Research Article

A New Approach on Helices in Pseudo-Riemannian Manifolds

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A proper curve α in the n-dimensional pseudo-Riemannian manifold (M, g) is called a V_n-slant helix if the function $g(V_n, X)$ is a nonzero constant along α, where X is a parallel vector field along α and V_n is nth Frenet frame. In this work, we study such curves and give important characterizations about them.

1. Introduction

Curves theory is an essential structure in the differential geometry works. Helix is one of the most spectacular curves because of its helical structure in nature and science. Helices are used in the field of imitation of kinematic motion or the shape of DNA and carbon nanotubes. Moreover, the helical structure can be seen in fractal geometry, especially in hyperhelices [1, 2].

Furthermore, "helices share common origins in the geometries of the platonic solids, with inherent hierarchical potential that is typical of biological structures. The helices provide an energy-efficient solution to close-packing in molecular biology, a common motif in protein construction, and a readily observable pattern at many size levels throughout the body. The helices are described in a variety of anatomical structures, suggesting their importance to structural biology and manual therapy" [3].

A curve of constant slope or general helix in Euclidean 3-space E^3 is defined by the property that its tangent vector field makes a constant angle with a fixed straight line (the axis of general helix). A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 [4, 5] as follows: A necessary and sufficient condition for a curve to be a general helix is that the ratio of the first curvature to the second curvature should be constant ;that is, κ/τ is constant along the curve, where κ and τ denote the first and second curvatures of the curve, respectively. In [6], Özdamar and Hacisalihoğlu defined harmonic curvature functions H_i $(1 \leq i \leq n-2)$

of a curve α and generalized helices in E^3 to those in n-dimensional Euclidean space E^n. Moreover, they gave a characterization for the inclined curves in E^n:

"A curve is an inclined curve

$$\text{if and only if } \sum_{i=1}^{n} H_i^2 = \text{constant."} \tag{1}$$

Harmonic curvature functions have important role in characterizations of general helices in higher dimensions, because the notion of a general helix can be generalized to higher dimension in different ways. However, these ways are not easy to show which curves are general helices and finding the axis of a general helix is complicated in higher dimension. Thanks to harmonic curvature functions, we can easily obtain the axis of such curves. Moreover, this way is confirmed in 3-dimensional spaces.

Izumiya and Takeuchi defined a new kind of helix (slant helix) and they gave a characterization of slant helices in Euclidean 3-space E^3 [7]. In 2008, Önder et al. defined a new kind of slant helix in Euclidean 4-space E^4 which is called B_2-slant helix and they gave some characterizations of these slant helices in Euclidean 4-space E^4 [8]. And then in 2009 Gök et al. generalized B_2-slant helix in E^4 to E^n, $n > 3$, called V_n-slant helix in Euclidean and Minkowski n-space [9, 10]. Lots of authors in their papers have investigated inclined curves and slant helices using the harmonic curvature functions in Euclidean and Minkowski n-space [11–15]. But,

Şenol et al. [16] see for the first time that the characterization of inclined curves and slant helices in (1) is true only for the case necessity but not true for the case sufficiency in Euclidean n-space. Then, they consider the precharacterizations about inclined curves and slant helices and restructure them with the necessary and sufficient condition [16].

Similar to the work in [16], in this work, we define V_n-slant helix and give characterizations about the helix with necessary and sufficient condition in n-dimensional pseudo-Riemannian manifolds for the first time.

2. Preliminaries

In this section, we give some basic definitions from differential geometry.

Definition 1. A metric tensor g on a smooth manifold M is a symmetric nondegenerate $(0, 2)$ tensor field on M.

In other words, $g(X, Y) = g(Y, X)$ for all $X, Y \in TM$ (tangent bundle) and at each point p of M if $g(X_p, Y_p) = 0$ for all $Y_p \in T_p(M)$; then $X_p = 0$ (nondegenerate), where $T_p(M)$ is the tangent space of M at the point p and $g : T_p(M) \times T_p(M) \rightarrow \mathbb{R}$ [17].

Definition 2. A pseudo-Riemannian manifold (or semi-Riemannian manifold) is a smooth manifold M furnished with a metric tensor g. That is, a pseudo-Riemannian manifold is an ordered pair (M, g) [17].

