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*Research Article*

# Parameter and State Estimator for State Space Models

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This paper proposes a parameter and state estimator for canonical state space systems from measured input-output data. The key is to solve the system state from the state equation and to substitute it into the output equation, eliminating the state variables, and the resulting equation contains only the system inputs and outputs, and to derive a least squares parameter identification algorithm. Furthermore, the system states are computed from the estimated parameters and the input-output data. Convergence analysis using the martingale convergence theorem indicates that the parameter estimates converge to their true values. Finally, an illustrative example is provided to show that the proposed algorithm is effective.

## 1. Introduction

Parameter estimation and identification have had important applications in system modelling, system control, and system analysis [1–5] and thus have received much research attention in recent decades [6–11]. Several identification methods have been developed for state space models, for example, the subspace identification methods [12]. Gibson and Ninness presented a robust maximum-likelihood estimation for fully parameterized linear time-invariant (LTI) state space models; the idea is to use the expectation maximization (EM) algorithm to estimate maximum-likelihood degrees [13]. Raghavan et al. studied the EM-based state space model identification problems with irregular output sampling [14].

The state space model includes not only the unknown parameter matrices/vectors, but also the unknown noise terms in the formation vector and unmeasurable state vector. Many algorithms can estimate the system states assuming that the system parameter matrices/vectors are available but such state estimation algorithm cannot work if the system parameters are unknown [15]. Recently, Ding presented a combined state and least squares parameter estimation algorithm for dynamic systems [16].

In the area of state space model identification, Ding and Chen proposed a hierarchical identification estimation algorithm for estimating the system parameters and states [17]. Li et al. assumed that the system states were available and used the measurable states and input-output data to estimate the parameters of lifted state space models for general dual-rate systems [18]. Recently, some identification methods have been developed, for example, the least squares methods [19, 20], the gradient-based methods [21, 22], the bias compensation methods [23, 24], and the maximum likelihood methods [25–30]. The objective of this paper is to present a new parameter and state estimation-based residual algorithm from the given input-output data and further to analyze the convergence of the proposed algorithm.

The convergence analysis of identification algorithms has always been one of the important projects in the field of control. By using the stochastic martingale theory, Ding et al. studied the properties of stochastic gradient identification algorithms under weak conditions [31]. Ding and Liu discussed the gradient-based identification approach and convergence for multivariable systems with output measurement noise [32]. Other identification methods for linear or nonlinear systems [33–42] include the auxiliary model identification methods [43–57], the hierarchical identification methods [58–73], and the two-stage or multistage identification methods [74–78].

This paper is organized as follows. Section 2 introduces the system description and its identification model paper. Section 3 derives a basic parameter identification algorithm for canonical state space systems and analyzes the performance of the proposed algorithm. Section 4 gives a state estimation algorithm. Section 5 provides an example for the proposed algorithm. Finally, concluding remarks are given in Section 6.

## 2. System Description and Identification Model

Let us introduce some notation [15]. "$A =: X$" or "$X := A$" stands for "$A$ is defined as $X$"; the symbol $\mathbf{I}(\mathbf{I}_n)$ stands for an identity matrix of appropriate size ($n \times n$); the superscript T denotes the matrix transpose; $|\mathbf{X}| = \det[\mathbf{X}]$ represents the determinant of a square matrix $\mathbf{X}$; the norm of a matrix $\mathbf{X}$ is defined by $\|\mathbf{X}\|^2 = \mathrm{tr}[\mathbf{X}\mathbf{X}^{\mathrm{T}}]$; $\mathbf{1}_n := \mathbf{1}_{n\times 1}$ represents an $n \times 1$ vector whose elements are all 1; $\lambda_{\min}[\mathbf{X}]$ represents the minimum eigenvalues of $\mathbf{X}$; for $g(t) \geqslant 0$, we write $f(t) = O(g(t))$ if there exists a positive constant $\delta_1$ such that $|f(t)| \leqslant \delta_1 g(t)$.

In order to study the convergence of the algorithm proposed in [15], here we simply give that algorithm in [15]. Consider a linear system described by the following observability canonical state space model [15]:

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \\ y(t) &= \mathbf{c}\mathbf{x}(t) + v(t), \end{aligned} \tag{1}$$

$$\begin{aligned} \mathbf{A} &:= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \in \mathbb{R}^{n\times n}, \\ \mathbf{b} &:= \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^{n}, \\ \mathbf{c} &:= [1, 0, 0, \dots, 0] \in \mathbb{R}^{1\times n}, \end{aligned} \tag{2}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the system input, $y(t) \in \mathbb{R}$ is the system output, and $v(t) \in \mathbb{R}$ is a random noise with zero mean. Assume that the order $n$ is known, and $u(t) = 0$, $y(t) = 0$ and $v(t) = 0$ for $t \leqslant 0$.

The system in (1) is an observability canonical form, and its observability matrix $\mathbf{Q}_o$ is an identity matrix; that is,

$$\mathbf{Q}_o := \begin{bmatrix} \mathbf{c} \\ \mathbf{c}\mathbf{A} \\ \vdots \\ \mathbf{c}\mathbf{A}^{n-1} \end{bmatrix} = \mathbf{I}_n. \tag{3}$$

For the system in (1), the objective of this paper is to develop a new algorithm to estimate the parameter matrix/vector $\mathbf{A}$ and $\mathbf{b}$ (i.e., the parameters $a_i$ and $b_i$) and the system state vector $\mathbf{x}(t)$ from the available measurement input-output data $\{u(t), y(t)\}$.

