Research Article

Long-Range Dependence in a Cox Process Directed by a Markov Renewal Process

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A Cox process N_{Cox} directed by a stationary random measure ξ has second moment var $N_{\text{Cox}}(0,t] = \text{E}(\xi(0,t]) + \text{var}\,\xi(0,t]$, where by stationarity $\text{E}(\xi(0,t]) = (\text{const.})t = \text{E}(N_{\text{Cox}}(0,t])$, so long-range dependence (LRD) properties of N_{Cox} coincide with LRD properties of the random measure ξ. When $\xi(A) = \int_A \nu_{J(u)}\,\mathrm{d}u$ is determined by a density that depends on rate parameters ν_i ($i \in \mathbb{X}$) and the current state $J(\cdot)$ of an $\mathbb{X}$-valued stationary irreducible Markov renewal process (MRP) for some countable state space $\mathbb{X}$ (so $J(t)$ is a stationary semi-Markov process on $\mathbb{X}$), the random measure is LRD if and only if each (and then by irreducibility, every) generic return time Y_{jj} ($j \in \mathbb{X}$) of the process for entries to state j has infinite second moment, for which a necessary and sufficient condition when $\mathbb{X}$ is finite is that at least one generic holding time X_j in state j, with distribution function (DF) H_j, say, has infinite second moment (a simple example shows that this condition is not necessary when $\mathbb{X}$ is countably infinite). Then, N_{Cox} has the same Hurst index as the MRP N_{MRP} that counts the jumps of $J(\cdot)$, while as $t \to \infty$, for finite $\mathbb{X}$, var $N_{\text{MRP}}(0,t] \sim 2\lambda^2 \int_0^t \mathcal{G}(u)\,\mathrm{d}u$, var $N_{\text{Cox}}(0,t] \sim 2\int_0^t \sum_{i \in \mathbb{X}} (\nu_i - \bar{\nu})^2 \varpi_i \mathcal{H}_i(t)\,\mathrm{d}u$, where $\bar{\nu} = \sum_i \varpi_i \nu_i = \text{E}[\xi(0,1]]$, $\varpi_j = \text{Pr}\{J(t) = j\}$, $1/\lambda = \sum_j \check{p}_j \mu_j$, $\mu_j = \text{E}(X_j)$, $\{\check{p}_j\}$ is the stationary distribution for the embedded jump process of the MRP, $\mathcal{H}_j(t) = \mu_i^{-1}\int_0^\infty \min(u,t)[1 - H_j(u)]\,\mathrm{d}u$, and $\mathcal{G}(t) \sim \int_0^t \min(u,t)[1 - G_{jj}(u)]\,\mathrm{d}u/m_{jj} \sim \sum_i \varpi_i \mathcal{H}_i(t)$ where G_{jj} is the DF and m_{jj} the mean of the generic return time Y_{jj} of the MRP between successive entries to the state j. These two variances are of similar order for $t \to \infty$ only when each $\mathcal{H}_i(t)/\mathcal{G}(t)$ converges to some $[0,\infty]$-valued constant, say, γ_i, for $t \to \infty$.

1. Introduction

This paper is a sequel to Daley [1] which arose from wanting to decide whether the detailed long-range dependent (LRD) behavior of a Cox process N_{Cox} directed by the ON phases of a stationary ON/OFF alternating renewal process N is the same as the LRD behavior of N. It was shown that both processes have the same Hurst index but that the ratio $\operatorname{var} N_{\text{Cox}}(0,t]/\operatorname{var} N(0,t]$ need not converge for $t \to \infty$.

Here, we examine the nature of these two variance functions for the case of a Cox process whose instantaneous rate ν_i is determined by the state $i \in \mathbb{X}$, with $\mathbb{X}$ being countable (sometimes it must be finite), of a LRD stationary Markov renewal process (MRP), of which our earlier example of an alternating renewal process (ARP) is the simplest. MRPs have long been an interest of Jeff Hunter (e.g., Hunter [2]), and it is a pleasure to contribute this paper to a volume that marks his contributions to the academic community both inside New Zealand and further afield where D. Daley in particular has enjoyed his company many times since first meeting him in Chapel Hill, NC, and T. Rolski at Cornell University.

In Section 2, we introduce the necessary notation and recall known results that are relevant to the problem at hand. Section 3 develops formulae for univariate and bivariate marginal probabilities for MRPs that take us into the realm of Markov renewal equations which enable us to address the questions we raise when $\mathbb{X}$ is finite. We conclude in Section 4 with remarks on the case where $\mathbb{X}$ is countably infinite. In the appendix, we prove an asymptotic convergence result due originally, we believe, to Sgibnev [3].

2. The setting and known results

A Cox process N_{Cox} driven by the random measure ξ is a point process which, conditional on the realization ξ, is a Poisson process with parameter measure ξ (e.g., Daley and Vere-Jones [4, Section 6.2]). Then, when N_{Cox} and ξ are located in the half-line $\mathbb{R}_+$, for Borel subsets A of $\mathbb{R}_+$,

$$\operatorname{E}[N_{\text{Cox}}(A)] = \operatorname{E}[\xi(A)], \qquad \operatorname{var} N_{\text{Cox}}(A) = \operatorname{E}[\xi(A)] + \operatorname{var} \xi(A) \tag{2.1}$$

[4, Proposition 6.2.II]. A stationary point process or random measure ξ on $\mathbb{R}$ is LRD when

$$\limsup_{t \to \infty} \frac{\operatorname{var} \xi(0,t]}{t} = \infty \tag{2.2}$$

[4, Section 12.7], and its Hurst index H is defined by

$$H = \inf \left\{ h : \limsup_{t \to \infty} \frac{\operatorname{var} \xi(0,t]}{t^{2h}} < \infty \right\}. \tag{2.3}$$

It follows from (2.1) and (2.2) that a Cox process is LRD if and only if the random measure driving it is LRD, and that they both have the same Hurst index (this is Daley [1, Proposition 1]).

