π is an epimorphism, so $\pi_{\mathrm{End}}(\operatorname{Mult}(\mathfrak{g}_1)) = \operatorname{Mult}(\mathfrak{g}_2)$. Now, we show that $\pi_{\mathrm{End}}(\Gamma(\mathfrak{g}_1) \cap \operatorname{End}_{\mathbf{F}}(\mathfrak{g}_1; \operatorname{Ker}\pi)) \subseteq \Gamma(\mathfrak{g}_2)$. Let $\phi \in \Gamma(\mathfrak{g}_1) \cap \operatorname{End}_{\mathbf{F}}(\mathfrak{g}_1; \operatorname{Ker}\pi)$. For any $x', y', z' \in \mathfrak{g}_2$, there exist $x, y, z \in \mathfrak{g}_1$ such that $\pi(x) = x'$, $\pi(y) = y'$, $\pi(z) = z'$. Then, we have

$$\begin{aligned} &\overline{\phi}([x', y', z']) \\ &\quad = \overline{\phi}(\pi([x, y, z])) = \pi(\phi([x, y, z])) \\ &\quad = \pi([x, \phi(y), z]) = [\pi(x), \overline{\phi}(\pi(y)), \pi(z)] \\ &\quad = [x', \overline{\phi}(y'), z'] = \pi([\phi(x), y, z]) \\ &\quad = [\overline{\phi}(\pi(x)), \pi(y), \pi(z)] = [\overline{\phi}(x'), y', z'] \\ &\quad = \pi([x, y, \phi(z)]) = [\pi(x), \pi(y), \overline{\phi}(\pi(z))] \\ &\quad = [x', y', \overline{\phi}(z')], \\ &\overline{\phi}(x'y') \\ &\quad = \overline{\phi}(\pi(x)\pi(y)) = \overline{\phi}(\pi(xy)) = \pi\phi(xy) \\ &\quad = (\overline{\phi}\pi x)(\pi(y)) = (\overline{\phi}(x'))\, y' = \pi(x(\phi(y))) \\ &\quad = \pi(x)(\pi\phi(y)) = x'(\overline{\phi}\pi(y)) = x'(\overline{\phi}(y')), \end{aligned} \tag{4}$$

which proves $\overline{\phi} \in \Gamma(\mathfrak{g}_2)$.

(2) If $\overline{\phi} = 0$ for, $\phi \in \Gamma(\mathfrak{g}_1) \cap \operatorname{End}_{\mathbf{F}}(\mathfrak{g}_1; \operatorname{Ker}\pi)$, then $\pi(\phi(\mathfrak{g}_1)) = \overline{\phi}(\pi(\mathfrak{g}_1)) = 0$, which means that $\phi(\mathfrak{g}_1) \subseteq \operatorname{Ker}\pi \subseteq Z(\mathfrak{g}_1)$. Hence, $\phi([x, y, z]) = [\phi(x), y, z] = 0$, $\phi(xy) = (\phi(x)y) = 0$, for all $x, y, z \in \mathfrak{g}_1$. Furthermore, since $\mathfrak{g}_1$ is a perfect ideal, we can get $\phi = 0$.

(3) We can see that $\pi(Z(\mathfrak{g}_1)) \subseteq Z(\mathfrak{g}_2) = 0$, which follows that $Z(\mathfrak{g}_1) \subseteq \operatorname{Ker}\pi$. So, $\operatorname{Ker}\pi \subseteq Z(\mathfrak{g}_1)$. From (1), we know that $\pi_\Gamma : \Gamma(\mathfrak{g}_1) \to \Gamma(\mathfrak{g}_2)$ is a well-defined homomorphism, which is an injection by (2). □

Proposition 17. *If the characteristic of $\mathbf{F}$ is not 2, then*

$$C(\mathfrak{g}) = \Gamma(\mathfrak{g}) \cap \operatorname{Der}(\mathfrak{g}). \tag{5}$$

Proof. If $\psi \in \Gamma(\mathfrak{g}) \cap \operatorname{Der}(\mathfrak{g})$, then by the definition of $\Gamma(\mathfrak{g})$ and $\operatorname{Der}(\mathfrak{g})$, for all $x, y, z \in \mathfrak{g}$, we have $\psi(xy) = (\psi(x))y + x(\psi(y))$, $\psi([x, y, z]) = [\psi(x), y, z] + [x, \psi(y), z] + [x, y, \psi(z)]$, and $\psi(xy) = (\psi(x))y = x(\psi(y))$, $\psi([x, y, z]) = [\psi(x), y, z] = [x, \psi(y), z] = [x, y, \psi(z)]$, so $\psi(\mathfrak{g}\mathfrak{g}) = \psi([\mathfrak{g}, \mathfrak{g}, \mathfrak{g}]) = 0$ and $\psi(\mathfrak{g}) \subseteq C(\mathfrak{g})$. It follows easily that $\Gamma(\mathfrak{g}) \cap \operatorname{Der}(\mathfrak{g}) \subseteq C(\mathfrak{g})$.

To show the inverse inclusion, let $\psi \in C(\mathfrak{g})$; then $0 = \psi([x,y,z]) = [\psi(x),y,z] = [x,\psi(y),z] = [x,y,\psi(z)], 0 = \psi(xy) = (\psi(x))y = x(\psi(y))$. Thus, $\psi \in \Gamma(\mathfrak{g}) \cap \operatorname{Der}(\mathfrak{g})$.

This implies that $C(\mathfrak{g}) = \Gamma(\mathfrak{g}) \cap \operatorname{Der}(\mathfrak{g})$. □

Lemma 18. *Let I be a nonzero $\Gamma(\mathfrak{g})$-invariant ideal of $\mathfrak{g}$, $V(I) = \{\psi \in \Gamma(\mathfrak{g})\psi(I) = 0\}$, and let $\operatorname{Hom}(\mathfrak{g}/I, Z_{\mathfrak{g}}(I))$ be the vector space of all linear maps from $\mathfrak{g}/I$ to $Z_{\mathfrak{g}}(I)$ over $\mathbf{F}$. Define $T(I) = \{f \in \operatorname{Hom}(\mathfrak{g}/I, Z_{\mathfrak{g}}(I)) \mid f([\overline{x},\overline{y},\overline{z}]) = [f(\overline{x}),y,z] = [x,f(\overline{y}),z] = [x,y,f(\overline{z})], f(\overline{x}\,\overline{y}) = (f(\overline{x}))y = x(f(\overline{y}))\}$, where $\overline{x},\overline{y},\overline{z} \in \mathfrak{g}/I$. Then one has the following.*