Since the available measurement input-output data $\{u(t), y(t)\}$ are known but the state vector $\mathbf{x}(t)$ is unknown, it is required to eliminate the state vector from (1) and obtain a new expression which only involves the input and output, in order to obtain the estimates of the parameters in (1). The following derives the identification model based on the method in [15].

Define some vectors/matrix,

$$\begin{aligned} \boldsymbol{\varphi}_y(t) &:= [y(t-n), y(t-n+1), \dots, y(t-1)]^{\mathrm{T}} \in \mathbb{R}^n, \\ \boldsymbol{\varphi}_u(t) &:= [u(t-n), u(t-n+1), \dots, u(t-1)]^{\mathrm{T}} \in \mathbb{R}^n, \\ \boldsymbol{\varphi}_v(t) &:= [v(t-n), v(t-n+1), \dots, v(t-1)]^{\mathrm{T}} \in \mathbb{R}^n, \\ \mathbf{M} &:= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mathbf{cb} & 0 & \cdots & 0 & 0 \\ \mathbf{cAb} & \mathbf{cb} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ \mathbf{c}\mathbf{A}^{n-2}\mathbf{b} & \mathbf{c}\mathbf{A}^{n-3}\mathbf{b} & \cdots & \mathbf{cb} & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}. \end{aligned} \tag{4}$$

From (1), we have

$$y(t) = \mathbf{c}\mathbf{x}(t) + v(t), \tag{5}$$

$$\begin{aligned} y(t+1) &= \mathbf{c}\mathbf{x}(t+1) + v(t+1) \\ &= \mathbf{c}[\mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)] + v(t+1) \\ &= \mathbf{cA}\mathbf{x}(t) + \mathbf{cb}u(t) + v(t+1), \end{aligned} \tag{6}$$

$$\begin{aligned} y(t+2) &= \mathbf{cA}\mathbf{x}(t+1) + \mathbf{cb}u(t+1) + v(t+2) \\ &= \mathbf{cA}[\mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)] + \mathbf{cb}u(t+1) + v(t+2) \\ &= \mathbf{c}\mathbf{A}^2\mathbf{x}(t) + \mathbf{cAb}u(t) + \mathbf{cb}u(t+1) + v(t+2), \\ &\vdots \end{aligned} \tag{7}$$

$$\begin{aligned} y(t+n-1) &= \mathbf{c}\mathbf{A}^{n-1}\mathbf{x}(t) + \mathbf{c}\mathbf{A}^{n-2}\mathbf{b}u(t) + \mathbf{c}\mathbf{A}^{n-3}\mathbf{b}u(t-1) \\ &\quad + \cdots + \mathbf{cb}u(t+n-2) + v(t+n-1), \end{aligned} \tag{8}$$

$$\begin{aligned} y(t+n) &= \mathbf{c}\mathbf{A}^{n}\mathbf{x}(t) + \mathbf{c}\mathbf{A}^{n-1}\mathbf{b}u(t) + \mathbf{c}\mathbf{A}^{n-2}\mathbf{b}u(t-1) \\ &\quad + \cdots + \mathbf{cb}u(t+n-1) + v(t+n). \end{aligned} \tag{9}$$

Combining (5) with (8) gives

$$\begin{aligned} \boldsymbol{\varphi}_y(t+n) &= \mathbf{Q}_o\mathbf{x}(t) + \mathbf{M}\boldsymbol{\varphi}_u(t+n) + \boldsymbol{\varphi}_v(t+n) \\ &= \mathbf{x}(t) + \mathbf{M}\boldsymbol{\varphi}_u(t+n) + \boldsymbol{\varphi}_v(t+n), \end{aligned} \tag{10}$$

or

$$\mathbf{x}(t) = \boldsymbol{\varphi}_y(t+n) - \mathbf{M}\boldsymbol{\varphi}_u(t+n) - \boldsymbol{\varphi}_v(t+n). \tag{11}$$

Define the parameter vector $\boldsymbol{\theta}$ and the information vector $\boldsymbol{\varphi}(t)$ as

$$\begin{gathered}\boldsymbol{\theta} := \begin{bmatrix}\boldsymbol{\theta}_a\\ \boldsymbol{\theta}_b\end{bmatrix} \in \mathbb{R}^{2n},\\ \boldsymbol{\theta}_a := \left[\mathbf{c}\mathbf{A}^n\right]^{\mathrm{T}} \in \mathbb{R}^n,\\ \boldsymbol{\theta}_b := \left[-\mathbf{c}\mathbf{A}^n\mathbf{M} + \left[\mathbf{c}\mathbf{A}^{n-1}\mathbf{b}, \mathbf{c}\mathbf{A}^{n-1}\mathbf{b}, \ldots, \mathbf{c}\mathbf{b}\right]\right]^{\mathrm{T}} \in \mathbb{R}^n,\\ \boldsymbol{\varphi}(t+n) := \left[\boldsymbol{\varphi}_y^{\mathrm{T}}(t+n) - \boldsymbol{\varphi}_v^{\mathrm{T}}(t+n), \boldsymbol{\varphi}_u^{\mathrm{T}}(t+n)\right]^{\mathrm{T}} \in \mathbb{R}^{2n}.\end{gathered} \tag{12}$$

