Research Article

Invariant Solutions and Conservation Laws of the (2 + 1)-Dimensional Boussinesq Equation

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Invariant solutions and conservation laws of the (2 + 1)-dimensional Boussinesq equation are studied. The Lie symmetry approach is used to obtain the invariant solutions. Conservation laws for the underlying equation are derived by utilizing the new conservation theorem and the partial Lagrange approach.

1. Introduction

In recent years, searching for explicit solutions of nonlinear evolution equations (NEEs) has attracted the attention of many mathematicians and physicists. Particularly, various effective methods have been used to explore different kinds of solutions of NEEs, such as the inverse scattering method [1], the Darboux transformation [2] and the Bäcklund transformation [3], the Hirota method [4], the homogeneous balance method [5, 6], the similarity reduced method [7, 8], the tanh method [9], and the sine-cosine method [10]. But up to now a unified method that can be used to deal with all types of NEEs has not been discovered. Among the above mentioned methods, the Lie symmetry method is one of the most effective methods to determine solutions of differential equations. In the past decades, there have been considerable developments in symmetry methods for differential equations [11, 12].

In the present paper we consider the (2 + 1)-dimensional Boussinesq equation:

$$u_{tt} - u_{xx} - u_{yy} - \left(u^2\right)_{xx} - u_{xxxx} = 0. \tag{1}$$

A well-known soliton equation is the Boussinesq equation:

$$u_{tt} - u_{xx} - 3\left(u^2\right)_{xx} - u_{xxxx} = 0. \tag{2}$$

Equation (2) was introduced by Boussinesq to describe motions of long waves in shallow water [13, 14]. It also appears in a wide variety of physical systems such as nonlinear lattice waves, iron sound waves in plasma, and vibrations in a nonlinear string. For the transonic speed perturbations, by neglecting the interaction of waves moving in the opposite directions, the Boussinesq equation (2) can be reduced to the KdV equation. Equation (2) itself is also a dimensional reduction of the KP equation in the moving frame. Moreover, the Boussinesq equation (2) is completely integrable and admits inverse scattering. Owing to its profound importance and nice mathematical properties, a great deal of research work has been invested in recent years for the study of the Boussinesq equation. Krishnan et al. [15] studied the dynamics of shallow water waves that are governed by the Boussinesq equations. Yang et al. [16] obtained solutions of homogeneous and inhomogeneous dissipative Boussinesq equation by using the modified Jacobi elliptic function expansion method and the pseudospectral method. However, there has also been a growing interest in the study of NEEs in multidimensions, especially in (1 + 2) and (1 + 3) dimensions. To find some exact soliton solutions in higher dimensions is much more difficult than in 1 + 1 dimensions. Recently, El-Sayed and Kaya [17] used the decomposition method to obtain the exact solitary-wave solutions of (1). Senthilvelan [18] obtained the travelling wave solutions for (2 + 1)-dimensional Boussinesq equation and (3 + 1)-dimensional KP equation by homogeneous balance method and explored certain new solutions of the equations. Chen et al. [19]

obtained many explicit exact solutions of (1) by using the new generalized transformation in homogeneous balance method. More new double periodic and multiple soliton solutions are obtained for the generalized $(2+1)$-dimensional Boussinesq equation [20].

The main goal of this paper is to use the Lie symmetry method [21, 22] to obtain the invariant solutions. In addition to this, conservation laws will be derived for (1) by using the new conservation theorem [23, 24] and the partial Lagrange approach [25, 26].

The outline of this paper is as follows. In Section 2, we present symmetry group analysis and group-invariant solutions of (1). In Section 3, the conservation laws for (1) are established. Finally, some conclusions are given in Section 4.