We choose to describe a Markov renewal process (see, e.g., Çinlar [5] or Kulkarni [6] for a textbook account) both in terms of the sequence $\{(X_n, J_n)\}$ of successive intervals

X_n between jumps of a Markov chain $\{J_n\}$ on a countable state space $\mathbb{X}$ with one-step transition probabilities $(p_{ij}, \ i,j \in \mathbb{X})$, and the $\mathbb{X}$-valued semi-Markov process $\{J(t) : t \in \mathbb{R}\}$ which can be related via the time epochs $T_n = T_0 + X_1 + \cdots + X_n$ subsequent to some initial epoch T_0, as $J_{n+1} = J(T_n+)$ and

$$
\begin{aligned}
J(t) = J_n \quad & (T_{n-1} \le t < T_n, \ n = 1,2,\dots) \\
= & \sum_{n=1}^{\infty} J_n I_{T_{n-1} \le t < T_n}(t).
\end{aligned}
\tag{2.4}
$$

We use the random measure

$$
\xi(A) \equiv \int_A \nu_{J(u)} \mathrm{d}u,
\tag{2.5}
$$

where $\{\nu_i\}$ is a family of nonnegative constants defined over $\mathbb{X}$, as the driving measure of the Cox process N_{Cox} that we consider. This means that if $\sigma_i(0,t]$ is the (Lebesgue) measure of that part of the interval $(0,t]$ during which $J(u) = i$ for $i \in \mathbb{X}$ (mnemonically, the sojourn time in i during $(0,t]$), then

$$
\xi(0,t] = \sum_{i \in \mathbb{X}} \nu_i \sigma_i(0,t] \quad (t \in \mathbb{R}_+),
\tag{2.6}
$$

and N_{Cox} consists of points evolving as a Poisson process at rate ν_i on the disjoint sets of support of σ_i for $i \in \mathbb{X}$. Equation (2.1) shows that in order to evaluate the variance of the Cox process, we must find

$$
\mathrm{var}\,\xi(0,t] = \sum_{i,j \in \mathbb{X}} \nu_i \nu_j \,\mathrm{cov}\left(\sigma_i(0,t], \sigma_j(0,t]\right).
\tag{2.7}
$$

When $\mathbb{X}$ is a finite set, the finiteness conditions we impose are automatically satisfied, but for the sake of completeness, we allow the countably infinite case of $\mathbb{X}$ except where we know of proof only in the finite case (see (2.20) and Section 4). For N_{Cox} to be well defined, we want $\xi(0,t] < \infty$ a.s. for finite $t > 0$, which is the case when $\overline{\nu} \equiv \sum_{i \in \mathbb{X}} \nu_i \varpi_i < \infty$, where for stationary $J(\cdot)$, we set

$$
\varpi_i = \mathrm{Pr}\{J(t) = i\} = \mathrm{E}[\sigma_i(0,1]] \quad \text{(all } t\text{)}.
\tag{2.8}
$$

Then, $\mathrm{E}[\xi(0,t]] = \overline{\nu} t$ for all $t > 0$. Assuming (as we must for the conditions of stationarity to hold) that the chain $\{J_n\}$ is irreducible and has a stationary distribution $\{\check{p}_i\}$ (so $\check{p}_j = \sum_{i \in \mathbb{X}} \check{p}_i p_{ij}$), this is related to the distribution $\{\varpi_i\}$ via the mean holding times $\mu_i = \int_0^{\infty} \overline{H}_i(u)\mathrm{d}u = \mathrm{E}(X_n \mid J_n = i)$ as at (2.9). When $F_{ij}(t) = \mathrm{Pr}\{X_n \le t \mid J_n = i, \ J_{n+1} = j\}$, the process of termination of sojourns in state i is governed by the (in general) dishonest DFs $Q_{ij}(t) = p_{ij} F_{ij}(t)$ but such that the holding time DFs $H_i(t) = \sum_j Q_{ij}(t)$ are honest. We make the simplifying assumption that $p_{ii} = 0$ (all i).

Assume that the point process defined by such an MRP (i.e., the sequence of epochs $\{T_n\}$) can and does exist in a stationary state; in which case, its intensity λ is given by $\lambda^{-1} = \sum_{i \in \mathbb{X}} \check{p}_i \mu_i$, and

$$
\varpi_i = \lambda \check{p}_i \mu_i = \mathrm{Pr}\{J(t) = i\} \quad \text{(all } t\text{)},
\tag{2.9}
$$

with the semi-Markov process $J(\cdot)$ here being stationary also. Since the rate of entry epochs into state i equals $\lambda \check{p}_i$, it follows that the mean time m_{ii} between successive entries into state i is given by

$$m_{ii} = \frac{1}{\lambda \check{p}_i} = \frac{\mu_i}{\varpi_i} \quad (i \in \mathbb{X}). \tag{2.10}$$

We assume that our MRP is irreducible (i.e., the Markov chain $\{J_n\}$ is irreducible), and therefore it can be studied via first passage distributions $G_{ij}(\cdot)$ (it is here that the assumption $p_{ii} = 0$ simplifies the discussion); define for every $i \in \mathbb{X}$ and $j \in \mathbb{X}$ except $j = i$

$$G_{ji}(t) = \Pr\{\text{entry to } i \text{ occurs in } (0,t] \mid \text{state } j \neq i \text{ entered at } 0\},$$
$$G_{ii}(t) = \Pr\{\text{second entry to } i \text{ occurs in } (0,t] \mid \text{state } i \text{ entered at } 0\}. \tag{2.11}$$

Then, for example,

$$G_{ii}(t) = \sum_{k \in \mathbb{X} \setminus \{i\}} p_{ik} \int_0^t F_{ik}(du) G_{ki}(t-u) = \sum_k (Q_{ik} * G_{ki})(t), \tag{2.12}$$

where our convention in writing the convolution $(A * B)(t)$ of a nonnegative function $B(\cdot)$ (like G_{ki}) with respect to a measure $A(\cdot)$ (like Q_{ik}) is that $(A * B)(t) = \int_0^t A(du)B(t-u)$, or in vector algebra notation when $\mathbf{A} = (A_{ij}(\cdot))$ and $\mathbf{B} = (B_{ij}(\cdot))$ are compatible,

$$((\mathbf{A} * \mathbf{B})(t))_{ij} = \sum_{k \in \mathbb{X}} \int_0^t A_{ik}(du)B_{kj}(t-u). \tag{2.13}$$

When we consider only the point process N_{MRP} of epochs where entrances into states occur, for which we should count the number of entries N_i into state i and therefore have $N_{\text{MRP}} = \sum_{i \in \mathbb{X}} N_i$, Sgibnev [7] has shown (under the condition of irreducibility) that there is a solidarity result; it implies that $m_{ii}^{-2} \operatorname{var} N_i(0,t] \sim m_{11}^{-2} \operatorname{var} N_1(0,t]$ as $t \to \infty$ when the number of visits to any one state has LRD behavior (and hence, that the point process of visits to any other given state is LRD also, and moreover the asymptotic behavior of the variance function $m_{ii}^{-2} \operatorname{var} N_i(0,t]$ is the same irrespective of the state i). Given this solidarity property, it is seemingly extraordinary that the variance of the amount of time spent in the various states need not have the same asymptotic behavior. The major aim now in considering a Cox process directed by a stationary MRP is to show that this asymptotic behavior is determined, as in the ARP case, by a linear mixture of integrals of certain functions that are crucial in Sgibnev [3, 7] (see also Appendix A), namely,

$$\mathcal{H}_i(t) \equiv \frac{1}{\mu_i} \int_0^\infty \min(u,t) \overline{H}_i(u) du. \tag{2.14}$$

We also write $\tilde{H}_i(t) = (1/\mu_i) \int_0^t \overline{H}_i(u) du$; this equals $\mathcal{H}_i'(t)$. Write $\mathcal{H}(t)$ and $\tilde{\mathbf{H}}(t)$ for vectors with components $\mathcal{H}_i(t)$ and $\tilde{H}_i(t)$, respectively.