Substituting (11) into (9) gives

$$\begin{aligned} y(t+n) &= \mathbf{c}\mathbf{A}^n\left[\boldsymbol{\varphi}_y(t+n) - \mathbf{M}\boldsymbol{\varphi}_u(t+n) - \boldsymbol{\varphi}_v(t+n)\right]\\ &\quad + \mathbf{c}\mathbf{A}^{n-1}\mathbf{b}u(t) + \mathbf{c}\mathbf{A}^{n-2}\mathbf{b}u(t-1)\\ &\quad + \cdots + \mathbf{c}\mathbf{b}u(t+n-1) + v(t+n)\\ &= \mathbf{c}\mathbf{A}^n\left[\boldsymbol{\varphi}_y(t+n) - \mathbf{M}\boldsymbol{\varphi}_u(t+n) - \boldsymbol{\varphi}_v(t+n)\right]\\ &\quad + \left[\mathbf{c}\mathbf{A}^{n-1}\mathbf{b}, \mathbf{c}\mathbf{A}^{n-2}\mathbf{b}, \ldots, \mathbf{c}\mathbf{b}\right]\begin{bmatrix}u(t)\\ u(t-1)\\ \vdots\\ u(t+n-1)\end{bmatrix} + v(t+n)\\ &= \mathbf{c}\mathbf{A}^n\left[\boldsymbol{\varphi}_y(t+n) - \boldsymbol{\varphi}_v(t+n)\right] - \mathbf{c}\mathbf{A}^n\mathbf{M}\boldsymbol{\varphi}_u(t+n)\\ &\quad + \left[\mathbf{c}\mathbf{A}^{n-1}\mathbf{b}, \mathbf{c}\mathbf{A}^{n-2}\mathbf{b}, \ldots, \mathbf{c}\mathbf{b}\right]\boldsymbol{\varphi}_u(t+n) + v(t+n)\\ &= \left[\boldsymbol{\varphi}_y^{\mathrm{T}}(t+n) - \boldsymbol{\varphi}_v^{\mathrm{T}}(t+n), \boldsymbol{\varphi}_u^{\mathrm{T}}(t+n)\right]\begin{bmatrix}\boldsymbol{\theta}_a\\ \boldsymbol{\theta}_b\end{bmatrix} + v(t+n)\\ &= \boldsymbol{\varphi}^{\mathrm{T}}(t+n)\boldsymbol{\theta} + v(t+n). \end{aligned} \tag{13}$$

Replacing $t$ in (13) with $t-n$ yields

$$y(t) = \boldsymbol{\varphi}^{\mathrm{T}}(t)\boldsymbol{\theta} + v(t), \tag{14}$$

which is called the identification model or identification expression of the state-space model.

## 3. The Parameter Estimation Algorithm and Its Convergence

The recursive least squares algorithm for estimating $\boldsymbol{\theta}$ is expressed as

$$\hat{\boldsymbol{\theta}}(t) = \hat{\boldsymbol{\theta}}(t-1) + \mathbf{P}(t)\hat{\boldsymbol{\varphi}}(t)\left[y(t) - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\hat{\boldsymbol{\theta}}(t-1)\right], \tag{15}$$

$$\mathbf{P}^{-1}(t) = \mathbf{P}^{-1}(t-1) + \hat{\boldsymbol{\varphi}}(t)\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t), \qquad \mathbf{P}(0) = p_0\mathbf{I}, \tag{16}$$

$$\hat{v}(t) = y(t) - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\hat{\boldsymbol{\theta}}(t), \tag{17}$$

$$\begin{aligned}\hat{\boldsymbol{\varphi}}(t) = [&y(t-n) - \hat{v}(t-n),\\ &y(t-n+1) - \hat{v}(t-n+1), \ldots,\\ &y(t-1) - \hat{v}(t-1), u(t-n),\\ &u(t-n+1), \ldots, u(t-1)]^{\mathrm{T}}.\end{aligned} \tag{18}$$

This algorithm is commonly used for convergence analysis. To avoid computing the matrix inversion, this algorithm is equivalently expressed as

$$\hat{\boldsymbol{\theta}}(t) = \hat{\boldsymbol{\theta}}(t-1) + \mathbf{L}(t)\left[y(t) - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\hat{\boldsymbol{\theta}}(t-1)\right], \tag{19}$$

$$\mathbf{L}(t) = \mathbf{P}(t)\hat{\boldsymbol{\varphi}}(t) = \frac{\mathbf{P}(t-1)\hat{\boldsymbol{\varphi}}(t)}{1 + \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\mathbf{P}(t-1)\hat{\boldsymbol{\varphi}}(t)}, \tag{20}$$

$$\mathbf{P}(t) = \left[\mathbf{I} - \mathbf{L}(t)\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\right]\mathbf{P}(t-1), \qquad \mathbf{P}(0) = p_0\mathbf{I}, \tag{21}$$

$$\hat{v}(t) = y(t) - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\hat{\boldsymbol{\theta}}(t), \tag{22}$$

$$\begin{aligned}\hat{\boldsymbol{\varphi}}(t) = \big[&\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t-n)\hat{\boldsymbol{\theta}}(t-n),\\ &\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t-n+1)\hat{\boldsymbol{\theta}}(t-n+1), \ldots, \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t-1)\hat{\boldsymbol{\theta}}(t-1),\\ &u(t-n), u(t-n+1), \ldots, u(t-1)\big]^{\mathrm{T}},\end{aligned} \tag{23}$$

where $\mathbf{L}(t) \in \mathbb{R}^{2n}$ is the gain vector.

Define the parameter estimation error vector $\tilde{\boldsymbol{\theta}}(t) := \hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}$ and the nonnegative function $T(t) := \tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\mathbf{P}^{-1}(t)\tilde{\boldsymbol{\theta}}(t)$.