(1) *$T(I)$ is a subspace of $\operatorname{Hom}(\mathfrak{g}/I, Z_{\mathfrak{g}}(I))$ and $V(I) \cong T(I)$ as vector spaces.*

(2) *If $\Gamma(I) = \mathbf{F}\mathrm{id}_I$, then $\Gamma(\mathfrak{g}) = \mathbf{F}\mathrm{id}_{\mathfrak{g}} \oplus V(I)$ as vector spaces.*

Proof. (1) It is easily seen that $V(I)$ is an ideal of the associative algebra $\Gamma(\mathfrak{g})$. To prove (1) consider the following map $\alpha : V(I) \to T(I)$ given by

$$\alpha(\psi)(\overline{y}) \longmapsto \psi(y), \tag{6}$$

where $\psi \in V(I)$ and $\overline{y} = y + I \in \mathfrak{g}/I$. The map α is well defined. If $\overline{y} = \overline{y_1}$, then $y - y_1 \in I$, and so $\psi(y - y_1) = 0$. It follows easily that α is injective. We now show that α is onto. For every $f \in T(I)$, set $\psi_f : \mathfrak{g} \to \mathfrak{g}, \psi_f(x) = f(\overline{x})$, for all $x \in \mathfrak{g}$. It follows from the definition of $T(I)$ that, for all $x,y,z \in \mathfrak{g}$, $\psi_f(xy) = f(\overline{x}\,\overline{y}) = (f(\overline{x}))y = x(f\overline{y}), \psi_f([x,y,z]) = f([\overline{x},\overline{y},\overline{z}]) = [f(\overline{x}),y,z] = [x,f(\overline{y}),z] = [x,y,f(\overline{z})]$. Thus, $\psi_f \in \Gamma(\mathfrak{g})$, and so $\psi_f \in V(I), \psi_f(I) = 0$. But, $\alpha(\psi_f) = f$ implies that α is onto. It is fairly easy to see that α preserves operations on vector spaces from $\mathfrak{g}/I$ to $Z_{\mathfrak{g}}(I)$. This proves (1).

We now prove (2). If $\Gamma(I) = \mathbf{F}\mathrm{id}_I$, then for all $\psi \in \Gamma(\mathfrak{g}), \psi|_I = \lambda \mathrm{id}_I$, for some $\lambda \in \mathbf{F}$. If $\psi \neq \lambda \mathrm{id}_{\mathfrak{g}}$, let $\varphi(x) = \lambda x$, for all $x \in \mathfrak{g}$; then $\psi \in \Gamma(\mathfrak{g})$ and $\psi - \varphi \in V(I)$. Clearly, $\psi = \varphi + (\psi - \varphi)$. Furthermore, $\mathbf{F}\mathrm{id}_{\mathfrak{g}} \cap V(I) = 0$, and so (2) is proved. □

Lemma 19. *Let $\mathfrak{g}$ be a Lie triple algebra; then φD is a derivation for $\varphi \in \Gamma(\mathfrak{g}), D \in \operatorname{Der}(\mathfrak{g})$.*

Proof. If $x,y,z \in \mathfrak{g}$, then

$$\begin{aligned} \varphi D(xy) &= \varphi((D(x))y + x(D(y))) = (\varphi D(x))y \\ &\quad + x(\varphi D(y)), \\ \varphi D([x,y,z]) &= \varphi([D(x),y,z] \\ &\quad + [x,D(y),z] + [x,y,D(z)]) \\ &= [\varphi D(x),y,z] + [x,\varphi D(y),z] \\ &\quad + [x,y,\varphi D(z)]. \end{aligned} \tag{7}$$

Thus, φD is a derivation. □

Theorem 20. *Let $\mathfrak{g}$ be a Lie triple algebra, and for any $\varphi \in \Gamma(\mathfrak{g}), D \in \operatorname{Der}(\mathfrak{g})$, then one has the following.*

(1) $\operatorname{Der}(\mathfrak{g})$ *is contained in the normalizer of $\Gamma(\mathfrak{g})$ in* End $(\mathfrak{g})$.

(2) *$D\varphi$ is contained in $\Gamma(\mathfrak{g})$ if and only if φD is a central derivation of $\mathfrak{g}$.*

(3) *φD is a derivation of $\mathfrak{g}$ if and only if $[D,\varphi]$ is a central derivation of $\mathfrak{g}$.*

Proof. For any $\varphi \in \Gamma(\mathfrak{g}), D \in \operatorname{Der}(\mathfrak{g})$, and $x,y,z \in \mathfrak{g}$,

$$\begin{aligned} D\varphi(xy) &= D(\varphi(x)y) = (D\varphi(x))y + (\varphi(x))(Dy) \\ &= (D\varphi(x))y + \varphi D(xy) - (\varphi D(x))y. \end{aligned} \tag{8}$$

Then, we get $(D\varphi - \varphi D)(xy) = ((D\varphi - \varphi D)(x))y$. On the other hand, we have

$$\begin{aligned} D\varphi([x,y,z]) &= D([\varphi(x),y,z]) \\ &= [D\varphi(x),y,z] + [\varphi(x),D(y),z] \\ &\quad + [\varphi(x),y,D(z)] \\ &= [D\varphi(x),y,z] + \varphi D([x,y,z]) \\ &\quad - [\varphi D(x),y,z]. \end{aligned} \tag{9}$$

So, $(D\varphi - \varphi D)([x,y,z]) = [(D\varphi - \varphi D)(x),y,z]$; this is $[D,\varphi] = D\varphi - \varphi D \in \Gamma(\mathfrak{g})$. This proves (1). From Lemma 19 and (1), $D\varphi$ is an element of $\Gamma(\mathfrak{g})$ if and only if $\varphi D \in \Gamma(\mathfrak{g}) \cap \operatorname{Der}(\mathfrak{g})$. Thanks to Proposition 17, we get the result (2). It follows from (1), Proposition 17, and Lemma 19 that the result holds. □

Now, we study the relationship between the centroid of a decomposable Lie triple algebra and the centroid of its factors.