Definition 3. One will recall the notion of a proper curve of order n in n-dimensional pseudo-Riemannian manifold M with the metric tensor g. Let $\alpha : I \rightarrow M$ be a nonnull curve in M parametrized by the arc length s, where I is an open interval of the real line $\mathbb{R}$. One denotes the tangent vector field of α by V_1. One assumes that α satisfies the following Frenet formula:

$$\nabla_{V_1} V_1 = k_1 V_2,$$
$$\nabla_{V_1} V_i = -\varepsilon_{i-2}\varepsilon_{i-1} k_{i-1} V_{i-1} + k_i V_{i+1}, \quad 1 < i < n, \quad (2)$$
$$\nabla_{V_1} V_n = -\varepsilon_{n-2}\varepsilon_{n-1} k_{n-1} V_{n-1},$$

where

$$k_1 = \left\| \nabla_{V_1} V_1 \right\| > 0,$$
$$k_i = \left\| \nabla_{V_1} V_i + \varepsilon_{i-2}\varepsilon_{i-1} k_{i-1} V_{i-1} \right\| > 0, \quad 2 \le i \le n - 1, \quad (3)$$
$$\varepsilon_{j-1} = g\left(V_j, V_j\right) (= \pm 1), \quad 1 \le j \le n, \text{ on } I,$$

and ∇ is Levi-Civita connection of M.

One calls such a curve a proper curve of order n, k_i ($1 \le i \le n - 1$) its ith curvature, and $V_1, \dots, V_n$ its Frenet Frame field.

Moreover, let one recall that $\|X\| = \sqrt{|g(X, X)|}$ for $X \in TM$, where TM is the tangent bundle of M [18].

3. V_n-Slant Helices and Their Harmonic Curvature Functions

In this section, we give definition of a V_n-slant helix curve in a n-dimensional pseudo-Riemannian manifold. Furthermore, we give characterizations by using harmonic curvatures for V_n-slant helices.

Definition 4. Let M be a n-dimensional pseudo-Riemannian manifold and let $\alpha(s)$ be a proper curve of order n (nonnull) with the curvatures k_i ($i = 1, \dots, n-1$) in M. Then, harmonic curvature functions of α are defined by

$$H_i^* : I \subset \mathbb{R} \longrightarrow \mathbb{R} \quad (4)$$

along α in M, where

$$H_0^* = 0,$$
$$H_1^* = \varepsilon_{n-3}\varepsilon_{n-2} \frac{k_{n-1}}{k_{n-2}},$$
$$H_i^* = \left(k_{n-i} H_{i-2}^* - \nabla_{V_1} H_{i-1}^*\right) \frac{\varepsilon_{n-(i+2)}\varepsilon_{n-(i+1)}}{k_{n-(i+1)}}, \quad 2 \le i \le n - 2. \quad (5)$$

Note that $\nabla_{V_1} H_{i-1}^* = V_1(H_{i-1}^*) = H_{i-1}^{*\prime}$.

Definition 5. Let (M, g) be a n-dimensional pseudo-Riemannian manifold and let $\alpha(s)$ be a proper curve of order n (nonnull). One calls α as a V_n-slant helix in M if the function

$$g\left(V_n, X\right) \quad (6)$$

is a nonzero constant along α and X is a parallel vector field along α in M; that is, $\nabla_{V_1} X = 0$. Here, V_n is nth Frenet frame field and $X \in TM$. Also, X is called the axis of α.

Lemma 6. *Let (M, g) be a n-dimensional pseudo-Riemannian manifold and let $\alpha(s)$ be a proper curve of order n (nonnull). Let one assume that $H_{n-2}^* \ne 0$ for $i = n - 2$. Then, $\varepsilon_{n-3} H_1^{*2} + \varepsilon_{n-4} H_2^{*2} + \cdots + \varepsilon_0 H_{n-2}^{*2}$ is nonzero constant if and only if $V_1(H_{n-2}^*) = H_{n-2}^{*\prime} = k_1 H_{n-3}^*$, where V_1 and $\{H_1^*, \dots, H_{n-2}^*\}$ are the unit tangent vector field and the harmonic curvatures of α, respectively.*

Proof. First, we assume that $\varepsilon_{n-3} H_1^{*2} + \varepsilon_{n-4} H_2^{*2} + \cdots + \varepsilon_0 H_{n-2}^{*2}$ is nonzero constant. Consider the following functions given in Definition 4:

$$H_i^* = \left(k_{n-i} H_{i-2}^* - H_{i-1}^{*\prime}\right) \frac{\varepsilon_{n-(i+2)}\varepsilon_{n-(i+1)}}{k_{n-(i+1)}} \quad (7)$$

for $3 \le i \le n - 2$. So, from the equality, we can write

$$k_{n-(i+1)} H_i^* = \varepsilon_{n-(i+2)}\varepsilon_{n-(i+1)} \left(k_{n-i} H_{i-2}^* - H_{i-1}^{*\prime}\right). \quad (8)$$