**Theorem 1.** *For the system in* (1) *and algorithm in* (15)–(18), *assume that* $\{v(t), \mathscr{F}_t\}$ *is a martingale difference sequence defined on a probability space* $\{\Omega, \mathscr{F}, P\}$, *where* $\{\mathscr{F}_t\}$ *is the* $\sigma$ *algebra sequence generated by the observations up to and including time* $t$. *The noise sequence* $\{v(t)\}$ *satisfies the following assumptions:*

(A1) $\mathrm{E}[v(t) \mid \mathscr{F}_{t-1}] = 0$, *a.s.,*

(A2) $\mathrm{E}[v^2(t) \mid \mathscr{F}_{t-1}] \leqslant \sigma^2 < \infty$, *a.s.,*

(A3) $A'(z) := A^{-1}(z) - 1/2$ *is strictly positive real.*

*Then the following inequality holds:*

$$\begin{aligned}&\mathrm{E}\left[T(t) + S(t) \mid \mathscr{F}_{t-1}\right]\\ &\quad \leqslant T(t-1) + S(t-1) + 2\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\mathbf{P}(t)\hat{\boldsymbol{\varphi}}(t)\sigma^2,\end{aligned} \tag{24}$$

*where*

$$S(t) := 2\sum_{i=1}^{t}\tilde{u}(i)\,\tilde{y}(i) \geqslant 0, \quad (25)$$

$$\tilde{u}(t) := -\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t), \quad (26)$$

$$\tilde{y}(t) := \frac{1}{2}\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t) + \left[y(t) - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\hat{\boldsymbol{\theta}}(t) - v(t)\right]. \quad (27)$$

*Proof.* Define the innovation vector $e(t) := y(t) - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\hat{\boldsymbol{\theta}}(t-1)$. Using (17), it follows that

$$\begin{aligned}\hat{v}(t) &= \left[1 - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\mathbf{P}(t)\hat{\boldsymbol{\varphi}}(t)\right]e(t)\\ &= \frac{e(t)}{1 + \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\mathbf{P}(t-1)\hat{\boldsymbol{\varphi}}(t)}.\end{aligned} \quad (28)$$

Subtracting $\boldsymbol{\theta}$ from both sides of (15) and using (14), we have

$$\begin{aligned}\tilde{\boldsymbol{\theta}}(t) &= \hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta} = \tilde{\boldsymbol{\theta}}(t-1) + \mathbf{P}(t)\hat{\boldsymbol{\varphi}}(t)e(t)\\ &= \tilde{\boldsymbol{\theta}}(t-1) + \mathbf{P}(t-1)\hat{\boldsymbol{\varphi}}(t)\hat{v}(t).\end{aligned} \quad (29)$$

According to the definition of $T(t)$ and using (16) and (29), we have

$$\begin{aligned}T(t) &= T(t-1) + \tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\hat{\boldsymbol{\varphi}}(t)\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t)\\ &\quad + 2\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t)\hat{v}(t) - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\mathbf{P}(t)\hat{\boldsymbol{\varphi}}(t)\\ &\quad \times\left[1 - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\mathbf{P}(t)\hat{\boldsymbol{\varphi}}(t)\right]e^2(t)\\ &\leqslant T(t-1) + \tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\hat{\boldsymbol{\varphi}}(t)\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t)\\ &\quad + 2\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t)\hat{v}(t)\\ &= T(t-1) + 2\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t)\\ &\quad \times\left[\frac{1}{2}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\hat{\boldsymbol{\varphi}}(t) + (\hat{v}(t) - v(t))\right] + 2\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t)\hat{v}(t).\end{aligned} \quad (30)$$

Using (26), (27), and (29), and $0 \leqslant \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\mathbf{P}(t)\hat{\boldsymbol{\varphi}}(t) \leqslant 1$, we have

$$\begin{aligned}T(t) &\leqslant T(t-1) - 2\tilde{u}(t)\tilde{y}(t) + 2\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\\ &\quad \times\left[\tilde{\boldsymbol{\theta}}(t-1) + \mathbf{P}(t)\hat{\boldsymbol{\varphi}}(t)e(t)\right]v(t)\\ &= T(t-1) - 2\tilde{u}(t)\tilde{y}(t) + 2\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t-1)v(t)\\ &\quad + 2\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\mathbf{P}(t)\hat{\boldsymbol{\varphi}}(t)[e(t) - v(t)]v(t) + v^2(t).\end{aligned} \quad (31)$$

Since $\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t-1)$, $e(t)-v(t)$, $\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\mathbf{P}(t)\hat{\boldsymbol{\varphi}}(t)$ are uncorrelated with $v(t)$ and are $\mathscr{F}_{t-1}$-measurable, taking the conditional expectation with respect to $\mathscr{F}_{t-1}$ and using (A1)-(A2) give

$$\begin{aligned}\mathrm{E}\left[T(t) \mid \mathscr{F}_{t-1}\right] &\leqslant T(t-1) - 2\mathrm{E}\left[\tilde{u}(t)\tilde{y}(t) \mid \mathscr{F}_{t-1}\right]\\ &\quad + 2\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\mathbf{P}(t)\hat{\boldsymbol{\varphi}}(t)\sigma^2, \quad \text{a.s.}\end{aligned} \quad (32)$$