Recall (2.7) writing alternatively

$$V(t) \equiv \mathrm{var}\, \xi(0,t] = \sum_{i,j \in \mathbb{X}} 2\nu_i \nu_j \int_0^t (t-u) \left[\varpi_{ij}(u) - \varpi_i \varpi_j \right] \mathrm{d}u, \qquad (2.15)$$

where $\varpi_{ij}(u) = \mathrm{Pr}\{J(0+) = i, J(u) = j\}$ for $u > 0$ (stationarity of $J(\cdot)$ is assumed as around (2.9)). In terms of the distribution of $J(\cdot)$, only the uni- and bivariate distributions $\{\varpi_i\}$ and $\varpi_{ij}(u)$ are involved in (2.15), and LRD behavior is therefore associated with the integral of $\varpi_{ij}(u) - \varpi_i \varpi_j$ over large intervals. Since these bivariate probabilities are those of a semi-Markov process, each $\varpi_{ij}(\cdot)$ has a representation as a convolution involving DFs of lifetimes on the state space $\mathbb{X}$, and this leads to renewal function representations and use of asymptotics of renewal functions as we shall demonstrate.

Write $U_{ij}(t) = \mathrm{E}(N_j[0,t] \mid \text{state } i \text{ entered at } 0) = \delta_{ij} + \sum_{k \in \mathbb{X}} \int_0^t Q_{ik}(\mathrm{d}u) U_{kj}(t-u)$. Then, $U_i(x) = \sum_{j \in \mathbb{X}} U_{ij}(x)$ satisfies the same backwards equation with δ_{ij} and U_{kj} replaced by 1 and U_k. Writing $\mathbf{Q} = (Q_{ij})_{i,j \in \mathbb{X}}$, define

$$\begin{aligned} \mathbf{U} = \left(U_{ij}(t) \right)_{i,j \in \mathbb{X}} &= \mathbf{I} + \mathbf{Q} + \mathbf{Q}^{2*} + \cdots \\ &= \mathbf{I} + \mathbf{Q} * \mathbf{U} = \mathbf{I} + \mathbf{U} * \mathbf{Q} \end{aligned} \qquad (2.16)$$

(note that $U_{ij}(t) = \mathrm{E}[N_j(0,t] \mid i \text{ entered at } 0]$ for $j \neq i$, while for $j = i$, since the N_j are orderly, $N_i[0,t] = 1 + N_i(0,t]$). Since $U_i = \mathbf{U}_i^T \mathbf{e}$, where $\mathbf{U}_i$ is the vector over $j \in \mathbb{X}$ of U_{ij} (all vectors are column vectors unless transposed as, e.g., $\mathbf{U}_i^T$),

$$U_{\mathrm{MRP}}(t) = \check{\mathbf{p}}^T \mathbf{U} \mathbf{e} = \mathrm{E}\left(N_{\mathrm{MRP}}[0,t] \mid \text{jump at } 0 \text{ of stationary } J(\cdot) \right). \qquad (2.17)$$

Now, substitute in the standard formula (e.g., Daley and Vere-Jones [4, page 62]) to give $\mathrm{var}\, N_{\mathrm{MRP}}(0,t]$ for the stationary point process generated by the jumps of a stationary MRP:

$$\mathrm{var}\, N_{\mathrm{MRP}}(0,t] = \lambda \int_0^t \left(2[U_{\mathrm{MRP}}(u) - \lambda u] - 1 \right) \mathrm{d}u, \qquad (2.18)$$

where in terms of the respective vectors $\check{\mathbf{p}}$ and $\boldsymbol{\mu}$ of the stationary jump distribution $\{\check{p}_i\}$ and mean sojourn times $\{\mu_i\}$ of the states $i \in \mathbb{X}$, $1/\lambda = \check{\mathbf{p}}^T \boldsymbol{\mu} = \sum_{i \in \mathbb{X}} \check{p}_i \mu_i$ as around (2.9). The integrand at (2.18) has uniformly bounded increments because $U_{\mathrm{MRP}}(t) \sim \lambda t$ ($t \to \infty$) and it is subadditive (see Appendix B), like the renewal function (e.g., Daley and Vere-Jones [4, Exercise 4.4.5(b)]).

Let G_{kk} be the return time DF for some given state $k \in \mathbb{X}$. Sgibnev [7] showed that for $t \to \infty$ and all other $i, j \in \mathbb{X}$ for the stationary irreducible LRD MRP,

$$m_{jj} U_{ij}(t) - t \sim \mathcal{G}(t) = \frac{1}{m_{kk}} \int_0^\infty \min(t,u) \overline{G}_{kk}(u) \mathrm{d}u. \qquad (2.19)$$

Then from (2.17), at least for a finite state space $\mathbb{X}$, it follows that

$$U_{\mathrm{MRP}}(t) - \lambda t = \sum_{i \in \mathbb{X}} \sum_{j \in \mathbb{X}} \check{p}_i \left(U_{ij}(t) - \lambda \check{p}_j t \right) \sim \sum_{i \in \mathbb{X}} \sum_{j \in \mathbb{X}} \check{p}_i \cdot \lambda \check{p}_j \cdot \mathcal{G}(t) = \lambda \mathcal{G}(t), \qquad (2.20)$$

and hence that

$$\operatorname{var} N_{\mathrm{MRP}}(t) \sim 2\lambda^2 \int_0^t \mathcal{G}(u)\mathrm{d}u. \tag{2.21}$$

Whether (2.21) holds for countably infinite state space remains a question for another place; the discussion in Section 4 is relevant to the nature of the return time distribution G_{kk} in (2.19).