Theorem 21. *Suppose that $\mathfrak{g}$ is a Lie triple algebra over $\mathbf{F}$ and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ being ideals of $\mathfrak{g}$; then*

$$\Gamma(\mathfrak{g}) \cong \Gamma(\mathfrak{g}_1) \oplus \Gamma(\mathfrak{g}_2) \oplus C(1) \oplus C(2), \tag{10}$$

as vector spaces, where $C(1) = \{\varphi \in \operatorname{Hom}(\mathfrak{g}_1,\mathfrak{g}_1) \mid \varphi(\mathfrak{g}_1) \subseteq Z(\mathfrak{g}_2), \varphi(\mathfrak{g}_2\mathfrak{g}_2) = \varphi[\mathfrak{g}_2,\mathfrak{g}_2,\mathfrak{g}_2] = 0\}$, $C(2) = \{\varphi \in \operatorname{Hom}(\mathfrak{g}_2,\mathfrak{g}_2) \mid \varphi(\mathfrak{g}_2) \subseteq Z(\mathfrak{g}_1), \varphi(\mathfrak{g}_1\mathfrak{g}_1) = \varphi[\mathfrak{g}_1,\mathfrak{g}_1,\mathfrak{g}_1] = 0\}$.

Proof. Let $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ be canonical projections for $i = 1,2$; then $\pi_1,\pi_2 \in \Gamma(\mathfrak{g})$ and $\pi_1 + \pi_2 = \mathrm{id}_{\mathfrak{g}}$. So we have for $\varphi \in \Gamma(\mathfrak{g})$,

$$\varphi = \pi_1\varphi\pi_1 + \pi_1\varphi\pi_2 + \pi_2\varphi\pi_1 + \pi_2\varphi\pi_2. \tag{11}$$

Note that $\pi_i\varphi\pi_j \in \Gamma(\mathfrak{g})$ for $i,j = 1,2$. We claim that

$$\Gamma(\mathfrak{g}) = \pi_1\Gamma(\mathfrak{g})\pi_1 + \pi_1\Gamma(\mathfrak{g})\pi_2 + \pi_2\Gamma(\mathfrak{g})\pi_1 + \pi_2\Gamma(\mathfrak{g})\pi_2. \tag{12}$$

It suffices to show that $\pi_1\Gamma(\mathfrak{g})\pi_1 \cap \pi_1\Gamma(\mathfrak{g})\pi_2$ (other cases are similar). For any $\varphi \in \pi_1\Gamma(\mathfrak{g})\pi_1 \cap \pi_1\Gamma(\mathfrak{g})\pi_2$, there exist $f_i \in \Gamma(\mathfrak{g}), i = 1,2$, such that $\varphi = \pi_1 f_1 \pi_1 = \pi_1 f_2 \pi_2$. Then, $\varphi(x) = \pi_1 f_2 \pi_2(x) = \pi_1 f_2 \pi_2(\pi_2(x)) = \pi_1 f_1(0) = 0$, for all $x \in \mathfrak{g}$, and so $\varphi = 0$. Let $\Gamma(\mathfrak{g})_{ij} = \pi_i\Gamma(\mathfrak{g})\pi_j, i,j = 1,2$. We now prove that

$$\begin{aligned} \Gamma(\mathfrak{g})_{11} &\cong \Gamma(\mathfrak{g}_1), & \Gamma(\mathfrak{g})_{22} &\cong \Gamma(\mathfrak{g}_2), \\ \Gamma(\mathfrak{g})_{12} &\cong C_2, & \Gamma(\mathfrak{g})_{21} &\cong C_1. \end{aligned} \tag{13}$$

Since $\varphi(\mathfrak{g}_2) = 0$ for $\varphi \in \Gamma(\mathfrak{g})_{11}$, we have $\varphi|_{\mathfrak{g}_1} \in \Gamma(\mathfrak{g}_1)$. On the other hand, one can regard $\Gamma(\mathfrak{g}_1)$ as a subalgebra of $\Gamma(\mathfrak{g})$ by extending any $\varphi_0 \in \Gamma(\mathfrak{g}_1)$ on $\mathfrak{g}_2$ being equal to zero; that is, $\varphi_0(x_1) = \varphi_0(x_1), \varphi_0(x_2) = 0$, for all $x_1 \in \mathfrak{g}_1, x_2 \in \mathfrak{g}_2$. Then $\varphi_0 \in \Gamma(\mathfrak{g})$ and $\varphi_0 \in \Gamma(\mathfrak{g})_{11}$. Therefore, $\Gamma(\mathfrak{g})_{11} \cong \Gamma(\mathfrak{g}_1)$ with isomorphism $\sigma : \Gamma(\mathfrak{g})_{11} \to \Gamma(\mathfrak{g}_1), \sigma(\varphi) = \varphi|_{\mathfrak{g}_1}$, for all $\varphi \in \Gamma(\mathfrak{g})_{11}$. Similarly, we have $\Gamma(\mathfrak{g})_{22} \cong \Gamma(\mathfrak{g}_2)$.

Next, we prove that $\Gamma(\mathfrak{g})_{12} \cong C_2$. If $\varphi \in \Gamma(\mathfrak{g})_{12}$ there exists φ_0 in $\Gamma(\mathfrak{g})$ such that $\varphi = \pi_1\varphi_0\pi_2$. For $x_k = x_k^1 + x_k^2 \in \mathfrak{g}$, where $x_k^i \in \mathfrak{g}_i, i = 1, 2$ and $k = 1, 2, 3$, we have

$$\begin{aligned}
\varphi(x_1x_2) &= \pi_1\varphi_0\pi_2(x_1x_2) \\
&= \pi_1\varphi_0\left(x_1^2x_2^2\right) \\
&= \pi_1\left(\left(\varphi_0x_1^2\right)x_2^2\right) = 0, \\
\varphi([x_1, x_2, x_3]) &= \pi_1\varphi_0\pi_2([x_1, x_2, x_3]) \\
&= \pi_1\varphi_0\left(\left[x_1^2, x_2^2, x_3^2\right]\right) \\
&= \pi_1\left(\left[\varphi_0x_1^2, x_2^2, x_3^2\right]\right) = 0, \\
(\varphi(x_1))x_2 &= \varphi(x_1x_2) = 0, \\
[\varphi(x_1), x_2, x_3] &= \varphi([x_1, x_2, x_3]) = 0.
\end{aligned} \tag{14}$$

Then, $\varphi(\mathfrak{g}) \subseteq Z(\mathfrak{g})$ and $\varphi(\mathfrak{g}\mathfrak{g}) = \varphi[\mathfrak{g}, \mathfrak{g}, \mathfrak{g}] = 0$. It follows that $\varphi|_{\mathfrak{g}_2}(\mathfrak{g}_2) \subseteq Z(\mathfrak{g}_1), \varphi|_{\mathfrak{g}_2}(\mathfrak{g}_2\mathfrak{g}_2) = \varphi|_{\mathfrak{g}_2}([\mathfrak{g}_2, \mathfrak{g}_2, \mathfrak{g}_2]) = 0$, and so $\varphi|_{\mathfrak{g}_2} \in C_2$.