The state space model in (1) can be transformed into an input-output representation,

$$\begin{aligned}y(t) &= \mathbf{c}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}u(t) + v(t)\\ &= \frac{\mathbf{c}\,\mathrm{adj}[z\mathbf{I} - \mathbf{A}]\,\mathbf{b}}{\det[z\mathbf{I} - \mathbf{A}]}u(t) + v(t)\\ &=: \frac{B(z)}{A(z)}u(t) + v(t),\end{aligned} \quad (33)$$

where $\mathrm{adj}[z\mathbf{I} - \mathbf{A}]$ is the adjoint matrix of $[z\mathbf{I} - \mathbf{A}]$, $A(z)$ and $B(z)$ are polynomials in a unit backward shift operator $z^{-1}[z^{-1}y(t) = y(t-1)]$, and

$$\begin{aligned}A(z) &:= z^{-n}\det[z\mathbf{I} - \mathbf{A}],\\ B(z) &:= z^{-n}\mathbf{c}\,\mathrm{adj}[z\mathbf{I} - \mathbf{A}]\,\mathbf{b}.\end{aligned} \quad (34)$$

Referring to the proof of Lemma 3 in [43], using (33), we have

$$\begin{aligned}A(z)[\hat{v}(t) - v(t)] &= A(z)\hat{v}(t) - A(z)y(t) + B(z)u(t)\\ &= -A(z)\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\hat{\boldsymbol{\theta}}(t) + B(z)u(t)\\ &= -\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t) = \tilde{u}(t).\end{aligned} \quad (35)$$

Using (17), (26), and (35), from (27), we get

$$\begin{aligned}\tilde{y}(t) &= \frac{1}{2}\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\tilde{\boldsymbol{\theta}}(t) + [\hat{v}(t) - v(t)]\\ &= \left[A^{-1}(z) - \frac{1}{2}\right]\tilde{u}(t).\end{aligned} \quad (36)$$

Since $A'(z)$ is a strictly positive real function, referring to Appendix C in [79], we can obtain the conclusion $S(t) \geqslant 0$. Adding both sides of (32) by $S(t)$ gives the conclusion of Theorem 1. □

**Theorem 2.** *For the system in* (1) *and the algorithm in* (15)–(18), *assume that (A1)–(A3) hold and that $A(z)$ is stable; that is, all zeros of $A(z)$ are inside the unit circle; then the parameter estimation error satisfies*

$$\left\|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\right\|^2 = O\left(\frac{[\ln r(t)]^c}{\lambda_{\min}[\mathbf{P}^{-1}(t)]}\right), \quad \textit{a.s., for any } c > 1. \quad (37)$$

*Proof.* Using the formula $\lambda_{\min}[\mathbf{Q}]\|\mathbf{x}\|^2 \leqslant \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \leqslant \lambda_{\max}[\mathbf{Q}]\|\mathbf{x}\|^2$, and from the definition of $T(t)$, we have

$$\left\|\tilde{\boldsymbol{\theta}}(t)\right\|^2 \leqslant \frac{\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(t)\mathbf{P}^{-1}(t)\tilde{\boldsymbol{\theta}}(t)}{\lambda_{\min}[\mathbf{P}^{-1}(t)]} = \frac{T(t)}{\lambda_{\min}[\mathbf{P}^{-1}(t)]}. \quad (38)$$

Let

$$W(t) := \frac{T(t) + S(t)}{[\ln|\mathbf{P}^{-1}(t)|]^c}, \quad c > 1. \quad (39)$$

Since $\ln|\mathbf{P}^{-1}(t)|$ is nondecreasing, using Theorem 1 yields

$$\begin{aligned}\mathrm{E}\left[W(t)\mid\mathscr{F}_{t-1}\right] &\leqslant \frac{T(t-1)+S(t-1)}{\left[\ln\left|\mathbf{P}^{-1}(t)\right|\right]^{c}}+\frac{2\widehat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\mathbf{P}(t)\widehat{\boldsymbol{\varphi}}(t)}{\left[\ln\left|\mathbf{P}^{-1}(t)\right|\right]^{c}}\sigma^{2}\\ &\leqslant V(t-1)+\frac{2\widehat{\boldsymbol{\varphi}}^{\mathrm{T}}(t)\mathbf{P}(t)\widehat{\boldsymbol{\varphi}}(t)}{\left[\ln\left|\mathbf{P}^{-1}(t)\right|\right]^{c}}\sigma^{2},\quad \text{a.s.}\end{aligned}\tag{40}$$

Referring to the proof of Theorem 2 in [43], we have

$$\begin{aligned}\left\|\widetilde{\boldsymbol{\theta}}(t)-\boldsymbol{\theta}\right\|^{2} &= O\left(\frac{\left[\ln\left|\mathbf{P}^{-1}(t)\right|\right]^{c}}{\lambda_{\min}\left[\mathbf{P}^{-1}(t)\right]}\right)\\ &= O\left(\frac{\left[\ln r(t)\right]^{c}}{\lambda_{\min}\left[\mathbf{P}^{-1}(t)\right]}\right),\quad \text{a.s. for any } c>1.\end{aligned}\tag{41}$$

□

Assume that there exist positive constants $\gamma$, $c_1$, $c_2$, and $t_0$ such that the following generalized persistent excitation condition (unbounded condition number) holds:

$$c_{1}\mathbf{I}\leqslant\frac{1}{t}\sum_{j=1}^{t}\boldsymbol{\varphi}(j)\boldsymbol{\varphi}^{\mathrm{T}}(j)\leqslant c_{2}t^{\gamma}\mathbf{I},\quad \text{a.s., for } t\geqslant t_{0}.\tag{42}$$