3. Recurrence relations for bivariate probabilities and asymptotics

In this section, we establish the result that extends the simpler conclusion of Daley [1] from an alternating renewal process to a Markov renewal process on a finite state space $\mathbb{X}$. So far, we do not know the nature of any extension to the case that $\mathbb{X}$ is countably infinite.

THEOREM 3.1. *Let the Cox process N_{Cox} be driven by a long-range dependent random measure $\xi(A) = \int_A \nu_{J(u)}\mathrm{d}u$ determined by a stationary semi-Markov process $J(\cdot)$ on a finite state space $\mathbb{X}$. Then, N_{Cox} has the same Hurst index as the Markov renewal process N_{MRP} underlying $J(\cdot)$. Both $\operatorname{var} N_{\mathrm{Cox}}(0,x]$ and $\operatorname{var} N_{\mathrm{MRP}}(0,x]$ are asymptotically determined by the holding time DFs $\{H_i(\cdot) : i \in \mathbb{X}\}$ in the MRP, at least one of which must have infinite second moment. Under these conditions, for $t \to \infty$,*

$$\operatorname{var} N_{\mathrm{Cox}}(0,t] \sim 2\int_0^t \sum_{i\in\mathbb{X}} (\nu_i - \bar{\nu})^2 \varpi_i \mathcal{H}_i(u)\mathrm{d}u, \tag{3.1}$$

while $\operatorname{var} N_{\mathrm{MRP}}(0,t]$ is given by (2.21) in which

$$\mathcal{G}(u) \sim \sum_{i\in\mathbb{X}} \varpi_i \mathcal{H}_i(u) \quad (u \longrightarrow \infty), \tag{3.2}$$

where $\{\varpi_i\}$ is the stationary distribution for $J(\cdot)$ and the truncated second moment functions $\mathcal{H}_i(\cdot)$ and $\mathcal{G}(\cdot)$ are given by (2.14) and (2.19).

In general, $\operatorname{var} N_{\mathrm{MRP}}(0,t] \sim \lambda^2 \int_0^t \mathcal{G}(u)\mathrm{d}u \not\propto (\mathrm{const.})\operatorname{var} N_{\mathrm{Cox}}(0,t]$, but if for some j, all the ratios $\mathcal{H}_i(t)/\mathcal{H}_j(t)$ $(i \in \mathbb{X}\setminus\{j\})$ converge as $t \to \infty$ to limits in $[0,\infty]$, then

$$\operatorname{var} N_{\mathrm{MRP}}(0,t] \sim (\mathrm{const.})\operatorname{var} N_{\mathrm{Cox}}(0,t] \quad (t \longrightarrow \infty). \tag{3.3}$$

Proof. If all holding time DFs H_i have finite second moments, then because $\mathbb{X}$ is finite, so do all return time DFs G_{kk}, and the MRP cannot be LRD.

The last part of the theorem, given (3.1)–(3.2), is proved in the same way as the analogous statement for the alternating renewal case, so for the rest, we concentrate on demonstrating (3.1)–(3.2).

We develop expressions involving the bivariate probabilities $\varpi_{ij}(t)$ (see around (2.15)) for the stationary irreducible semi-Markov process $J(\cdot)$. The variance function $V(t) = \operatorname{var}\xi(0,t]$ at (2.15) describes the variance of the Cox process via (2.1). Equation (2.15) shows that $V(\cdot)$ is differentiable, with derivative

$$V'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{var}\xi(0,t] = \sum_{i,j\in\mathbb{X}} 2\nu_i\nu_j \int_0^t [\varpi_{ij}(u) - \varpi_i\varpi_j]\mathrm{d}u, \tag{3.4}$$

which is already simpler to evaluate than (2.15) itself. In particular, when $\xi(\cdot)$ is LRD, $V(t)$ is larger than $O(t)$ for $t \to \infty$, so that when $V'(t) \sim g(t) \to \infty$ as $t \to \infty$ for some $g(\cdot)$ that is ultimately monotone, the asymptotic behavior of $V(t)$ for large t is the same as for $\int_0^t g(u)\mathrm{d}u$.

For a stationary irreducible semi-Markov process $J(\cdot)$ on $\mathbb{X}$ as we are considering, the joint distribution on $\mathbb{X} \times \mathbb{X} \times \mathbb{R}_+$ of the current state i, the state k next entered, and the forward recurrence time x for that next entry, is determined by the density function

$$\frac{\varpi_i \overline{Q}_{ik}(x)\mathrm{d}x}{\mu_i} \qquad (i \in \mathbb{X}, \; k \in \mathbb{X} \setminus \{i\}, \; 0 < x < \infty). \tag{3.5}$$

In (3.13), we use $\widetilde{\mathbf{Q}}(t)$ to denote the array with elements $(1/\mu_i)\int_0^t \overline{Q}_{ij}(u)\mathrm{d}u$. Note that the vector $\widetilde{\mathbf{H}}(t)$ as below (2.14) satisfies $\widetilde{\mathbf{H}}(t) = \widetilde{\mathbf{Q}}(t)\mathbf{e}$.

Define

$$\Pi_{j|i}(t) = \int_0^t \mathrm{E}\big(\delta_{j,J(u)} \mid J(0+) = i\big)\mathrm{d}u, \tag{3.6}$$

so that

$$\mathrm{E}\left[\delta_{i,J(0)}\int_0^t \delta_{j,J(u)}\mathrm{d}u\right] = \varpi_i \Pi_{j|i}(t) = \int_0^t \varpi_{ij}(u)\mathrm{d}u. \tag{3.7}$$

Setting $\mathbf{\Pi}(t) = (\Pi_{j|i}(t))_{i,j\in\mathbb{X}}$, it follows that (3.4) is expressible as

$$V'(t) = \sum_{i,j\in\mathbb{X}} 2\nu_i\nu_j\big[\varpi_i(\Pi_{j|i}(t) - \varpi_j t)\big] = 2\nu^T \operatorname{diag}(\boldsymbol{\varpi})\big(\mathbf{\Pi}(t) - \mathbf{e}\boldsymbol{\varpi}^T t\big)\nu. \tag{3.8}$$

We now develop expressions for $\Pi_{j|i}$ in terms of the truncated second moment functions at (2.14) and the related functions, discussed in Lemma 3.3,

$$M_{ij}(t) = \mathrm{E}\left[\int_0^t \delta_{j,J(u)}\mathrm{d}u \;\Big|\; \text{state } i \text{ entered at } 0\right]. \tag{3.9}$$