Conversely, for $\varphi \in C_2$, expending φ on $\mathfrak{g}$ (also denoted by φ) or by $\varphi(\mathfrak{g}_1\mathfrak{g}_1) = \varphi[\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1] = 0$, we have $\pi_1\varphi\pi_2 = \varphi$ and $\varphi \in \Gamma(\mathfrak{g})_{12}$. This proves that $\Gamma(\mathfrak{g})_{12}$ is isomorphic to C_2 with the following isomorphism:

$$\tau : \Gamma(\mathfrak{g})_{12} \longrightarrow C_2, \qquad \tau(\varphi) = \varphi \mid \Gamma(\mathfrak{g})_{12}, \tag{15}$$

for all $\varphi \in \Gamma(\mathfrak{g})_{12}$. Similarly, we can prove that $\Gamma(\mathfrak{g})_{21} \cong C_1$. Summarizing the aforesaid discussion, we have

$$\Gamma(\mathfrak{g}) \cong \Gamma(\mathfrak{g}_1) \oplus \Gamma(\mathfrak{g}_2) \oplus C(1) \oplus C(2). \tag{16}$$

The proof is completed. □

A generalized version of Theorem 21 is stated next without proof.

Theorem 22. *Suppose that $\mathfrak{g}$ is a Lie triple algebra over $\mathbf{F}$ and with a decomposition of ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m$. Then, one has*

$$\Gamma(\mathfrak{g}) \cong \Gamma(\mathfrak{g}_1) \oplus \cdots \oplus \Gamma(\mathfrak{g}_m) \oplus \left(\oplus_{1\le i\ne j\le m} C_{ij}\right), \tag{17}$$

as vector spaces, where $C_{ij} = \{\varphi \in \mathrm{Hom}(\mathfrak{g}_i, \mathfrak{g}_j) \mid \varphi(\mathfrak{g}_i) \subseteq Z(\mathfrak{g}_j), \varphi(\mathfrak{g}_i\mathfrak{g}_i) = \varphi[\mathfrak{g}_i, \mathfrak{g}_i, \mathfrak{g}_i] = 0$, for $1 \le i \ne j \le m\}$.

4. The Centroids of Tensor Product of Lie Triple Algebras

Definition 23. Let A be an associative algebra over a field $\mathbf{F}$, where the centroid of A is the space of $\mathbf{F}$-linear transforms on A given by $\Gamma(A) = \{\varphi \in \mathrm{End}_{\mathbf{F}}(A)\varphi(ab) = a\varphi(b) = \varphi(a)b$, for all $a, b \in A\}$; then $\Gamma(A)$ is an associative algebra of $\mathrm{End}_{\mathbf{F}}(A)$. If $\mathfrak{g}$ is a finite-dimensional Lie triple algebra over $\mathbf{F}$, let $\mathfrak{g} \otimes A$ be a tensor product of the underlying vector spaces A and $\mathfrak{g}$. Then, $\mathfrak{g} \otimes A$ over a field $\mathbf{F}$ with respect to the following 2-ary and 3-ary multilinear operation:

$$\begin{aligned}
[x_1 \otimes a_1, x_2 \otimes a_2, x_3 \otimes a_3] &= [x_1, x_2, x_3] \otimes a_1a_2a_3, \\
(x_1 \otimes a_1)(x_2 \otimes a_2) &= (x_1x_2) \otimes a_1a_2,
\end{aligned} \tag{18}$$

where $x_i \in \mathfrak{g}$, $a_i \in A$, $i = 1, 2, 3$. This Lie triple algebra $\mathfrak{g} \otimes A$ is called the tensor product of A and $\mathfrak{g}$. For $f \in \mathrm{End}_{\mathbf{F}}(\mathfrak{g})$ and $\varphi \in \mathrm{End}_{\mathbf{F}}(A)$, there exists a unique map $f \tilde{\otimes} \varphi \in \mathrm{End}_{\mathbf{F}}(\mathfrak{g} \otimes A)$ such that

$$(f \tilde{\otimes} \varphi)(x \otimes a) = f(x) \otimes \varphi(a), \quad \forall x \in \mathfrak{g},\ a \in A. \tag{19}$$

The map should not be confused with the element $f \otimes \varphi$ of the tensor product $\mathrm{End}_{\mathbf{F}}(\mathfrak{g}) \otimes \mathrm{End}_{\mathbf{F}}(A)$. Of course, we have a canonical map as the following:

$$\begin{aligned}
\omega : \mathrm{End}_{\mathbf{F}}(\mathfrak{g}) \otimes \mathrm{End}_{\mathbf{F}}(A) \\
\longrightarrow \mathrm{End}_{\mathbf{F}}(\mathfrak{g} \otimes A) : f \otimes \varphi \longmapsto f \tilde{\otimes} g.
\end{aligned} \tag{20}$$

It is easy to see that if $\phi \in \Gamma(\mathfrak{g})$ and $\psi \in \Gamma(A)$, then $\phi \tilde{\otimes} \psi \in \Gamma(\mathfrak{g} \otimes A)$. Hence, $\Gamma(\mathfrak{g}) \tilde{\otimes} \Gamma(A) \subseteq \Gamma(\mathfrak{g} \otimes A)$, where $\Gamma(\mathfrak{g}) \tilde{\otimes} \Gamma(A)$ is the $\mathbf{F}$-span of all endomorphisms $\phi \tilde{\otimes} \psi$.

Definition 24. The transformation $\phi \in \Gamma(\mathfrak{g} \otimes A)$ is said to have finite $\mathfrak{g}$-image, if for any $a \in A$, there exist finitely many $a_1, \ldots, a_n \in A$ such that

$$\phi(\mathfrak{g} \otimes \mathbf{F}a) \subseteq \mathfrak{g} \otimes \mathbf{F}a_1 + \cdots + \mathfrak{g} \otimes \mathbf{F}a_n. \tag{21}$$

It is easy to see that $\Gamma(\mathfrak{g}) \tilde{\otimes} \Gamma(A) \subseteq \{\phi \in \Gamma(\mathfrak{g} \otimes A) \mid \phi$ has finite $\mathfrak{g}$-image$\}$. In addition, $\phi \in \Gamma(\mathfrak{g} \otimes A)$ has finite $\mathfrak{g}$-image, if $\phi(\mathfrak{g} \otimes 1) \subseteq \mathfrak{g} \otimes \mathbf{F}a_1 + \cdots + \mathfrak{g} \otimes \mathbf{F}a_n$ for suitable $a_i \in A$.