Then for any $c>1$, we have

$$\left\|\widehat{\boldsymbol{\theta}}(t)-\boldsymbol{\theta}\right\|^{2}=O\left(\frac{[\ln t]^{c}}{t}\right)\longrightarrow 0,\quad \text{a.s. for any } c>1.\tag{43}$$

## 4. The State Estimation Algorithm

Referring to the method in [15], the state estimate $\widehat{\mathbf{x}}(t)$ of the state vector $\mathbf{x}(t)$ can be expressed as

$$\widehat{\mathbf{x}}(t-n)=\boldsymbol{\varphi}_{y}(t)-\widehat{\mathbf{M}}(t)\boldsymbol{\varphi}_{u}(t)-\widehat{\boldsymbol{\varphi}}_{v}(t),\tag{44}$$

$$\boldsymbol{\varphi}_{y}(t)=\left[y(t-n),y(t-n+1),\ldots,y(t-1)\right]^{\mathrm{T}},\tag{45}$$

$$\boldsymbol{\varphi}_{u}(t)=\left[u(t-n),u(t-n+1),\ldots,u(t-1)\right]^{\mathrm{T}},\tag{46}$$

$$\widehat{\boldsymbol{\varphi}}_{v}(t)=\left[\widehat{v}(t-n),\widehat{v}(t-n+1),\ldots,\widehat{v}(t-1)\right]^{\mathrm{T}},\tag{47}$$

$$\widehat{\mathbf{M}}(t)=\begin{bmatrix}0&0&\cdots&0&0\\ \widehat{b}_{1}(t)&0&\cdots&0&0\\ \widehat{b}_{2}(t)&\widehat{b}_{1}(t)&\ddots&\vdots&\vdots\\ \vdots&\vdots&\ddots&0&0\\ \widehat{b}_{n-1}(t)&\widehat{b}_{n-2}(t)&\cdots&\widehat{b}_{1}(t)&0\end{bmatrix},\tag{48}$$

$$\begin{bmatrix}\widehat{b}_{1}(t)\\ \widehat{b}_{2}(t)\\ \vdots\\ \widehat{b}_{n-1}(t)\\ \widehat{b}_{n}(t)\end{bmatrix}=\begin{bmatrix}\widehat{a}_{n-1}(t)&\widehat{a}_{n-2}(t)&\cdots&\widehat{a}_{1}(t)&1\\ \widehat{a}_{n-2}(t)&\widehat{a}_{n-3}(t)&\cdots&1&0\\ \vdots&\vdots&&\vdots&\vdots\\ \widehat{a}_{1}(t)&1&\cdots&0&0\\ 1&0&\cdots&0&0\end{bmatrix}^{-1}\widehat{\boldsymbol{\theta}}_{b}(t),\tag{49}$$

$$\widehat{\boldsymbol{\theta}}(t)=\begin{bmatrix}\widehat{\boldsymbol{\theta}}_{a}(t)\\ \widehat{\boldsymbol{\theta}}_{b}(t)\end{bmatrix},\tag{50}$$

$$\widehat{\boldsymbol{\theta}}_{a}(t)=\left[-\widehat{a}_{n}(t),-\widehat{a}_{n-1}(t),\ldots,-\widehat{a}_{1}(t)\right]^{\mathrm{T}}.\tag{51}$$

To summarize, we list the steps involved in the algorithm in (19)–(23) and (44)–(51) to compute the parameter estimate $\widehat{\boldsymbol{\theta}}(t)$ and the state estimate $\widehat{\mathbf{x}}(t-n)$.

(1) Let $t=1$; set the initial values $\widehat{\boldsymbol{\theta}}(i)=\mathbf{1}_n/p_0$, $\mathbf{P}(0)=p_0\mathbf{I}$, $u(i)=0$, $y(i)=0$, $\widehat{v}(i)=0$, or $\widehat{v}(i)=1/p_0$ for $i\leqslant 0$, $p_0=10^6$. Give a small positive number $\varepsilon$.

(2) Collect the input-output data $u(t)$ and $y(t)$; form $\widehat{\boldsymbol{\varphi}}(t)$ using (23), $\boldsymbol{\varphi}_y(t)$ using (45), and $\boldsymbol{\varphi}_u(t)$ using (46).

(3) Compute the gain vector $\mathbf{L}(t)$ using (20) and the covariance matrix $\mathbf{P}(t)$ using (21).

(4) Update the parameter estimation vector $\widehat{\boldsymbol{\theta}}(t)$ using (19).

(5) Compute $\widehat{v}(t)$ using (22), and form $\widehat{\boldsymbol{\varphi}}_v(t)$ using (47).

(6) Determine $\widehat{a}_i(t)$ using (51) and compute $\widehat{b}_i(t)$ using (49); then form $\widehat{\mathbf{M}}(t)$ using (48).

(7) Compute the state estimate $\widehat{\mathbf{x}}(t-n)$ using (44).

(8) If they are sufficiently close, if $\|\widehat{\boldsymbol{\theta}}(t)-\widehat{\boldsymbol{\theta}}(t-1)\|\leqslant\varepsilon$, then terminate the procedure and obtain the estimate $\widehat{\boldsymbol{\theta}}(t)$; otherwise, increase $t$ by 1 and go to step 2.