2. Method of Lie Symmetries

2.1. Preliminaries. In this section we briefly present the notation and pertinent results used in this paper [21, 22].

Consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$:

$$E_\alpha\left(x, u, u_{(1)}, \ldots, u_{(k)}\right) = 0, \quad \alpha = 1, \ldots, m, \quad (3)$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second,..., kth-order partial derivatives; that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$, respectively, with the total derivative operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, \ldots, n, \quad (4)$$

where the summation convention is used whenever appropriate.

The Euler-Lagrange operator, for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \cdots i_s}^\alpha}, \quad (5)$$

and the Lie-Bäcklund operator is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathscr{A}, \quad (6)$$

where $\mathscr{A}$ is the space of differential functions. The operator (6) is an abbreviated form of infinite formal sum:

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \cdots i_s}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \cdots i_s}^\alpha}, \quad (7)$$

where the additional coefficients are determined uniquely by the prolongation formulae:

$$\zeta_i^\alpha = D_i(W^\alpha) + \xi^j u_{ij}^\alpha,$$
$$\zeta_{i_1 \cdots i_s}^\alpha = D_{i_1} \cdots D_{i_s}(W^\alpha) + \xi^j u_{ji_1 \cdots j_s}^\alpha, \quad s > 1, \quad (8)$$

in which W^α is the Lie characteristic function:

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \quad (9)$$

One can write the Lie-Bäcklund operator (7) in characteristic form as follows:

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1 i_2 \cdots i_s}(W^\alpha) \frac{\partial}{\partial u_{i_1 i_2 \cdots i_s}^\alpha}. \quad (10)$$

The Noether operators associated with a Lie-Bäcklund symmetry operator X are given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1 i_2 \cdots i_s}(W^\alpha) \frac{\delta}{\delta u_{i i_1 i_2 \cdots i_s}^\alpha}, \quad (11)$$
$$i = 1, 2, \ldots, n,$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (5) by replacing u^α by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1}(-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial u_{ij_1 j_2 \cdots j_s}^\alpha}, \quad (12)$$
$$i = 1, \ldots, n, \quad \alpha = 1, \ldots, m,$$

and the Euler-Lagrange, Lie-Bäcklund, and Noether operators are connected by the operator identity:

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \quad (13)$$

The n-tuple vector $T = (T^1, T^2, \ldots, T^n), T^j \in \mathscr{A}, \ j = 1, \ldots, n$ is a conserved vector of (3) if T^i satisfies

$$D_i T^i|_{(3)} = 0. \quad (14)$$

Equation (14) defines a local conservation law of system (3).

2.2. Lie Point Symmetries. Now, we consider the following Lie group of transformations with independent variables x, y, t and dependent variable u:

$$x^* = x^*(x, y, t, \varepsilon), \qquad y^* = y^*(x, y, t, \varepsilon),$$
$$t^* = t^*(x, y, t, \varepsilon), \qquad u^* = u^*(x, y, t, \varepsilon), \quad (15)$$

where ε is the group parameter. The infinitesimal generator for the Lie group transformations can be expressed in the following form:

$$X = \xi \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}. \quad (16)$$

Applying the fourth prolongation $\mathrm{pr}^{(4)}X$ to (1), we obtain the following determining equations:

$$\xi_y = \xi_t = \xi_u = \phi_x = \phi_u = \tau_x = \tau_u = 0,$$
$$\xi_{xx} = \eta_{xx} = \eta_{xu} = \eta_{uu} = 0,$$
$$\tau_t - \phi_y = 0, \qquad \tau_t - 2\xi_x = 0, \qquad \phi_t - \tau_y = 0,$$
$$\tau_{tt} + 2\eta_{ut} + \tau_{yy} = 0, \qquad \phi_{tt} - 2\eta_{ut} + \phi_{yy} = 0, \quad (17)$$
$$\eta_{xx} + \eta_{yy} - \eta_{tt} + \eta_{xxxx} = 0,$$
$$\eta - \xi_x + \tau_t + 2u(\tau_t - \xi_x) = 0,$$
$$\eta_u + 2\tau_t - 2\xi_x = 0, \qquad 4\eta_x + \xi_{tt} + 2\eta_{xu} - \xi_{yy} = 0.$$