Lemma 25. *Let $\mathfrak{g}$ be a Lie triple algebra over $\mathbf{F}$ and A a unitary commutative associative over $\mathbf{F}$. Let $\{a_r\}_{r\in\mathfrak{R}}$ be a basis of A, and let $\phi \in \Gamma(\mathfrak{g} \otimes A)$. Define $\phi_r \in \mathrm{End}_{\mathbf{F}}(\mathfrak{g})$ by*

$$\phi(x \otimes 1) = \sum_{r\in\mathfrak{R}} \phi_r(x) \otimes a_r, \tag{22}$$

then all $\phi_r \in \Gamma(\mathfrak{g})$.

Proof. For $x_1, x_2, x_3 \in \mathfrak{g}$, we have

$$\begin{aligned}
\phi([x_1,x_2,x_3]\otimes 1) &= \sum_{r\in\mathfrak{R}} \phi_r([x_1,x_2,x_3])\otimes a_r, \phi(x_1x_2) \\
&= \sum_{r\in\mathfrak{R}} \phi_r(x_1x_2)\otimes a_r, \\
\phi([x_1,x_2,x_3]\otimes 1) &= \phi([x_1\otimes 1, x_2\otimes 1, x_3\otimes 1]) \\
&= [\phi(x_1\otimes 1), x_2\otimes 1, x_3\otimes 1] \\
&= \sum_{r\in\mathfrak{R}} [\phi_r(x_1), x_2, x_3]\otimes a_r \\
&= [x_1\otimes 1, \phi(x_2\otimes 1), x_3\otimes 1] \\
&= \sum_{r\in\mathfrak{R}} [x_1, \phi_r(x_2), x_3]\otimes a_r \\
&= [x_1\otimes 1, x_2\otimes 1, \phi(x_3\otimes 1)] \\
&= \sum_{r\in\mathfrak{R}} [x_1, x_2, \phi_r(x_3)]\otimes a_r, \\
(\phi(x_1x_2)\otimes 1) &= \phi((x_1\otimes 1)(x_2\otimes 1)) \\
&= \sum_{r\in\mathfrak{R}} \phi_r(x_1x_2)\otimes a_r \\
&= (\phi(x_1\otimes 1))(x_2\otimes 1) \\
&= \sum_{r\in\mathfrak{R}} (\phi_r(x_1)x_2)\otimes a_r \\
&= (x_1\otimes 1)(\phi(x_2\otimes 1)) \\
&= \sum_{r\in\mathfrak{R}} (x_1(\phi_r(x_1)))\otimes a_r.
\end{aligned} \tag{23}$$

Hence, one has

$$\begin{aligned}
\phi_r([x_1,x_2,x_3]) &= [\phi_r(x_1), x_2, x_3] = [x_1, \phi_r(x_2), x_3] \\
&= [x_1, x_2, \phi_r(x_3)], \\
\phi_r(x_1x_2) &= (\phi_r(x_1))x_2 = x_1(\phi_r(x_2)),
\end{aligned} \tag{24}$$

for all $x_i \in \mathfrak{g}$. So all $\phi_r \in \Gamma(\mathfrak{g})$. □

Proposition 26. *Let $\mathfrak{g}$ be a perfect Lie triple algebra over $\mathbf{F}$, and let A be a unitary commutative associative over $\mathbf{F}$ which is free as a $\mathbf{F}$-module. Then, one has the following.*

(1) *If $\mathfrak{g}$ is perfect, then $\mathfrak{g}\otimes A$ is perfect too.*

(2) *If $\mathfrak{g}$ is finitely generated as a $\mathrm{Mult}(\mathfrak{g})$-module (or as a $\Gamma(\mathfrak{g})$-module), or if $\mathfrak{g}$ is a central and a torsion-free $\mathbf{F}$-module, then every $\phi \in \Gamma(\mathfrak{g}\otimes A)$ has finite $\mathfrak{g}$-image.*

(3) *If $\Gamma(\mathfrak{g})$ is a free $\mathbf{F}$-module and the map ω of Definition 23 is injective, then $\Gamma(\mathfrak{g})\,\widetilde{\otimes}\,\Gamma(A) = \{\phi \in \Gamma(\mathfrak{g}\otimes A) \mid \phi$ has finite $\mathfrak{g}$-image$\}$.*

Proof. The definition of perfect shows that (1) holds. Let $M = \mathrm{Mult}(\mathfrak{g})$. It follows that $M\widetilde{\otimes}\mathrm{id} \subseteq \mathrm{Mult}(\mathfrak{g}\otimes A)$ from A is unital. Since $\mathfrak{g}$ is finitely generated as a $\mathrm{Mult}(\mathfrak{g})$-module, we can suppose $\mathfrak{g} = Mx_1 + \cdots + Mx_n$ for $x_1, \ldots, x_n \in \mathfrak{g}$. Fix $\phi \in \Gamma(\mathfrak{g}\otimes A)$ and $a \in A$. There exist finite families $\{x_{ij}\} \subseteq \mathfrak{g}$ and $\{a_{ij}\} \subseteq A$ such that

$$\phi(x_i\otimes a) = \sum_j x_{ij}\otimes a_{ij}, \tag{25}$$

for $1 \le i \le n$. Hence,

$$\begin{aligned}
\phi(\mathfrak{g}\otimes a) &= \sum_i \phi((M\widetilde{\otimes}\mathrm{id})(x_i\otimes a)) \\
&= \sum_{i,j} (M\widetilde{\otimes}\mathrm{id})(x_{ij}\otimes a_{ij}) \subseteq \sum_{i,j} \mathfrak{g}\otimes a_{ij}.
\end{aligned} \tag{26}$$

Since $\mathfrak{g}\otimes A$ is perfect, the centroid $\Gamma(\mathfrak{g}\otimes A)$ is commutative. By replacing M with $\Gamma(\mathfrak{g})$ we can use the same argument aforementione to show that every $\phi \in \Gamma(T\otimes A)$ has finite $\mathfrak{g}$-image, if $\mathfrak{g}$ is a finitely generated $\Gamma(\mathfrak{g})$-module.

Now, suppose that $\Gamma(\mathfrak{g}) = \mathbf{F}\mathrm{id}$ and that $\mathfrak{g}$ is a torsion-free $\mathbf{F}$-module. Then there exist scalars $k_r \in \mathbf{F}$ such that $\phi_r = k_r\mathrm{id}$. Hence,

$$\phi(x\otimes 1) = \sum_{r\in\mathfrak{R}} k_r x\otimes a_r. \tag{27}$$

Fix $x \in \mathfrak{g}$; then almost all $k_r x = 0$ and almost all $k_r = 0$, which in turn implies that ϕ has finite $\mathfrak{g}$-image.