Lemma 3.2

$$\Pi_{j|i}(t) = \delta_{ji}\mathcal{H}_i(t) + \sum_{k\in\mathbb{X}}\int_0^t \frac{\overline{Q}_{ik}(v)}{\mu_i}M_{kj}(t-v)\mathrm{d}v, \tag{3.10a}$$

equivalently, with $\mathbf{M}(t) = (M_{ij}(t))_{i,j\in\mathbb{X}}$,

$$\mathbf{\Pi}(t) = \operatorname{diag}\big(\mathcal{H}(t)\big) + \big(\widetilde{\mathbf{Q}} * \mathbf{M}\big)(t). \tag{3.10b}$$

Proof. For $j \neq i$, we use the joint distribution at (3.5) and a backwards decomposition to write

$$\Pi_{j|i}(t) = \sum_{k\in\mathbb{X}}\int_0^t \frac{\overline{Q}_{ik}(x)}{\mu_i}\mathrm{d}x\, M_{kj}(t-x), \tag{3.11}$$

which is (3.10a) for $j \neq i$. For $j = i$,

$$\Pi_{i|i}(t) = \sum_{k \in \mathbb{X}} t \int_t^\infty \frac{\overline{Q}_{ik}(x)}{\mu_i}\,dx + \sum_{k \in \mathbb{X}} \int_0^t \frac{\overline{Q}_{ik}(x)}{\mu_i}[x + M_{ki}(t - x)]\,dx. \tag{3.12}$$

Grouping terms according to whether they involve any $M_{ki}(\cdot)$ or not leads to (3.10). □

LEMMA 3.3 (Recurrence relations for $M_{ij}(\cdot)$)

$$M_{ij}(t) = \delta_{ji} \int_0^t \overline{H}_i(u)\,du + \sum_{k \in \mathbb{X}} \int_0^t Q_{ik}(du) M_{kj}(t - u) \quad (i, j \in \mathbb{X}), \tag{3.13}$$

hence $\mathbf{M}(t) = \mathrm{diag}(\tilde{\mathbf{H}}(t))\,\mathrm{diag}(\boldsymbol{\mu}) + (\mathbf{Q} * \mathbf{M})(t)$, so

$$M_{ij}(t) = (\mathbf{U} * \mathrm{diag}(\tilde{\mathbf{H}})\,\mathrm{diag}(\boldsymbol{\mu})(t))_{ij} = \int_0^t \overline{H}_j(u) U_{ij}(t - u)\,du. \tag{3.14}$$

Proof. Equation (3.13) is established by a standard backwards decomposition. The equation is written more usefully in the form of a generalized Markov renewal equation as shown, from which the rest of the lemma follows. □

In the second term of (3.10b), substituting for $\mathbf{M}$ from (3.14) yields

$$(\tilde{\mathbf{Q}} * \mathbf{M})(t) = (\tilde{\mathbf{Q}} * \mathbf{U} * \mathrm{diag}(\tilde{\mathbf{H}}))(t). \tag{3.15}$$

Since $U_{ij}(t) \leq U_{jj}(t)$ for all $t > 0$ and all $i, j \in \mathbb{X}$, a dominated convergence argument involving $U_{ij}(t - u)/U_{jj}(t)$ in (3.14) implies that $\lim_{t \to \infty} M_{ij}(t)/U_{jj}(t) = \int_0^\infty \overline{H}_j(u)\,du = \mu_j$, and since $U_{jj}(t) \sim \lambda \check{p}_j t$ for $t \to \infty$, this implies, with (2.9), that

$$M_{ij}(t) \sim (\lambda \check{p}_j \mu_j) t = \varpi_j t \quad (\text{all } i). \tag{3.16}$$

The same arguments applied to (3.10a) show that $\Pi_{j|i}(t) \sim \varpi_j t$ for every i so that every element of $\boldsymbol{\Pi}(t) - \mathbf{e}\bar{\varpi}^T t$ in (3.8) is at most $o(t)$ for $t \to \infty$. We now find the exact asymptotics of these elements.

The components of $(\tilde{\mathbf{Q}} * \mathbf{M})(t)$ in (3.10b) can be written as

$$\begin{aligned}((\tilde{\mathbf{Q}} * \mathbf{M})(t))_{ij} &= \sum_{k \in \mathbb{X}} \int_0^t \frac{\overline{Q}_{ik}(u)}{\mu_i} M_{kj}(t - u)\,du \\ &= \sum_{k \in \mathbb{X}} \int_0^t \frac{\overline{Q}_{ik}(u)}{\mu_i}(M_{kj}(t - u) - \varpi_j(t - u))\,du + \sum_{k \in \mathbb{X}} \int_0^t \frac{\overline{Q}_{ik}(u)}{\mu_i} \varpi_j(t - u)\,du.\end{aligned} \tag{3.17}$$

The last term equals

$$\varpi_j \int_0^t \frac{\overline{H}_i(u)}{\mu_i}(t - u)\,du = \varpi_j \int_0^\infty \frac{\overline{H}_i(u)}{\mu_i}(t - u)_+\,du. \tag{3.18}$$

Consequently, from (3.10a), $\Pi_{j|i}(t) - \varpi_j t$ equals

$$\delta_{ji}\mathcal{H}_i(t) + \sum_{k\in\mathbb{X}}\int_0^t \frac{\overline{Q}_{ik}(u)}{\mu_i}\left(M_{kj}(t-u) - \varpi_j(t-u)\right)\mathrm{d}u - \varpi_j\int_0^\infty \frac{\overline{H}_i(u)}{\mu_i}\left(t - (t-u)_+\right)\mathrm{d}u,$$

(3.19)

and the last term equals $\varpi_j\mathcal{H}_i(t)$; so finally

$$\Pi_{j|i}(t) - \varpi_j t = \left(\delta_{ji} - \varpi_j\right)\mathcal{H}_i(t) + \sum_{k\in\mathbb{X}}\int_0^t \frac{\overline{Q}_{ik}(u)}{\mu_i}\left(M_{kj}(t-u) - \varpi_j(t-u)\right)\mathrm{d}u. \qquad (3.20)$$

In vector algebra notation, writing $L(t) = t_+$, this reads

$$\mathbf{\Pi}(t) - \mathbf{e}\boldsymbol{\varpi}^T t = \mathrm{diag}\left(\boldsymbol{\mathcal{H}}(t)\right) - \boldsymbol{\mathcal{H}}(t)\boldsymbol{\varpi}^T + \left(\tilde{\mathbf{Q}} * \left(\mathbf{M} - \mathbf{e}\boldsymbol{\varpi}^T L\right)\right)(t). \qquad (3.21)$$