Hence, in (8), if we take $i + 1$ instead of i, we get

$$\varepsilon_{n-(i+3)}\varepsilon_{n-(i+2)} H_i^{*\prime} = \varepsilon_{n-(i+3)}\varepsilon_{n-(i+2)} k_{n-(i+1)} H_{i-1}^*$$
$$- k_{n-(i+2)} H_{i+1}^*, \quad 2 \le i \le n - 3 \quad (9)$$

together with

$$H_1^{*\prime} = -\frac{1}{\varepsilon_{n-4}\varepsilon_{n-3}}k_{n-3}H_2^* \tag{10}$$

or

$$H_1^{*\prime} = -\varepsilon_{n-4}\varepsilon_{n-3}k_{n-3}H_2^*. \tag{11}$$

On the other hand, since $\varepsilon_{n-3}H_1^{*2} + \varepsilon_{n-4}H_2^{*2} + \cdots + \varepsilon_0 H_{n-2}^{*2}$ is constant, we have

$$\varepsilon_{n-3}H_1^*H_1^{*\prime} + \varepsilon_{n-4}H_2^*H_2^{*\prime} + \cdots + \varepsilon_0 H_{n-2}^*H_{n-2}^{*\prime} = 0 \tag{12}$$

and so

$$\begin{aligned}\varepsilon_0 H_{n-2}^*H_{n-2}^{*\prime} &= -\varepsilon_{n-3}H_1^*H_1^{*\prime} \\ &\quad - \varepsilon_{n-4}H_2^*H_2^{*\prime} - \cdots - \varepsilon_1 H_{n-3}^*H_{n-3}^{*\prime}.\end{aligned} \tag{13}$$

By using (9) and (11), we obtain

$$H_1^*H_1^{*\prime} = -\varepsilon_{n-4}\varepsilon_{n-3}k_{n-3}H_1^*H_2^*, \tag{14}$$

$$\begin{aligned}\varepsilon_{n-(i+3)}\varepsilon_{n-(i+2)}H_i^*H_i^{*\prime} &= \varepsilon_{n-(i+3)}\varepsilon_{n-(i+2)}k_{n-(i+1)}H_{i-1}^*H_i^* \\ &\quad - k_{n-(i+2)}H_i^*H_{i+1}^*, \quad 2 \le i \le n-3.\end{aligned} \tag{15}$$

Therefore, by using (13), (14), and (15), an algebraic calculus shows that

$$\varepsilon_0 H_{n-2}^*H_{n-2}^{*\prime} = \varepsilon_0 k_1 H_{n-3}^*H_{n-2}^* \tag{16}$$

or

$$H_{n-2}^*H_{n-2}^{*\prime} = k_1 H_{n-3}^*H_{n-2}^*. \tag{17}$$

Since $H_{n-2}^* \ne 0$, we get the relation

$$H_{n-2}^{*\prime} = k_1 H_{n-3}^*. \tag{18}$$

Conversely, we assume that

$$H_{n-2}^{*\prime} = k_1 H_{n-3}^*. \tag{19}$$

By using (19) and $H_{n-2}^* \ne 0$, we can write

$$H_{n-2}^*H_{n-2}^{*\prime} = k_1 H_{n-2}^*H_{n-3}^*. \tag{20}$$

From (15), we have the following equation system:

for $i = n-3$, $\quad \varepsilon_1 H_{n-3}^*H_{n-3}^{*\prime} = \varepsilon_1 k_2 H_{n-4}^*H_{n-3}^*$
$$- \varepsilon_0 k_1 H_{n-3}^*H_{n-2}^*,$$

for $i = n-4$, $\quad \varepsilon_2 H_{n-4}^*H_{n-4}^{*\prime} = \varepsilon_2 k_3 H_{n-5}^*H_{n-4}^*$
$$- \varepsilon_1 k_2 H_{n-4}^*H_{n-3}^*,$$

for $i = n-5$, $\quad \varepsilon_3 H_{n-5}^*H_{n-5}^{*\prime} = \varepsilon_3 k_4 H_{n-6}^*H_{n-5}^*$
$$- \varepsilon_2 k_3 H_{n-5}^*H_{n-4}^*,$$

$$\vdots$$

for $i = 2$, $\quad \varepsilon_{n-4}H_2^*H_2^{*\prime} = \varepsilon_{n-4}k_{n-3}H_1^*H_2^* - \varepsilon_{n-5}k_{n-4}H_2^*H_3^*. \tag{21}$