From the aforementione discussion above, we get

$$\Gamma(\mathfrak{g})\,\widetilde{\otimes}\,\Gamma(A) \subseteq \{\phi \in \Gamma(\mathfrak{g}\otimes A) \mid \phi \text{ has finite } \mathfrak{g}\text{-image}\}. \tag{28}$$

So, it suffices to prove that $\Gamma(\mathfrak{g})\widetilde{\otimes}\Gamma(A) \supseteq \{\phi \in \Gamma(\mathfrak{g}\otimes A) \mid \phi$ has finite $\mathfrak{g}$-image$\}$. We suppose that $\phi \in \Gamma(\mathfrak{g}\otimes A)$ has finite $\mathfrak{g}$-image, and then there exists a finite subset $\mathfrak{J} \subseteq \mathfrak{R}$ such that equation in Lemma 25 becomes as follows:

$$\phi(x\otimes 1) = \sum_{r\in\mathfrak{J}} \phi_r(x)\otimes a_r. \tag{29}$$

For $x_1, x_2, x_3 \in \mathfrak{g}$ and $a \in A$, we get

$$\begin{aligned}
\phi([x_1,x_2,x_3]\otimes a) &= \phi([x_1\otimes 1, x_2\otimes 1, x_3\otimes a]) \\
&= [\phi(x_1\otimes 1), x_2\otimes 1, x_3\otimes a] \\
&= \sum_{r\in\mathfrak{J}} [\phi_r(x_1), x_2, x_3]\otimes a_r a \\
&= \sum_{r\in\mathfrak{J}} \phi_r([x_1,x_2,x_3])\otimes a_r a, \\
\phi((x_1x_2)\otimes a) &= \phi((x_1\otimes 1)(x_2\otimes a)) \\
&= (\phi(x_1\otimes 1))(x_2\otimes a) \\
&= \sum_{r\in\mathfrak{J}} (\phi_r(x_1)x_2)\otimes a_r a \\
&= \sum_{r\in\mathfrak{J}} \phi_r(x_1x_2)\otimes a_r a.
\end{aligned} \tag{30}$$

Since $\mathfrak{g}$ is perfect, we can get

$$\phi = \sum_{r\in\mathfrak{J}} \phi_r \tilde{\otimes} \lambda_r, \quad (31)$$

where λ_r is the left multiplication in A by a_r. Let $\{\psi_s \mid s \in \mathfrak{S}\}$ be a basis of $\Gamma(\mathfrak{g})$. Then, there exist a finite subset $\mathfrak{B} \subseteq \mathfrak{S}$ and scalars $k_{rs} \in \mathbf{F}\,(r \in \mathfrak{J}, s \in \mathfrak{B})$ such that

$$\phi_r = \sum_{s\in\mathfrak{B}} k_{rs}\psi_s. \quad (32)$$

We then get

$$\phi = \sum_{r\in\mathfrak{J}}\sum_{s\in\mathfrak{B}} k_{rs}\psi_s \tilde{\otimes} \lambda_r = \sum_{s\in\mathfrak{B}} \psi_s \tilde{\otimes} \phi_s, \quad \text{for } \phi_s = \sum_{r\in\mathfrak{J}} k_{rs}\lambda_r. \quad (33)$$

Now we show that $\phi_s \in \Gamma(A)$. For any $x_1, x_2, x_3 \in \mathfrak{g}, a_1, a_2 \in A$, we have

$$\begin{aligned}
\sum_{s\in\mathfrak{B}} \psi_s([x_1, x_2, x_3]) \otimes \phi_s(a_1 a_2)
&= \phi([x_1, x_2, x_3] \otimes a_1 a_2) \\
&= [\phi(x_1 \otimes a_1), x_2 \otimes a_2, x_3 \otimes 1] \\
&= \left[\sum_{s\in\mathfrak{B}} \psi_s(x_1) \otimes \phi_s(a_1), x_2 \otimes a_2, x_3 \otimes 1\right] \\
&= \sum_{s\in\mathfrak{B}} [\psi_s(x_1), x_2, x_3] \otimes \phi_s(a_1) a_2 \\
&= \sum_{s\in\mathfrak{B}} \psi_s([x_1, x_2, x_3]) \otimes \phi_s(a_1) a_2, \\
\sum_{s\in\mathfrak{B}} \psi_s(x_1 x_2) \otimes \phi_s(a_1)
&= \phi((x_1 x_2) \otimes a_1) = (\phi(x_1 \otimes a_1)(x_2 \otimes 1)) \\
&= \sum_{s\in\mathfrak{B}} (\psi_s(x_1))(x_2) \otimes \phi_s(a_1) 1 \\
&= \sum_{s\in\mathfrak{B}} \psi_s(x_1 x_2) \otimes \phi_s(a_1) 1.
\end{aligned} \quad (34)$$

Since $[\mathfrak{g}\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}\mathfrak{g} = \mathfrak{g}$, we have

$$\sum_{s\in\mathfrak{B}} \psi_s(x) \otimes (\phi_s(a_1 a) - \phi_s(a_1) a) = 0, \quad (35)$$

for all $x \in \mathfrak{g}$ and $a_1, a_2 \in A$. It follows that

$$\sum_{s\in\mathfrak{B}} \psi_s \tilde{\otimes} \mu_s = 0, \quad (36)$$

where $\mu_s \in \mathrm{End}_{\mathbf{F}}(A)$ is defined by $\mu_s(a_2) = \phi_s(a_1 a_2) - \phi_s(a_1) a_2$, for all $a_2 \in A$. Since ω is injective, we also have

$$\sum_{s\in\mathfrak{B}} \psi_s \otimes \mu_s = 0. \quad (37)$$

So by the linear independence of the ψ_s, we get that $\mu_s = 0$. Then $\psi_s \in \Gamma(A)$. Hence $\phi \in \Gamma(\mathfrak{g}) \tilde{\otimes} \Gamma(A)$. □

Next, we will determine the centroid of the tensor product of a simple Lie triple algebra and a polynomial ring. Here after we study the centroid of Lie triple algebra over a filed $\mathbf{F}$. Let $\{e_k\}$ be a basis of the central derivation $C(\mathfrak{g})$ and $\{\varphi_j\}$ a maximal subset of $\Gamma(\mathfrak{g})$ such that $\{\varphi_j|_{\mathfrak{g}\mathfrak{g}}\}$ and $\{\varphi_j|_{[\mathfrak{g},\mathfrak{g},\mathfrak{g}]}\}$ are linear independent. Then, we have the following result.