Solving the above system (17) we reach that the symmetry of (1) is spanned by the five vector fields:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial t},$$

$$X_4 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \tag{18}$$

$$X_5 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} + (2u+1)\frac{\partial}{\partial u}.$$

2.3. Group-Invariant Solutions.

In this section we present the reduction forms of (1) by using symmetry group method. To do this, particular linear combinations of infinitesimals are considered and their corresponding invariants are determined. For example, for the symmetry operator $X = X_1 + X_2 + X_3$, we can compute the invariants by integrating the characteristic equations:

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dt}{1}. \tag{19}$$

The corresponding invariants are $q = x - y$, $p = t - y$, $\omega = u$. Now treating p, q as the new independent variables and ω as the new dependent variable, we obtain the ordinary differential equation:

$$(2\omega(q) + 1)\omega''(q) + 2(\omega'(q))^2 + \omega^{(4)}(q) = 0. \tag{20}$$

Integrating (20) twice with respect to q we obtain

$$\omega''(q) + \omega^2(q) + \omega(q) = c_0 q + c_1, \quad c_0, c_1 \in R. \tag{21}$$

Taking $c_0 = 0$ and $\omega'(q) = \psi(\omega)$ we get

$$\psi'(\omega)\psi(\omega) + \omega^2 + \omega = c_1. \tag{22}$$

The solution of (22) is

$$\psi(\omega) = \pm\frac{1}{3}\sqrt{-6\omega^3 + 18c_1\omega - 9\omega^2 + 9c_2}, \tag{23}$$

where c_2 is a constant. Notice that $\omega'(q) = \psi(\omega)$; we have

$$\omega'(q) = \pm\frac{1}{3}\sqrt{-6\omega^3 + 18c_1\omega - 9\omega^2 + 9c_2}, \tag{24}$$

so we get

$$q = \pm 3\int \frac{1}{\sqrt{-6\omega^3 + 18c_1\omega - 9\omega^2 + 9c_2}}d\omega. \tag{25}$$

Recall $q = x - y$, $\omega = u$; we get the solution of (1):

$$x - y = \pm 3\int \frac{1}{\sqrt{-6u^3 + 18c_1u - 9u^2 + 9c_2}}du. \tag{26}$$

3. Conservation Law

Here we briefly present the pertinent results of the two variational methods we utilize below.

3.1. Variational Method for a System and Its Adjoint.

The system of adjoint equations to the system of kth-order differential equation (3) is defined by [23]

$$E_\alpha^*\left(x, u, v, \ldots, u_{(k)}, v_{(k)}\right) = 0, \quad \alpha = 1, \ldots, m, \tag{27}$$

where

$$E_\alpha^*\left(x, u, v, \ldots, u_{(k)}, v_{(k)}\right) = \frac{\delta\left(v^\beta E_\beta\right)}{\delta u^\alpha}, \tag{28}$$

$$\alpha = 1, \ldots, m, \quad v = v(x),$$

and $v = (v^1, v^2, \ldots, v^m)$ are new dependent variables.

We recall here the following results as given in Ibragimov [23].

A system of (3) is said to be self-adjoint if the substitution of $v = u$ into the system of adjoint equation (27) yields the same system (3).