Moreover, from (14) and (20), we obtain

$$\begin{aligned}\varepsilon_{n-3}H_1^*H_1^{*\prime} &= -\varepsilon_{n-4}k_{n-3}H_1^*H_2^*, \\ \varepsilon_0 H_{n-2}^*H_{n-2}^{*\prime} &= \varepsilon_0 k_1 H_{n-2}^*H_{n-3}^*.\end{aligned} \tag{22}$$

So, by using the above equation system and considering (22), an algebraic calculus shows that

$$\varepsilon_{n-3}H_1^*H_1^{*\prime} + \varepsilon_{n-4}H_2^*H_2^{*\prime} + \cdots + \varepsilon_0 H_{n-2}^*H_{n-2}^{*\prime} = 0. \tag{23}$$

And, by integrating (23), we can easily say that

$$\varepsilon_{n-3}H_1^{*2} + \varepsilon_{n-4}H_2^{*2} + \cdots + \varepsilon_0 H_{n-2}^{*2} \tag{24}$$

is a nonzero constant. This completes the proof. □

Proposition 7. *Let (M,g) be a n-dimensional pseudo-Riemannian manifold and let $\alpha(s)$ be a proper curve of order n (nonnull). If α is V_n-slant helix in M, then we have*

$$g(V_{n-(i+1)}, X) = H_i^* g(V_n, X), \quad i = 0, 1, \ldots, n-2, \tag{25}$$

where X is the axis of α. Here, $\{V_1, V_2, \ldots, V_n\}$ denote the Frenet frame of α and $\{H_1^*, H_2^*, \ldots, H_{n-2}^*\}$ denote the harmonic curvature functions of the curve α.

Proof. We will use the induction method. Let $i = 1$. Since X is the axis of the V_n-slant helix α, we get

$$X = \lambda_1 V_1 + \cdots + \lambda_n V_n. \tag{26}$$

From the definition of V_n-slant helix, we have

$$g(V_n, X) = \lambda_n \varepsilon_{n-1} = \text{constant}. \tag{27}$$

A differentiation in (27) and the Frenet formulas gives us

$$g(V_{n-1}, X) = 0. \tag{28}$$

Again, differentiation in (28) and the Frenet formulas give

$$\begin{aligned}g(\nabla_{V_1}V_{n-1}, X) &= 0, \\ -\varepsilon_{n-3}\varepsilon_{n-2}k_{n-2}g(V_{n-2}, X) + k_{n-1}g(V_n, X) &= 0, \\ g(V_{n-2}, X) &= \varepsilon_{n-3}\varepsilon_{n-2}\frac{k_{n-1}}{k_{n-2}}g(V_n, X), \\ g(V_{n-2}, X) &= H_1^* g(V_n, X),\end{aligned} \tag{29}$$

respectively. Hence, it is shown that (25) is true for $i = 1$.

We now assume that (25) is true for the first $i - 1$. Then, we have

$$g(V_{n-i}, X) = H_{i-1}^* g(V_n, X). \tag{30}$$

A differentiation in (30) and the Frenet formulas give us that

$$\begin{aligned}&-\varepsilon_{n-i-2}\varepsilon_{n-i-1}k_{n-i-1}g(V_{n-i-1}, X) + k_{n-i}g(V_{n-i+1}, X) \\ &\quad = \nabla_{V_1}H_{i-1}^* g(V_n, X).\end{aligned} \tag{31}$$

Since we have the induction hypothesis, $g(V_{n-i+1}, X) = H_{i-2}^* g(V_n, X)$, we get

$$\left(k_{n-i}H_{i-2}^* - \nabla_{V_1}H_{i-1}^*\right) \frac{\varepsilon_{n-(i+2)}\varepsilon_{n-(i+1)}}{k_{n-(i+1)}} g(V_n, X) \tag{32}$$

$$= g(V_{n-(i+1)}, X),$$

which gives

$$g(V_{n-(i+1)}, X) = H_i^* g(V_n, X). \tag{33}$$

$\square$

Theorem 8. *Let (M, g) be a n-dimensional pseudo-Riemannian manifold and let $\alpha(s)$ be a proper curve of order n (nonnull). Then, α is a V_n-slant helix in M if and only if it satisfies that*

$$\sum_{i=1}^{n-2}\varepsilon_{n-(i+2)}H_i^{*2} \tag{34}$$

is equal to nonzero constant and $H_{n-2}^* \neq 0$.