This is not quite of the form we want; the first two terms on the right-hand side are expressed in terms of the truncated second moments of the sojourn time DFs H_i as at (2.14); it remains to consider the last term. Start by using the expression below (3.13) in writing

$$
\begin{aligned}
\left(\left(\mathbf{M} - \mathbf{e}\boldsymbol{\varpi}^T L\right)(t)\right)_{ij} & \\
&= \left(\left(\mathbf{U} * \mathrm{diag}(\tilde{\mathbf{H}})\,\mathrm{diag}(\boldsymbol{\mu})\right)(t)\right)_{ij} - \varpi_j t \\
&= \int_0^t U_{ij}(\mathrm{d}u)\int_0^{t-u}\overline{H}_j(v)\mathrm{d}v - \varpi_j t \\
&= \int_0^t U_{ij}(u)\overline{H}_j(t-u)\mathrm{d}u - \varpi_j t \\
&= \int_0^t\left[U_{ij}(u) - \frac{u}{m_{jj}}\right]\overline{H}_j(t-u)\mathrm{d}u + \frac{1}{m_{jj}}\int_0^t(t-v)\overline{H}_j(v)\mathrm{d}v - \varpi_j t \\
&= \int_0^t\left[U_{ij}(u) - \lambda\check{p}_j u\right]\overline{H}_j(t-u)\mathrm{d}u + \frac{\varpi_j}{\mu_j}\int_0^\infty(t-v)_+\overline{H}_j(v)\mathrm{d}v - \varpi_j t\int_0^\infty\frac{\overline{H}_j(v)}{\mu_j}\mathrm{d}v \\
&= \int_0^t\left[U_{ij}(u) - \lambda\check{p}_j u\right]\overline{H}_j(t-u)\mathrm{d}u - \varpi_j\mathcal{H}_j(t).
\end{aligned}
$$

(3.22)

By (2.19), the integral here $\sim \mu_j\mathcal{G}(t)/m_{jj} = \varpi_j\mathcal{G}(t)$, so

$$\left(\mathbf{M} - \mathbf{e}\boldsymbol{\varpi}^T L\right)(t) \sim \mathbf{e}\boldsymbol{\varpi}^T\mathcal{G}(t) - \mathbf{e}\boldsymbol{\varpi}^T\,\mathrm{diag}\left(\boldsymbol{\mathcal{H}}(t)\right). \qquad (3.23)$$

But in (3.21), $\tilde{\mathbf{Q}}$ is a stochastic kernel, so the last term there has this same asymptotic behavior and

$$\mathbf{\Pi}(t) - \mathbf{e}\boldsymbol{\varpi}^T t \sim \mathrm{diag}\left(\boldsymbol{\mathcal{H}}(t)\right) - \boldsymbol{\mathcal{H}}(t)\boldsymbol{\varpi}^T + \mathbf{e}\boldsymbol{\varpi}^T\left(\mathbf{I}\mathcal{G}(t) - \mathrm{diag}\left(\boldsymbol{\mathcal{H}}(t)\right)\right), \qquad (3.24)$$

at least in the case of a finite state space $\mathbb{X}$. Finally then (cf. (3.8)),

$$
\begin{aligned}
V'(t) &\sim 2 \sum_{i \in \mathbb{X}} \sum_{j \in \mathbb{X}} v_i v_j \varpi_i \left(\delta_{ji} \mathcal{H}_i(t) + \varpi_j \left[- \mathcal{H}_i(t) + \mathcal{G}(t) - \mathcal{H}_j(t) \right] \right) \\
&= 2 \left(\sum_{i \in \mathbb{X}} v_i^2 \varpi_i \mathcal{H}_i(t) - 2 \overline{v} \sum_i v_i \varpi_i \mathcal{H}_i(t) + (\overline{v})^2 \mathcal{G}(t) \right) \\
&= 2 \sum_{i \in \mathbb{X}} (v_i - \overline{v})^2 \varpi_i \mathcal{H}_i(t) + 2(\overline{v})^2 \left(\mathcal{G}(t) - \sum_{i \in \mathbb{X}} \varpi_i \mathcal{H}_i(t) \right) \quad (t \to \infty).
\end{aligned}
\tag{3.25}
$$

This establishes (3.1) except for showing that the coefficient of $(\overline{v})^2$ vanishes asymptotically, that is, (3.2) holds.

Recall (see above (2.16)) the function $U_i(x) = E(N_{\mathrm{MRP}}[0,x] \mid \text{state } i \text{ entered at } 0)$. Just as the functions $M_{ij}(\cdot)$ satisfy generalized Markov renewal equations (see Lemma 3.3), so too do the functions $U_i(x) - \lambda x$. Using a backwards decomposition, we have

$$
U_i(x) = 1 + \sum_{k \in \mathbb{X}} \int_0^x Q_{ik}(du) U_k(x - u),
\tag{3.26}
$$

and therefore

$$
\begin{aligned}
U_i(x) - \lambda x &= 1 - \lambda x + \sum_{k \in \mathbb{X}} \int_0^x Q_{ik}(du) \left[U_k(x - u) - \lambda(x - u) \right] + \lambda \int_0^x (x - u) H_i(du) \\
&= \sum_{k \in \mathbb{X}} \int_0^x Q_{ik}(du) \left(U_k(x - u) - \lambda(x - u) \right) + 1 - \lambda x + \lambda \int_0^x H_i(u) du \\
&= 1 - \lambda \mu_i + \lambda \int_x^\infty \overline{H}_i(v) dv + \sum_{k \in \mathbb{X}} \int_0^x Q_{ik}(du) \left(U_k(x - u) - \lambda(x - u) \right).
\end{aligned}
\tag{3.27}
$$

Write $\mathbf{Z}(x)$ and $\mathbf{z}(x)$ for the vectors with respective components $U_i(x) - \lambda x$ and $1 - \lambda \mu_i + \lambda \int_x^\infty \overline{H}_i(v) dv$ $(i \in \mathbb{X})$. Then, $\mathbf{Z} = \mathbf{z} + \mathbf{Q} * \mathbf{Z}$ is a generalized Markov renewal equation, and therefore it has solution (under the condition that it is unique, which is the case when $\mathbb{X}$ is finite) $\mathbf{Z}(x) = (\mathbf{U} * \mathbf{z})(x)$. In terms of the components, this gives

$$
\begin{aligned}
U_i(x) - \lambda x &= \sum_{j \in \mathbb{X}} \int_0^x U_{ij}(du) \left[1 - \lambda \mu_j + \lambda \int_{x-u}^\infty \overline{H}_j(v) dv \right] \\
&= U_i(x) - \lambda x - \sum_{j \in \mathbb{X}} \lambda \mu_j \left[U_{ij}(x) - \lambda \check{p}_j x \right] + \lambda \sum_{j \in \mathbb{X}} \int_0^x U_{ij}(du) \int_{x-u}^\infty \overline{H}_j(v) dv,
\end{aligned}
\tag{3.28}
$$

that is,

$$
\sum_{j \in \mathbb{X}} \frac{\lambda \mu_j}{m_{jj}} \left(m_{jj} U_{ij}(x) - x \right) = \lambda \sum_{j \in \mathbb{X}} \int_0^x U_{ij}(du) \int_{x-u}^\infty \overline{H}_j(v) dv.
\tag{3.29}
$$