Assume the system of (3) admits the symmetry generator:

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha}, \tag{29}$$

and then the system of adjoint equation (27) admits the operator:

$$Y = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \eta_*^\alpha\frac{\partial}{\partial v^\alpha},$$

$$\eta_*^\alpha = -\left[\lambda_\beta^\alpha v^\beta + v^\alpha D_i\left(\xi^i\right)\right], \tag{30}$$

where the operator (30) is an extension of (29) to the variable v^α and the λ_β^α are obtainable from

$$X\left(E_\alpha\right) = \lambda_\alpha^\beta E_\beta. \tag{31}$$

Theorem 1 (see [23]). *Every Lie point, Lie-Bäcklund, and nonlocal symmetry (29) admitted by the system of (3) gives rise to a conservation law for the system consisting of (3) and the adjoint equation (27), where the components T^i of the conserved vector $T = (T^1, \ldots, T^n)$ are determined by*

$$T^i = \xi^i L + W^\alpha\frac{\delta L}{\delta u_i^\alpha} + \sum_{s\geq 1}D_{i_1}\cdots D_{i_s}\left(W^\alpha\right)\frac{\delta L}{\delta u_{ii_1i_2\cdots i_s}^\alpha}, \tag{32}$$

$$i = 1, \ldots, n,$$

with Lagrangian given by

$$L = v^\alpha E_\alpha\left(x, u, \ldots, u_{(k)}\right). \tag{33}$$

3.2. Partial Noether Approach for a System of PDEs.

The following results are due to Kara and Mahomed [26] that are based on the partial Lagrangian approach to construct conservation laws for a system of PDEs.

Suppose that the system of (3) is written as

$$E_\alpha = E_\alpha^0 + E_\alpha^1 = 0, \quad \alpha = 1, \ldots, m. \tag{34}$$

If there exist a function $L = L(x, u, u_{(1)}, \ldots, u_{(l)}) \in \mathcal{A}$, $l \leq k$ and nonzero functions $f_\alpha^\beta \in \mathcal{A}$ such that (34) can be written as $\delta L / \delta u^\alpha = f_\alpha^\beta E_\beta^1$, provided $E_\beta^1 \neq 0$, then L is known as a partial Lagrangian of (34); otherwise it is the standard Lagrangian. The differential equations of the form

$$\frac{\delta L}{\delta u^\alpha} = f_\alpha^\beta E_\beta^1 \tag{35}$$

are called a system of partial Euler-Lagrange equations.

The operator X in (10) is a partial Noether operator corresponding to a partial Lagrangian $L \in \mathcal{A}$ of the system (35) if it can be determined from

$$X(L) + LD_i(\xi^i) = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i(B^i), \tag{36}$$

for some vector $B = (B^1, B^2, \ldots, B^n)$, $B^i \in \mathcal{A}$. Here $W = (W^1, W^2, \ldots, W^m)$, $W^\alpha \in \mathcal{A}$ is the characteristic of X.

Theorem 2 (see [26]). *If the operator X as in (10) is a partial Noether operator of a partial Lagrangian L corresponding to a partial Eule-Lagrange system of the form* (35), *the components T^i of the conserved vector T of (3) or (35) can be constructed by the following formula:*

$$T^i = B^i - \xi^i L - W^\alpha \frac{\delta L}{\delta u_i^\alpha} - \sum_{s \geq 1} D_{i_1} \cdots D_{i_s} (W^\alpha) \frac{\delta L}{\delta u_{ii_1 i_2 \cdots i_s}^\alpha},$$
$$i = 1, \ldots, n, \tag{37}$$

where the characteristic $W = (W^1, W^2, \ldots, W^m)$, $W^\alpha \in \mathcal{A}$ of X is also the characteristic of the conservation law $D_i T^i = 0$ of (3).

3.3. Conservation Law.

We now construct conservation law of (1) using the two approaches outlined above.

Application of the New Conservation Theorem. The $(2 + 1)$-dimensional Boussinesq equation is given by

$$E_\alpha = u_{tt} - (2u + 1) u_{xx} - 2u_x^2 - u_{yy} - u_{xxxx} = 0. \tag{38}$$

We recall that (38) admits the following five Lie point symmetry generators:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y},$$

$$X_3 = \frac{\partial}{\partial t}, \qquad X_4 = y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}, \tag{39}$$

$$X_5 = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t} + (2u + 1) \frac{\partial}{\partial u}.$$