Proof. Suppose α to be a V_n-slant helix. According to Definition 5 and the proof of Proposition 7,

$$g(V_n, X) = \lambda_n \varepsilon_{n-1} = \text{constant}, \tag{35}$$

where X is the axis of α. From Proposition 7, we have

$$g(V_{n-(i+1)}, X) = H_i^* g(V_n, X) \tag{36}$$

for $1 \le i \le n-2$. Moreover, from (35) and Frenet formulas, we can write

$$-\varepsilon_{n-2}\varepsilon_{n-1}k_{n-1}g(V_{n-1}, X) = 0. \tag{37}$$

Since $-\varepsilon_{n-2}\varepsilon_{n-1}k_{n-1}$ is different from zero, $g(V_{n-1}, X) = 0$. It is known that the system $\{V_1, \ldots, V_n\}$ is a basis of $\varkappa(M)$ (tangent bundle) along α. Hence, X can be expressed in the form

$$X = \sum_{i=1}^{n}\lambda_i V_i. \tag{38}$$

Moreover, from (38), we get the system

$$\varepsilon_0 \lambda_1 = g(X, V_1),$$

$$\varepsilon_1 \lambda_2 = g(X, V_2),$$

$$\vdots \tag{39}$$

$$\varepsilon_{n-3}\lambda_{n-2} = g(X, V_{n-2}),$$

$$\varepsilon_{n-2}\lambda_{n-1} = g(X, V_{n-1}) = 0,$$

$$\varepsilon_{n-1}\lambda_n = g(X, V_n)$$

by using the metric g. Therefore, from Proposition 7 and the above system, we can see that the following system is true:

$$\lambda_1 = g(X, V_1) = \varepsilon_0 H_{n-2}^* g(X, V_n),$$

$$\lambda_2 = g(X, V_2) = \varepsilon_1 H_{n-3}^* g(X, V_n),$$

$$\vdots \tag{40}$$

$$\lambda_{n-2} = g(X, V_{n-2}) = \varepsilon_{n-3}H_1^* g(X, V_n),$$

$$\lambda_{n-1} = g(X, V_{n-1}) = 0,$$

$$\lambda_n = \varepsilon_{n-1}g(X, V_n).$$

Thus, the axis of the curve α can be easily obtained as

$$X = g(X, V_n)\left\{\sum_{i=1}^{n-2}H_i^* V_{n-(i+1)}\varepsilon_{n-(i+2)} + (\varepsilon_{n-1}V_n)\right\} \tag{41}$$

by making use of equality (38) and the last system.

Therefore, from (41), we can write

$$g(X, X) = [g(X, V_n)]^2 \left(\varepsilon_0^3 H_{n-2}^{*2} + \cdots + \varepsilon_{n-3}^3 H_1^{*2} + \varepsilon_{n-1}^3\right). \tag{42}$$

Moreover, by the definition of metric tensor, we have

$$|g(X, X)| = \|X\|^2. \tag{43}$$

Since α is a V_n-slant helix, $\|X\|$ = constant and $g(X, V_n)$ is nonzero constant along α. Hence, from (42), we obtain that

$$\varepsilon_0^3 H_{n-2}^{*2} + \cdots + \varepsilon_{n-3}^3 H_1^{*2} + \varepsilon_{n-1}^3 \tag{44}$$

is constant. In other words,

$$\varepsilon_0 H_{n-2}^{*2} + \cdots + \varepsilon_{n-3}H_1^{*2} = \sum_{i=1}^{n-2}\varepsilon_{n-(i+2)}H_i^{*2} \tag{45}$$

is constant.

Now, we will show that $H_{n-2}^* \neq 0$. We assume that $H_{n-2}^* = 0$. Then, for $i = n-2$ in (36), we have

$$g(V_1, X) = H_{n-2}^* g(X, V_n) = 0. \tag{46}$$

If we take derivative in each part of (46) in the direction V_1 on M, then we have

$$g\left(\nabla_{V_1}V_1, X\right) + g\left(V_1, \nabla_{V_1}X\right) = 0. \tag{47}$$

On the other hand, $\nabla_{V_1}X = 0$ since α is a V_n-slant helix. Then, from (47), we have

$$g\left(\nabla_{V_1}V_1, X\right) = k_1 g(V_2, X) = 0 \tag{48}$$

by using the Frenet formulas. Since k_1 is positive, it must be $g(V_2, X) = 0$. Now, for $i = n-3$ in (36),

$$g(V_2, X) = H_{n-3}^* g(V_n, X). \tag{49}$$

Since $g(V_2, X) = 0$ and $g(V_n, X) \neq 0$, it must be $H_{n-3}^* = 0$. Continuing this process, we get $H_1^* = 0$. Let us recall that $H_1^* = \varepsilon_{n-3}\varepsilon_{n-2}(k_{n-1}/k_{n-2})$; thus we have a contradiction because all the curvatures are nowhere zero. Consequently, $H_{n-2}^* \neq 0$.