## 5. Example

Consider the following single-input single-output second-order system in canonical form:

$$\begin{aligned}\mathbf{x}(t+1)&=\begin{bmatrix}0&1\\ -0.70&1.35\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}1\\ 1\end{bmatrix}u(t),\\ y(t)&=[1,\,0]\,\mathbf{x}(t)+v(t).\end{aligned}\tag{52}$$

The simulation conditions are the same as in [15]. That is, the input $\{u(t)\}$ is taken as an independent persistent excitation signal sequence with zero mean and unit variances and $\{v(t)\}$ as a white noise sequence with zero mean and variances $\sigma^2=0.20^2$ and $\sigma^2=1.00^2$, respectively. Apply the proposed parameter and state estimation algorithm in (19)–(23) and (44)–(51) to estimate the parameters and states of this example system; the parameter estimates and their estimation errors are shown in Tables 1 and 2; the parameter estimation errors $\delta$ versus $t$ are shown in Figure 1; the states $x_i(t)$ and their estimates $\widehat{x}_i(t)$ versus $t$ are shown in Figures 2 and 3, where $\delta:=\|\widehat{\boldsymbol{\theta}}(t)-\boldsymbol{\theta}\|/\|\boldsymbol{\theta}\|$ ($\|\mathbf{x}\|^2=\mathbf{x}^{\mathrm{T}}\mathbf{x}$) is the parameter estimation error.

From the simulation results of Tables 1 and 2 and Figures 1–3, we can draw the following conclusions.

(1) A lower noise level leads to a faster rate of convergence of the parameter estimates to the true parameters.

(2) The parameter estimation errors $\delta$ become smaller (in general) as the data length $t$ increases; see

Table 1: The parameter estimates and errors ($\sigma^2 = 0.20^2$).

| $t$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\delta$ (%) |
|---|---|---|---|---|---|
| 100 | −0.70568 | 1.35339 | −0.39169 | 1.01645 | 2.44445 |
| 200 | −0.71515 | 1.38650 | −0.41330 | 1.00994 | 4.06207 |
| 500 | −0.70678 | 1.36377 | −0.38556 | 0.99926 | 2.08997 |
| 1000 | −0.70624 | 1.36483 | −0.37257 | 1.00253 | 1.50180 |
| 2000 | −0.70481 | 1.35720 | −0.35450 | 0.99836 | 0.53395 |
| 3000 | −0.70098 | 1.35107 | −0.34969 | 0.99994 | 0.08005 |
| True values | −0.70000 | 1.35000 | −0.35000 | 1.00000 | |

Table 2: The parameter estimates and errors ($\sigma^2 = 1.00^2$).

| $t$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\delta$ (%) |
|---|---|---|---|---|---|
| 100 | −0.32642 | 0.84623 | −0.03136 | 1.05653 | 38.07875 |
| 200 | −0.60245 | 1.38498 | −0.53220 | 1.01317 | 11.33189 |
| 500 | −0.72060 | 1.42162 | −0.52967 | 0.98306 | 10.53486 |
| 1000 | −0.70654 | 1.38791 | −0.42136 | 1.00912 | 4.40155 |
| 2000 | −0.71358 | 1.37357 | −0.35605 | 0.98820 | 1.63287 |
| 3000 | −0.70120 | 1.35114 | −0.34033 | 0.99742 | 0.54719 |
| True values | −0.70000 | 1.35000 | −0.35000 | 1.00000 | |

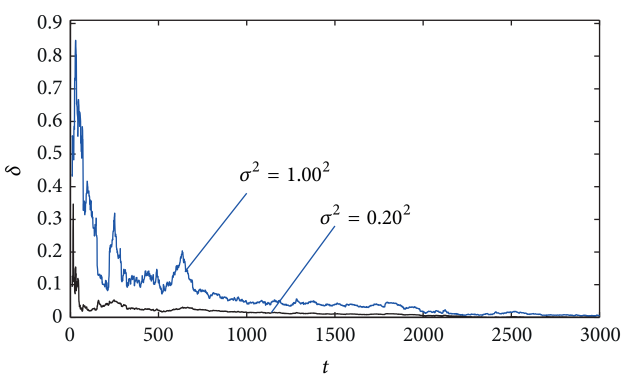


Figure 1: The parameter estimation errors $\delta$ versus $t$ ($\sigma^2 = 0.20^2$ and $\sigma^2 = 1.00^2$).

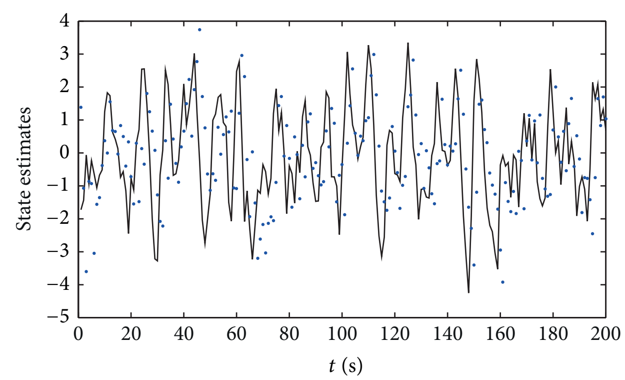


Figure 2: The state estimation errors $\delta$ versus $t$ ($\sigma^2 = 0.20^2$). Solid line: the true $x_1(t)$; dots: the estimated $\hat{x}_1(t)$.

Tables 1 and 2 and Figure 1. In other words, increasing data length generally results in smaller parameter estimation errors.

(3) The state estimates are close to their true values with $t$ increasing; see Figures 2 and 3. These indicate that the proposed parameter and state estimation algorithm are effective.