The adjoint equation of (38), by invoking (27), is

$$E_\alpha^* = \frac{\delta \left[v \left(u_{tt} - (2u + 1) u_{xx} - 2u_x^2 - u_{yy} - u_{xxxx} \right) \right]}{\delta u} = 0, \tag{40}$$

where $v = v(x, y, t)$ is a new dependent variable and (40) gives

$$v_{tt} - 2u_{xx}v - 4u_x v_x - (2u + 1) v_{xx} - v_{yy} - v_{xxxx} = 0. \tag{41}$$

By using Theorem 1, we obtain the following Lagrangian for the system of (38) and (40):

$$L = v \left(u_{tt} - (2u + 1) u_{xx} - 2u_x^2 - u_{yy} - u_{xxxx} \right). \tag{42}$$

(1) We first consider the Lie point symmetry generator $X_1 = \partial/\partial x$, and we have $W = -u_x$. Hence using (32), we obtain the following components of the conserved vector T:

$$T_x^1 = (2u + 2v - 1) u_x v_x - u_{xx} v_{xx} - u_x v_{xxx},$$
$$T_t^1 = -u_{xt} v + u_x v_t, \tag{43}$$
$$T_y^1 = u_{xy} v - u_x v_y.$$

(2) The Lie point symmetry generator $X_2 = \partial/\partial y$ has the Lie characteristic function $W = -u_y$. Thus by using (32), the components of the conserved vector are given by

$$T_x^2 = \left[2u_x u_y + (2u + 1) u_{xy} + u_{yxxx} \right] v$$
$$- \left(2uu_y + u_y \right) v_x - u_{xy} v_{xx} - u_y v_{xxx},$$
$$T_t^2 = -u_{yt} v + u_y v_t, \tag{44}$$
$$T_y^2 = \left[u_{tt} - (2u + 1) u_{xx} - 2u_x^2 \right] v - u_y v_y - v v_{xxxx}.$$

(3) Now the Lie point symmetry generator $X_3 = \partial/\partial t$ has the Lie characteristic function $W = -u_t$. Hence we can obtain the conserved vector whose components are

$$T_x^3 = \left[2u_x u_t + (2u + 1) u_{xt} + u_{xxxt} \right] v - (2u + 1) u_t v_x$$
$$- u_{xt} v_{xx} - u_t v_{xxx},$$
$$T_t^3 = -\left[2u_x^2 + (2u + 1) u_{xx} + u_{yy} \right] v + u_t v_t - v v_{xxxx},$$
$$T_y^3 = -u_{yt} v - u_t v_y. \tag{45}$$

(4) The Lie point symmetry generator $X_4 = y(\partial/\partial t) + t(\partial/\partial y)$ has the Lie characteristic function $W = -tu_y - yu_t$. Hence using (32), the components of the conserved vector are given by

$$T_x^4 = \left[2u_x \left(tu_y + yu_t \right) + (2u + 1) \left(tu_{xy} + yu_{xt} \right) \right.$$
$$\left. + tu_{xxxy} + yu_{xxxt} \right] v - (2u + 1) \left(tu_y + yu_t \right) v_x$$
$$- \left(tu_{xy} + yu_{xt} \right) v_{xx} - \left(tu_y + yu_t \right) v_{xxx},$$

$$T_t^4 = - \left[u_y + 2yu_x^2 + (2u + 1)\, yu_{xx} + yu_{yy} \right.$$

$$\left. + tu_{yt} + yu_{xxxx} \right] v + \left(tu_y + yu_t \right) v_t,$$

$$T_y^4 = \left[u_t - 2tu_x^2 + tu_{tt} - (2u + 1)\, tu_{xx} + yu_{yt} - tu_{xxxx} \right] v$$

$$- \left(tu_y + yu_t \right) v_y. \tag{46}$$

(5) Finally, we consider the Lie point symmetry generator $X_5 = x(\partial/\partial x) + 2y(\partial/\partial y) + 2t(\partial/\partial t) + (2u + 1)(\partial/\partial u)$ and the Lie characteristic function $W = (2u + 1) - xu_x - 2yu_y - 2tu_t$. Hence we can obtain the conserved vector whose components are