Conversely, we assume that $\sum_{i=1}^{n-2} \varepsilon_{n-(i+2)} H_i^{*2} = $ constant and $H_{n-2}^* \neq 0$. We take the vector field

$$X = \lambda_n V_n + \sum_{i=1}^{n-2} \lambda_n \varepsilon_{n-1} \varepsilon_{n-(i+2)} H_i^* V_{n-(i+1)} \tag{50}$$

or

$$X = \lambda_n V_n + \lambda_n \varepsilon_{n-1} \sum_{i=3}^{n} \varepsilon_{n-i} H_{i-2}^* V_{n-(i-1)}, \tag{51}$$

where λ_n is constant. We will show that it is parallel along α; that is, $\nabla_{V_1} X = 0$. By direct calculation, we have

$$\nabla_{V_1} X = \nabla_{V_1} (\lambda_n V_n) + \lambda_n \varepsilon_{n-1} \sum_{i=3}^{n} \varepsilon_{n-i} \nabla_{V_1} (H_{i-2}^* V_{n-(i-1)})$$

$$= \lambda_n \nabla_{V_1} V_n$$

$$+ \lambda_n \varepsilon_{n-1} \sum_{i=3}^{n} \varepsilon_{n-i} \left[H_{i-2}^{*\prime} V_{n-(i-1)} + H_{i-2}^* \nabla_{V_1} (V_{n-(i-1)}) \right]$$

$$= \lambda_n \varepsilon_{n-1} \left[-\varepsilon_{n-2} k_{n-1} V_{n-1} \right.$$

$$+ \left(\sum_{i=3}^{n-1} \varepsilon_{n-i} H_{i-2}^{*\prime} V_{n-(i-1)} \right.$$

$$- \varepsilon_{n-(i+1)} k_{n-i} V_{n-i} H_{i-2}^*$$

$$\left. + k_{n-(i-1)} V_{n-(i-2)} H_{i-2}^* \varepsilon_{n-i} \right)$$

$$\left. + \varepsilon_0 H_{n-2}^{*\prime} V_1 + \varepsilon_0 k_1 H_{n-2}^* V_2 \right]. \tag{52}$$

Here, in the case $n = 3$, we omit the term of sum.

On the other hand, by using (9), we can write

$$\varepsilon_{n-(i+1)} \varepsilon_{n-i} H_{i-2}^{*\prime} = \varepsilon_{n-(i+1)} \varepsilon_{n-i} k_{n-(i-1)} H_{i-3}^* - k_{n-i} H_{i-1}^* \tag{53}$$

for $4 \leq i \leq n-1$ together with (11). Moreover, from Lemma 6, we know that

$$H_{n-2}^{*\prime} = k_1 H_{n-3}^*. \tag{54}$$

Therefore, by using (11), (53), and (54) and by the definition of H_1^*, algebraic calculus shows that $\nabla_{V_1} X = 0$. Besides, $g(V_n, X) = \lambda_n \varepsilon_{n-1}$ is constant. Consequently, α is a V_n-slant helix in M. $\square$

Corollary 9. Let (M, g) be a n-dimensional pseudo-Riemannian manifold and let $\alpha(s)$ be a proper curve of order n (non-null). Then, α is a V_n-slant helix in M if and only if $H_{n-2}^{*\prime} = k_1 H_{n-3}^*$ and $H_{n-2}^* \neq 0$, where $\{H_1^*, H_2^*, \ldots, H_{n-2}^*\}$ denote the harmonic curvature functions of the curve α.

Proof. It is obvious by using Lemma 6. and Theorem 8. $\square$

4. Conclusion

In this work, V_n-slant helix is defined and new characterizations are given about the helix by using the harmonic curvature functions in pseudo-Riemannian manifolds. Links with other disciplines of the curve can be examined since the curve is important for other branches of science such as molecular biology and mathematical physics. On the other hand, the curve can be analyzed in different spaces of mathematics.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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