Theorem 27. *Let Ψ denote the subspace of $\Gamma(\mathfrak{g})$ spanned by $\{\varphi_j\}$. Then $\{e_k, \varphi_j\}$ is a basis of $\Gamma(\mathfrak{g})$ and $\Gamma(\mathfrak{g}) = \Psi \oplus C(\mathfrak{g})$ as vector spaces.*

Proof. Since $\{\varphi_j|_{\mathfrak{g}\mathfrak{g}}\}$ and $\{\varphi_j|_{[\mathfrak{g},\mathfrak{g},\mathfrak{g}]}\}$ are linear independent, $\{\varphi_j\}$ is linear independent in $\Gamma(\mathfrak{g})$. By definition of $\{e_k, \varphi_j\}$, the $\{e_k, \varphi_j\}$ is independent in $\Gamma(\mathfrak{g})$. For $\varphi \in \Gamma(\mathfrak{g})$ since $\{\varphi_j|_{\mathfrak{g}\mathfrak{g}}\}$ is a basis of vector spaces $\{\varphi|_{\mathfrak{g}\mathfrak{g}} \mid \varphi \in \Gamma(\mathfrak{g})\}$, and $\{\varphi_j|_{[\mathfrak{g},\mathfrak{g},\mathfrak{g}]}\}$ is a basis of vector spaces $\{\varphi|_{[\mathfrak{g},\mathfrak{g},\mathfrak{g}]} \mid \varphi \in \Gamma(\mathfrak{g})\}$, there exist $g_s \in \mathbf{F}$, $s \in J$ (J is a finite set of positive integers) such that

$$\begin{aligned}
\varphi|_{[\mathfrak{g},\mathfrak{g},\mathfrak{g}]} &= \sum_{s\in J} g_s \varphi_s|_{[\mathfrak{g},\mathfrak{g},\mathfrak{g}]}, \\
\varphi|_{\mathfrak{g}\mathfrak{g}} &= \sum_{s\in J} g_s \varphi_s|_{\mathfrak{g}\mathfrak{g}}.
\end{aligned} \quad (38)$$

We then have

$$\left(\varphi - \sum_{s\in J} g_s\varphi_s\right)|_{[\mathfrak{g},\mathfrak{g},\mathfrak{g}]} = 0, \qquad \left(\varphi - \sum_{s\in J} g_s\varphi_s\right)|_{\mathfrak{g}\mathfrak{g}} = 0. \quad (39)$$

If $y_1, y_2, y_3 \in \mathfrak{g}$, then

$$\begin{aligned}
0 &= \left(\varphi - \sum_{s\in J} g_s\varphi_s\right)([y_1, y_2, y_3]) \\
&= \left[\left(\varphi - \sum_{s\in J} g_s\varphi_s\right)(y_1), y_2, y_3\right], \\
0 &= \left(\varphi - \sum_{s\in J} g_s\varphi_s\right)(y_1 y_2) = \left(\left(\varphi - \sum_{s\in J} g_s\varphi_s\right)(y_1)\right) y_2.
\end{aligned} \quad (40)$$

It follows that

$$\begin{aligned}
&\left(\varphi - \sum_{s\in J} g_s\varphi_s\right)(\mathfrak{g}) \subseteq Z(\mathfrak{g}), \\
&\varphi - \sum_{s\in J} g_s\varphi_s,
\end{aligned} \quad (41)$$

is a central derivation. So there exist $r_i \in \mathbf{F}$, $i \in I$ (I is a finite set of positive integers) such that

$$\varphi - \sum_{s\in J} g_s\varphi_s = \sum_{i\in I} r_i c_i. \quad (42)$$

Therefore,

$$\varphi = \sum_{s\in J} g_s\varphi_s + \sum_{i\in I} r_i c_i. \quad (43)$$

The proof is completed. □

Lemma 28. *Let $\mathfrak{g}$ be a simple Lie triple algebra, $R = \mathbf{F}[x_1, \ldots, x_n]$, and $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes R$; then one has $\Gamma(\tilde{\mathfrak{g}}) \supseteq \Psi \otimes R + C(g) \otimes \operatorname{End}_{\mathbf{F}}(R)$.*

Proof. Let $\psi \in \Gamma(\mathfrak{g}), m \in R$. Then, we have

$$\begin{aligned}
(\psi \otimes m)([x \otimes p, y \otimes q, z \otimes r]) &= (\psi \otimes m)([x, y, z] \otimes pqr) \\
&= \psi([x, y, z]) \otimes mpqr \\
&= [\psi(x), y, z] \otimes mpqr \\
&= [x, \psi(y), z] \otimes mpqr \\
&= [x, y, \psi(z)] \otimes mpqr, \\
[(\psi \otimes m)(x \otimes p), y \otimes q, z \otimes r] &= [\psi(x), y, z] \otimes mpqr, \\
[x \otimes p, (\psi \otimes m)(y \otimes q), z \otimes r] &= [x, \psi(y), z] \otimes mpqr, \\
[x \otimes p, y \otimes q, (\psi \otimes m)(z \otimes r)] &= [x, y, \psi(z)] \otimes mpqr, \\
(\psi \otimes m)((x \otimes p)(y \otimes q)) &= (\psi \otimes m)((xy) \otimes pq) \\
&= \psi(xy) \otimes mpq \\
&= (\psi(x) y) \otimes mpq,
\end{aligned} \tag{44}$$

$$\begin{aligned}
((\psi \otimes m)(x \otimes p))(y \otimes q) &= (\psi(x) \otimes mp)(y \otimes q) \\
&= (\psi(x)) y \otimes mpq.
\end{aligned} \tag{45}$$

Thus $\psi \otimes m \in \Gamma(\tilde{\mathfrak{g}})$. For any $d \in C(g)$ and $f \in \operatorname{End}_{\mathbf{F}}(R)$, $d \otimes f \in \Gamma(\tilde{\mathfrak{g}})$, we have

$$\begin{aligned}
&(d \otimes f)([x \otimes p, y \otimes q, z \otimes r]) \\
&\quad = d([x, y, z]) \otimes f(pqr) = 0, \\
&[(d \otimes f)(x \otimes p), y \otimes q, z \otimes r] \\
&\quad = [d(x) \otimes f(p), y \otimes q, z \otimes r] \\
&\quad = [d(x), y, z] \otimes f(p) qr = 0, \\
&[(x \otimes p)(d \otimes f)(y \otimes q), z \otimes r] \\
&\quad = [x \otimes p, d(y) \otimes f(q), z \otimes r] \\
&\quad = [x, d(y), z] \otimes pf(q) r = 0, \\
&[x \otimes p, y \otimes q, (d \otimes f)(z \otimes r)] \\
&\quad = [(x \otimes p) y \otimes q, d(z) \otimes f(r)] \\
&\quad = [x, y, d(z)] \otimes pqf(r) = 0, \\
&(d \otimes f)((x \otimes p)(y \otimes q)) = d(xy) \otimes f(pq) = 0, \\
&((d \otimes f)(x \otimes p))(y \otimes q) = (d(x) \otimes f(p))(y \otimes q) \\
&\qquad = (d(x) y) \otimes f(p) q = 0, \\
&(x \otimes p)((d \otimes f)(y \otimes q)) = (x \otimes p)(d(y) \otimes f(q)) \\
&\qquad = (xd(y)) \otimes pf(q) = 0.
\end{aligned} \tag{46}$$