$$T_x^5 = \left[xu_{tt} - u_{yy} + 2u_x \left(2yu_y + 2tu_t - 2u - 1 \right) \right.$$

$$+ (4u + 2) \left(yu_{xy} + tu_{xt} \right) - u_{xxx}$$

$$\left. -2yu_{xxxy} - 2tu_{xxxt} \right] v$$

$$+ (2u + 1) \left[(2u + 1) - xu_x - 2yu_y - 2tu_t \right] v_x$$

$$- \left(xu_{xx} + 2yu_{xy} + 2tu_{xt} \right) v_{xx}$$

$$+ \left[(2u + 1) - xu_x - 2yu_y - 2tu_t \right] v_{xxx},$$

$$T_t^5 = v \left[-2 (2u + 1)\, tu_{xx} - 4tu_x^2 - 2tu_y y \right.$$

$$\left. -2tu_{xxxx} - xu_{xt} - 2yu_{yt} \right]$$

$$- v_t \left[(2u + 1) - xu_x - 2yu_y - 2tu_t \right],$$

$$T_y^5 = v \left[2yu_{tt} - 2y (2u + 1)\, u_{xx} - 4yu_x^2 \right.$$

$$\left. -2yu_{xxxx} + xu_{xy} + 2tu_{yt} \right]$$

$$+ v_y \left(2u + 1 - xu_x - 2yu_y - 2tu_t \right). \tag{47}$$

Remark 3. The conserved vector contains the arbitrary solution v of the adjoint equation (40) and hence one can obtain an infinite number of conservation laws.

Application of the Partial Lagrange Method. Consider the partial Lagrange approach given by Kara and Mahomed [26]. A partial Lagrange for (1) is

$$L = \frac{1}{2} u_t^2 - \left(u + \frac{1}{2} \right) u_x^2 - \frac{1}{2} u_y^2 - \frac{1}{2} u_{xx}^2. \tag{48}$$

The Euler-Lagrange-type equation is

$$\frac{\delta L}{\delta u} = -u_{tt} + (2u + 1)\, u_{xx} + u_x^2 + u_{yy} + u_{xxxx}. \tag{49}$$

So (1) can be written as

$$\frac{\delta L}{\delta u} = -u_x^2, \tag{50}$$

to determine the Noether-type operators. If we substitute (48) and (50) into the partial Noether-type operators determining (36), we obtain

$$\xi = \phi = \tau = \eta_u = 0, \qquad \eta = C(x, y, t),$$

$$B^1 = -C_t u + \alpha(x, y, t),$$

$$B^2 = -\left(u^2 + u \right) C_x + \beta(x, y, t), \tag{51}$$

$$B^3 = -C_y u + \gamma(x, y, t),$$

subject to the condition

$$B_t^1 + B_x^2 + B_y^3 = 0. \tag{52}$$

The conserved components, using (37), are

$$T^1 = C_t u - C u_t + \alpha,$$

$$T^2 = -\left(u^2 + u \right) C_x + C(2u + 1)\, u_x + C_x u_{xx} + \beta, \tag{53}$$

$$T^3 = -C_y u + C u_y + \gamma.$$

4. Conclusions

In this paper, we investigate Lie point symmetries, similarity reduction, invariant solutions, and conservation laws of the $(2 + 1)$-dimensional Boussinesq equation. One of the most important applications of theory of Lie groups is to obtain the conservation laws of differential equations. It is well known that the famous Noether theorem establishes a connection between symmetries and conservation laws of differential equations provided that the equations are Euler-Lagrange equations. However, (1) does not admit Lagrangians. We obtain conservation laws of (1) by utilizing the new conservation theorem and the partial Lagrange approach.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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