Now, our MRP is LRD, so by (2.10) and Sgibnev's [7] solidarity result quoted at (2.19), the left-hand side here $\sim \sum_j \lambda \varpi_j \mathcal{G}(x) = \lambda \mathcal{G}(x)$. For the right-hand side, we can apply

the asymptotic convergence lemma in Sgibnev [3] (see Appendix A), because $U_{ij}(x) \sim x/m_{jj} = \lambda \check{p}_j x$ $(x \to \infty)$, to deduce that the right-hand side of (3.29) $\sim \lambda^2 \sum_{j \in \mathbb{X}} \check{p}_j \mu_j \mathscr{H}_j(x) = \lambda \sum_{j \in \mathbb{X}} \varpi_j \mathscr{H}_j(x)$; so (3.2) holds.   □

In the setting in Daley [1] for the case of an alternating renewal process, we should have in our general notation above that $\nu_1 = 1$ for the ON state, 1, say, and $\nu_0 = 0$ for the OFF state, $\bar{\nu} = \varpi = \varpi_1 = \mathrm{E}(X_1)/\mathrm{E}(Y)$, where $Y = X_1 + X_0$ is a generic cycle time, $\check{p}_0 = \check{p}_1 = 1/2$, and $\varpi_0 = 1 - \varpi$. An ARP can be studied via cycle times (with generic duration Y), with return time distribution $G(x) = \Pr\{Y \le x\}$ for which $\mathscr{G}(\cdot)$ emerges naturally for (2.19) and (3.2). The right-hand side of (3.1) equals $(1 - \varpi)^2 \varpi \mathscr{H}_1(t) + \varpi^2(1 - \varpi)\mathscr{H}_0(t)$, so our theorem above is consistent with Daley [1].

4. Discussion

Our proof of the asymptotic relation at (2.21) for the behavior of $\mathrm{var}\, N_{\mathrm{MRP}}(0,t]$ when the MRP is LRD depends on Sgibnev's [7] solidarity result and, lacking any uniform convergence result over the state space $\mathbb{X}$, it is confined to the case that $\mathbb{X}$ is finite. Whether or not a relation like (2.21) persists in the countable case is not clear. We indicate one difficulty.

Consider a realization of our MRP. Let a "tour" consist of the successive states $\{j_n\}$ visited on a path starting from $j_0 = k$ until a first return to k, consisting of say, N_{tour} transitions in all, so $j_{N_{\mathrm{tour}}} = k$ and $j_n \ne k$ for $n = 1, \ldots, N_{\mathrm{tour}} - 1$; for such a path, represent the first return time Y_{kk}, with DF G_{kk}, and in self-evident notation, as

$$Y_{kk} = \sum_{\{j_n\}} X_{j_n, j_{n+1}}^{\mathrm{tour}} = \sum_{n=0}^{N_{\mathrm{tour}}-1} X_{j_n, j_{n+1}}^{\mathrm{tour}}. \tag{4.1}$$

Then, Y_{kk} has infinite second moment if and only if either (or both) of some X_{ij} and N_{tour} has infinite second moment. For a Markov chain in discrete time, only the latter is possible (because whenever $p_{ij} > 0$, $X_{ij} = 1$ a.s.). Trivially, a Markov chain in discrete time is also a Markov renewal process, and thus, in a LRD MRP with all holding times being equal to 1, say, a relation like (3.2) would be impossible because the left-hand side would be infinite but the right-hand side would be finite.

Appendices

A. An asymptotic convergence lemma

The result given below is the essence of Sgibnev [3, Theorem 4], used to establish the asymptotic behavior of the difference between a renewal function $U(t)$ and its asymptote λt when a generic lifetime r.v. has infinite second moment. Sgibnev's proof assumes that $U(\cdot)$ is a renewal function, but this is not needed in our proof below.

LEMMA A.1. *Let the nonnegative function $z(x)$ $(x > 0)$ be monotonic decreasing and such that $L(t) \equiv \int_0^t z(u)\mathrm{d}u \to \infty$ for $t \to \infty$. Let the monotonic increasing nonnegative function $U(t)$ have uniformly bounded increments $U(x+1) - U(x) \le K < \infty$ (all $x > 0$) and let it be*

asymptotically linear, so that $U(t) \sim \lambda t$ $(t \to \infty)$ for some finite positive λ. Then,

$$L_1(t) \equiv (U * z)(t) \equiv \int_0^t z(t-u)U(du) \sim \lambda L(t) \quad (t \longrightarrow \infty). \tag{A.1}$$

Proof. Given $\epsilon > 0$, the asymptotic linearity of $U(\cdot)$ implies that there exists finite positive t_ϵ such that

$$|U(t) - \lambda t| \le \epsilon t \quad (\text{all } t \ge t_\epsilon). \tag{A.2}$$

Write

$$
\begin{aligned}
L_1(t) &= \int_0^t \left[z(t) + \int_{t-u}^t |dz(v)| \right] U(du) \\
&= [U(t) - U(0+)]z(t) + \int_0^t [U(t) - U(t-v)]|dz(v)| \\
&= [U(t) - U(0+)]z(t) + \left(\int_0^{t_\epsilon} + \int_{t_\epsilon}^t \right) \int_0^t [U(t) - U(t-v)]|dz(v)| \\
&= [U(t) - U(0+)]z(t) + A_\epsilon(t) + \int_{t_\epsilon}^t [U(t) - U(t-v)]|dz(v)|,
\end{aligned}
\tag{A.3}
$$

where $0 < A_\epsilon(t) \le [z(0) - z(t_\epsilon)]Kt_\epsilon$, uniformly in t. Then,

$$L_1(t) - A_\epsilon(t) = U(t)z(t_\epsilon) - U(0+)z(t) - \int_{t_\epsilon}^t U(t-v)|dz(v)|. \tag{A.4}$$