## 6. Conclusions

In this paper, the identification problems for linear systems based on the canonical state space models with unknown parameters and states are studied. A new parameter and state estimation algorithm has been presented directly from input-output data. The analysis using the martingale convergence

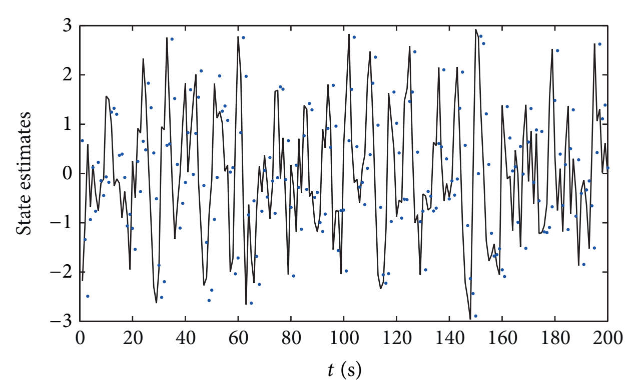


Figure 3: The state estimation errors $\delta$ versus $t$ ($\sigma^2 = 0.20^2$). Solid line: the true $x_2(t)$; dots: the estimated $\hat{x}_2(t)$.

theorem indicates that the proposed algorithms can give consistent parameter estimation. The simulation results show that the proposed algorithms are effective. The method in this paper can combine the multiinnovation identification methods [80–92], the iterative identification methods [93–100], and other identification methods [101–111] to present new identification algorithms or to study adaptive control problems for linear or nonlinear, single-rate or dual-rate, scalar or multivariable systems [112–117].

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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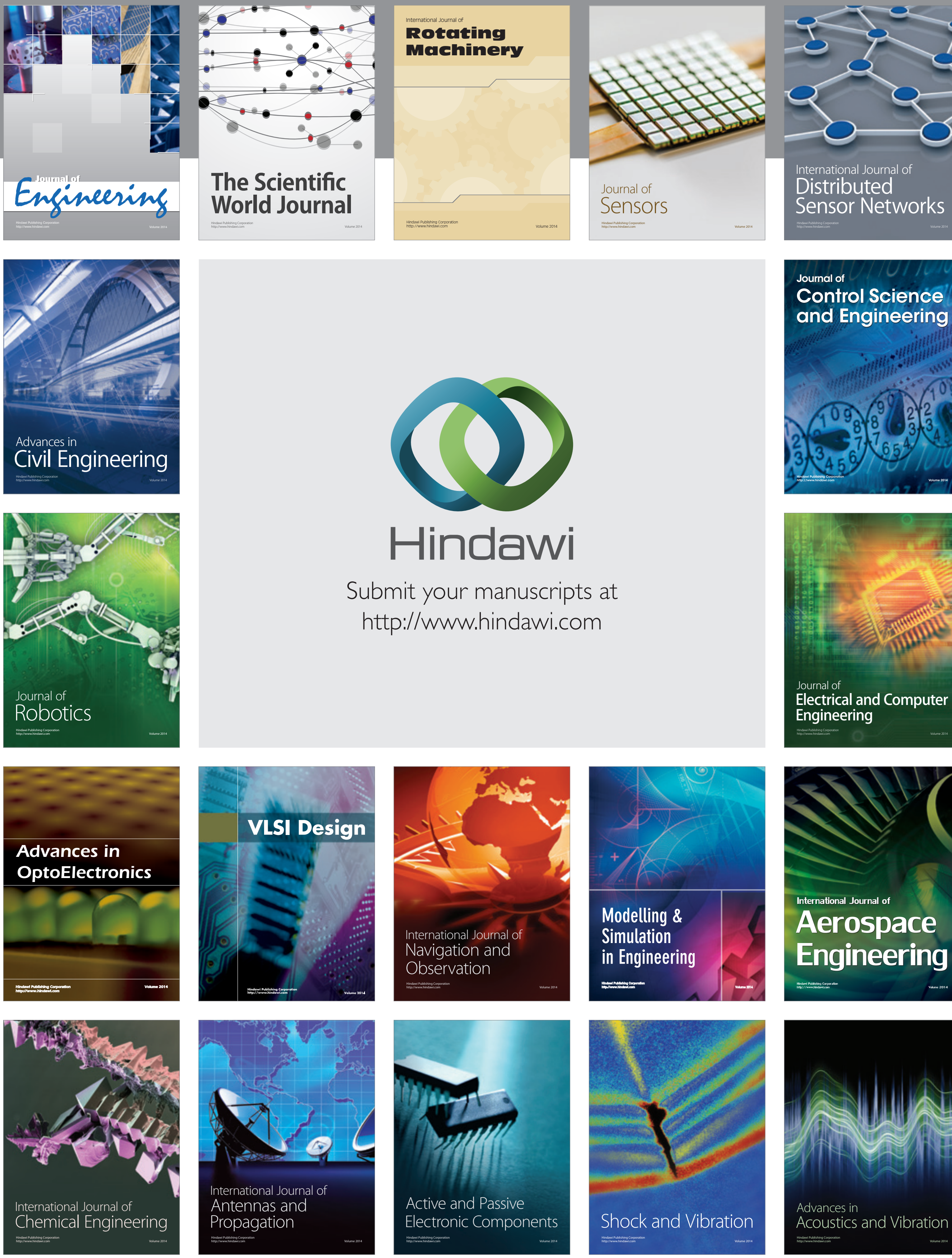
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