Hence $\Psi \otimes R + C(g) \otimes \operatorname{End}_{\mathbf{F}}(R) \subseteq \Gamma(\tilde{\mathfrak{g}})$. □

Theorem 29. *Let $\mathfrak{g}$ be a simple Lie triple algebra, $R = \mathbf{F}[x_1, \ldots, x_n]$, and $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes R$. Then $\Gamma(\tilde{\mathfrak{g}}) = \Gamma(\mathfrak{g}) \otimes R$.*

Proof. From Lemma 28, we get $\Gamma(\mathfrak{g}) \otimes R \subseteq \Gamma(\tilde{\mathfrak{g}})$. Now we prove the opposite inclusion. Let $\{m_i\}$ be a basis for R, $\psi \in \Gamma(\tilde{\mathfrak{g}})$, $p \in R$, $x, y, z \in \mathfrak{g}$, and let $\eta_i(-, p)$ be suitable maps in $\operatorname{End}_{\mathbf{F}}(\mathfrak{g})$ such that

$$\psi(x \otimes p) = \sum_i \eta_i(x, p) \otimes m_i. \tag{47}$$

Then,

$$\begin{aligned}
\psi([x \otimes p, y \otimes 1, z \otimes 1]) &= [\psi(x \otimes p), y \otimes 1, z \otimes 1] \\
&= \left[\sum_i \eta_i(x, p) \otimes m_i, y \otimes 1, z \otimes 1\right] \\
&= \sum_i [\eta_i(x, p), y, z] \otimes m_i \\
&= \psi([x \otimes 1, y \otimes p, z \otimes 1]) \\
&= [x \otimes 1, \psi(y \otimes p), z \otimes 1] \\
&= \left[x \otimes 1, \sum_i \eta_i(y, p) \otimes m_i, z \otimes 1\right] \\
&= \sum_i [x, \eta_i(y, p), z] \otimes m_i \\
&= \psi([x \otimes 1, y \otimes 1, z \otimes p]) \\
&= [x \otimes 1, y \otimes 1, \psi(z \otimes p)] \\
&= \left[x \otimes 1, y \otimes 1, \sum_i \eta_i(z, p) \otimes m_i\right] \\
&= \sum_i [x, y, \eta_i(z, p)] \otimes m_i, \\
\psi((x \otimes p)(y \otimes 1)) &= \sum_i (\eta_i(x, p) \otimes m_i)(y \otimes 1) \\
&= \sum_i (\eta_i(x, p) y) \otimes m_i \\
&= (x \otimes p)(\psi(y \otimes 1)) \\
&= \sum_i (x, \eta_i(y, p)) \otimes m_i,
\end{aligned} \tag{48}$$

while

$$\psi([x,y,z]\otimes p)=\sum_i \eta_i([x,y,z],p)\otimes m_i,$$
$$\psi((xy)\otimes p)=\sum_i \eta_i((xy),p)\otimes m_i. \tag{49}$$

So, for each i and p, we have $\eta_i(p[x,y,z],)=[\eta_i(x,p),y,z]=[x,\eta_i(y,p),z]=[x,y,\eta_i(z,p)]$, $\eta_i(p,(xy))=(\eta_i(x,p),y)=(x,\eta_i(y,p))$, for all $x,y,z\in\mathfrak{g}$. Hence, $\eta_i(-,p)\in\Gamma(\mathfrak{g})$. But $\Gamma(\mathfrak{g})=\mathbf{F}\mathrm{id}$. Therefore, $\eta_i(x,p)=\lambda_i(p)x$, for all $x\in\mathfrak{g}$, for suitable scalars $\lambda_i(p)$.

Thus, we may write

$$\psi(x\otimes p)=\sum_i \lambda_i(p)\,x\otimes m_i. \tag{50}$$

Since the right-hand side is in $\mathfrak{g}\otimes R$, for each p, we get $\lambda_i(p)=0$ for all except for a finite number of i. That is,

$$\psi(x\otimes p)=\sum_{i=1}^{n} x\otimes\lambda_i(p)\,m_i. \tag{51}$$

Then, the map

$$\rho: p\longmapsto\sum_{i=1}^{n}\lambda_i(p)\,m_i \tag{52}$$

is well defined. Hence, we have $\psi(x\otimes p)=x\otimes\rho(p)$. Thus,

$$\begin{aligned}
[\psi(x\otimes p),y\otimes 1,z\otimes 1]&=[x\otimes\rho(p),y\otimes 1,z\otimes 1]\\
&=[x,y,z]\otimes\rho(p),\\
[x\otimes p,\psi(y\otimes 1),z\otimes 1]&=[x\otimes p,y\otimes\rho(1),z\otimes 1]\\
&=[x,y,z]\otimes p\rho(1),\\
(\psi(x\otimes p)(y\otimes 1))&=(x\otimes\rho(p))(y\otimes 1)\\
&=(xy)\otimes\rho(p)\\
&=(x\otimes p)(\psi(y\otimes 1))\\
&=(x\otimes p)(y\otimes\rho(1))\\
&=(xy)\otimes p\rho(1).
\end{aligned} \tag{53}$$

Choose $x,y,z\in\mathfrak{g}$ such that $xy\neq 0$, $[x,y,z]\neq 0$. Then we can conclude that $\rho(p)=p\rho(1)$, for all $p\in R$. So ρ is determined by its action on 1. Therefore, $\rho\in R$. Thus,

$$\psi(x\otimes p)=(\mathrm{id}\otimes\rho)(x\otimes p),\quad \forall p\in R,\ x\in\mathfrak{g}. \tag{54}$$

That is, $\psi\in\mathrm{id}\otimes R\cong\Gamma(\mathfrak{g})\otimes R$. So, we have $\Gamma(\tilde{\mathfrak{g}})=\Gamma(\mathfrak{g})\otimes R$. □

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