For $t > 2t_\epsilon$, this integral equals $(\int_{t_\epsilon}^{t-t_\epsilon} + \int_{t-t_\epsilon}^t)U(t-v)|dz(v)|$, in which the latter integral, $B_\epsilon(t)$, say, satisfies

$$0 \le B_\epsilon(t) \equiv \int_{t-t_\epsilon}^t U(t-v)|dz(v)| \le U(t_\epsilon)z(t-t_\epsilon) \le (\lambda + \epsilon)t_\epsilon z(t - t_\epsilon), \tag{A.5}$$

which for a given ϵ is uniformly bounded, independently of t. The integral that remains equals $\int_{t_\epsilon}^{t-t_\epsilon} U(t-v)|dz(v)|$ which by (A.1) is bounded above and below by

$$
\begin{aligned}
(\lambda \pm \epsilon) \int_{t_\epsilon}^{t-t_\epsilon} (t-v)|dz(v)| &= (\lambda \pm \epsilon) \int_{t_\epsilon}^{t-t_\epsilon} |dz(v)| \int_0^{t-v} dw \\
&= (\lambda \pm \epsilon) \int_0^{t-t_\epsilon} dw \int_{t_\epsilon}^{\min(t-w,t-t_\epsilon)} |dz(v)| \\
&= (\lambda \pm \epsilon) \int_0^{t-t_\epsilon} [z(t_\epsilon) - z(\min(t-w,t-t_\epsilon))]dw \\
&= (\lambda \pm \epsilon) \int_{t_\epsilon}^t [z(t_\epsilon) - z(\min(w,t-t_\epsilon))]dw.
\end{aligned}
\tag{A.6}
$$

Using the upper bound, we can therefore write

$$
L_1(t) - A_\epsilon(t) - B_\epsilon(t) + U(0+)z(t)
$$

$$
< [U(t) - (\lambda+\epsilon)(t-t_\epsilon)]z(t_\epsilon) + (\lambda+\epsilon)\int_{t_\epsilon}^t z(\min(w, t-t_\epsilon))\mathrm{d}w \tag{A.7}
$$

$$
\leq (\lambda+\epsilon)t_\epsilon z(t_\epsilon) + (\lambda+\epsilon)\int_{t_\epsilon}^t z(\min(w, t-t_\epsilon))\mathrm{d}w,
$$

in which the second inequality comes from (A.1) because $t > t_\epsilon$. Divide each extreme of this inequality by $L(t)$, and observe that in the limit $t \to \infty$, the only term on the left-hand side that does not vanish is $L_1(t)/L(t)$, while the right-hand side (after division) converges to $\lambda + \epsilon$. It then follows that, because ϵ is arbitrary, $\limsup_{t\to\infty} L_1(t)/L(t) \leq \lambda$.

Using the lower bound at (A.6) leads instead to

$$
L_1(t) - A_\epsilon(t) - B_\epsilon(t) + U(0+)z(t) \geq (\lambda-\epsilon)t_\epsilon z(t_\epsilon) + (\lambda-\epsilon)\int_{t_\epsilon}^t z(\min(w, t-t_\epsilon))\mathrm{d}w, \tag{A.8}
$$

and a similar argument as in using the upper bound gives $\liminf_{t\to\infty} L_1(t)/L(t) \geq \lambda$. $\square$

B. Subadditivity of the renewal function $U_{\mathrm{MRP}}(\cdot)$

LEMMA B.1. *The renewal function $U_{\mathrm{MRP}}(\cdot)$ defined on jump epochs of a stationary MRP is subadditive.*

Proof. For a stationary MRP on state space $\mathbb{X}$, recall the Palm expectations (see around (2.16) and (3.26) above)

$$
U_i(x) = \mathrm{E}(N_{\mathrm{MRP}}[0,x] \mid \text{state } i \text{ entered at } 0) = \sum_j U_{ij}(x), \tag{B.1}
$$

$$
U_{\mathrm{MRP}}(x) = \sum_i \check{p}_i U_i(x). \tag{B.2}
$$

For a stationary MRP, the stationary distribution $\{\check{p}_i\}$ for the embedded jump process $\{J_n\}$ satisfies both $\check{p}_j = \sum_{i\in\mathbb{X}} \check{p}_i p_{ij}$ and the equation for the state probability at the epoch of the first jump after any fixed time interval thereafter, that is, the semi-Markov process $J(t)$ satisfies $\check{p}_k = \Pr\{\text{first jump of } J(x+t) \text{ in } t > 0 \text{ is to } k \mid J(\cdot) \text{ has jump at } 0\}$, namely,

$$
\check{p}_k = \sum_i \check{p}_i \sum_j \int_0^x U_{ij}(\mathrm{d}u)\overline{H}_j(x-u)\int_0^\infty \frac{Q_{jk}(x-u+\mathrm{d}z)}{\overline{H}_j(x-u)} = \sum_i \check{p}_i \sum_j \int_0^x U_{ij}(\mathrm{d}u)\int_{x-u}^\infty Q_{jk}(\mathrm{d}z). \tag{B.3}
$$

Now,

$$U_i(x+y) - U_i(x) = \mathrm{E}\big(N(x,x+y) \mid \text{state } i \text{ entered at } 0\big)$$

$$= \sum_j \sum_k \int_0^x U_{ij}(\mathrm{d}u) \int_{x-u}^{x+y-u} Q_{jk}(\mathrm{d}z) U_k(x+y-u-z) \tag{B.4}$$

$$\leq \sum_j \sum_k \int_0^x U_{ij}(\mathrm{d}u) \int_{x-u}^{x+y-u} Q_{jk}(\mathrm{d}z) U_k(y),$$

because every $U_i(\cdot)$ is nondecreasing and every $Q_{jk}(\cdot)$ is a measure, and then, again because every $Q_{jk}(\cdot)$ is a measure and using (B.2), $U_{\mathrm{MRP}}(x+y) - U_{\mathrm{MRP}}(x)$ equals

$$\sum_i \check{p}_i\big[U_i(x+y) - U_i(x)\big] \leq \sum_i \check{p}_i \sum_j \sum_k \int_0^x U_{ij}(\mathrm{d}u) \int_{x-u}^{x+y-u} Q_{jk}(\mathrm{d}z) U_k(y)$$

$$\leq \sum_k U_k(y) \sum_i \check{p}_i \sum_j \int_0^x U_{ij}(\mathrm{d}u) \int_{x-u}^{\infty} Q_{jk}(\mathrm{d}z) \tag{B.5}$$

$$= \sum_k U_k(y)\check{p}_k = U_{\mathrm{MRP}}(y).$$

$\square$

Acknowledgments

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