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Research Article

LMI-Based Stability Criterion for Impulsive CGNNs via Fixed Point Theory

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Linear matrices inequalities (LMIs) method and the contraction mapping theorem were employed to prove the existence of globally exponentially stable trivial solution for impulsive Cohen-Grossberg neural networks (CGNNs). It is worth mentioning that it is the first time to use the contraction mapping theorem to prove the stability for CGNNs while only the Leray-Schauder fixed point theorem was applied in previous related literature. An example is given to illustrate the effectiveness of the proposed methods due to the large allowable variation range of impulse.

1. Introduction and Preliminaries

Dynamics of Cohen-Grossberg neural networks (CGNNs) has been extensively investigated due to its immense potentials of application perspective in various areas such as pattern recognition, parallel computing, associate memory, combinational optimization, and signal and image processing. However, these successful applications always depend on the stability of the equilibrium solution for CGNNs. All the time, the method of Lyapunov theory has usually been employed to solve the stability problem of dynamical systems [1–11]. In studying the stability of neural networks, Lyapunov-Krasovskii functional method can always be combined with other methods in a perfect way, such as the linear matrix inequality (LMI) optimization approach, M-Matrix theory, and nonsmooth analysis technique (see, e.g., [6–8]). Of course, as one of stability analysis methods, the Lyapunov method has its limitations. In fact, a stability criterion is regarded as an effective and efficient method for impulsive neural networks if a larger variation range of impulse is allowable (see, e.g., [12, Remark 11]). Thereby, using the methods different from Lyapunov direct method may obtain a more efficient stability criterion for impulsive systems. Indeed, fixed point theories have always been considered by many authors. Burton [13, 14], Rao and Pu [15], Jung [16], Luo [17], Zhang [18], and Wu et al. [19] studied the stability by using the fixed point theory which solved the difficulties encountered in the study of stability by means of Lyapunov's direct method. Contraction mapping theorem was the usual method to study the stability of neural networks, except CGNNs. Owing to some difficulties, only the Leray-Schauder fixed point theorem was considered in investigating the stability of CGNNs [20, 21]. In this paper, contraction mapping theorem is applied to the stability analysis for CGNNs. We wish that our newly obtained stability criterion will allow a larger variation range of impulses against a series of previously related literatures. This is the main purpose of this paper.

Consider the following CGNNs:

$$\frac{dx_j(t)}{dt} = -a_j(x_j(t))\Big\{b_j(x_j(t)) - \sum_{k=1}^{n}\left[c_{jk}f_k(x_k(t)) + d_{jk}g_k(x_k(t-\tau_j(t)))\right]\Big\}, \quad t \geqslant 0,\ t \neq t_i,$$

$$\Delta x_j(t_i) = x_j(t_i^+) - x_j(t_i) = \rho_{ji}(x_j(t_i)), \quad i = 1, 2, \ldots,$$
$$x_j(\theta) = \phi_j(\theta), \quad -\tau \leqslant \theta \leqslant 0, \tag{1}$$

where $j \in \mathcal{N} \triangleq \{1, 2, \ldots, n\}$. $a_j(x_j(t))$ and $b_j(x_j(t))$ represent an amplification function at time t and an appropriately behaved function at time t. $f_k(\cdot)$ and $g_k(\cdot)$ are the activation functions of the neurons, and c_{jk} and d_{jk} are connection parameters. The time-varying delays $\tau_j(t) \in [0, \tau]$. The impulsive moments t_i $(i = 1, 2, \ldots)$ satisfy $0 = t_0 < t_1 < t_2 < \cdots$, and $\lim_{i\to+\infty} t_i = +\infty$. $x_j(t_i^+)$ and $x_j(t_i^-)$ stand for the right-hand and left-hand limit of $x_j(t)$ at time t_i, respectively. $\rho_{ji}(x_j(t_i))$ means the abrupt change of $x_j(t)$ at the impulsive moment t_i and $\rho_{ji}(\cdot) \in C[R, R]$.

Throughout this paper, we assume that $b_j(0) = f_j(0) = g_j(0) = \rho_{ji}(0) = 0$ for $j \in \mathcal{N}$ and $i = 1, 2, \ldots$. Denote by $x(t) \triangleq x(t; \theta, \phi) = (x_1(t; \theta, \phi_1), \ldots, x_n(t; \theta, \phi_n))^T \in R^n$ the solution for system (1) with the initial condition $x_j(\theta) = \phi_j(\theta)$, $-\tau \leqslant \theta \leqslant 0$, $j \in \mathcal{N}$, where $\phi(\theta) = (\phi_1(\theta), \ldots, \phi_n(\theta))^T \in R^n$ and $\phi_j(\theta) \in C[[-\tau, 0], R]$. The solution $x(t) \triangleq x(t; \theta, \phi) \in R^n$ of system (1) is, for the time variable t, a continuous vector-valued function.

Throughout this paper, we assume the following.

(H1) For any $j \in \mathcal{N}$, there exist constants $\overline{F}_j > 0$, $\overline{G}_j > 0$, $F_j > 0$, and $G_j > 0$ such that

$$\begin{gathered} |f_j(r)| \leqslant \overline{F}_j, \\ |g_j(r)| \leqslant \overline{G}_j, \\ |f_j(r) - f_j(s)| \leqslant F_j |r - s|, \\ |g_j(r) - g_j(s)| \leqslant G_j |r - s|, \\ \forall r, s \in R. \end{gathered} \tag{2}$$

(H2) For any $j \in \mathcal{N}$, $a_j(\cdot)$ is differentiable, and there exists a constant $\tilde{a}_j$ such that

$$\begin{gathered} 0 < a_j(r) \leqslant \overline{a}_j, \\ |a_j'(r)| \leqslant \tilde{a}_j, \\ r \in R. \end{gathered} \tag{3}$$

(H3) There exist nonnegative constants h_{ji} such that

$$|\rho_{ji}(r) - \rho_{ji}(s)| \leqslant h_{ji} |r - s|, \quad \forall r, s \in R, \; j \in \mathcal{N}, \; i = 1, 2, \ldots \tag{4}$$

(H4) For $j \in \mathcal{N}$, there exist $\Upsilon_j(t)$ and a constant $\gamma_j > 0$ such that

$$\begin{gathered} \Upsilon_j(t) \geqslant \gamma_j, \quad \forall t \geqslant 0, \\ a_j(r) b_j(r) = \Upsilon_j(t) r, \quad r \in R. \end{gathered} \tag{5}$$

Remark 1. $a_j(\cdot)$ represents an amplification function; $b_j(\cdot)$ is an appropriate behavior function. There exist a lot of functions satisfying the above assumptions. For example, let $j \in \mathcal{N} \triangleq \{1, 2\}$, $b_1(x_1(t)) = 2x_1(t)$, $b_2(x_2(t)) = 2x_2(t)$, $a_1(x_1(t)) = a_2(x_2(t)) = 2 + \cos t$, and then $a_j(r) b_j(r) = 2(2 + \cos t) r = \Upsilon_j(t) r$; hence $\Upsilon_j(t) = 2(2 + \cos t)$. Now we let $\gamma_j = 2$, and obviously $\Upsilon_j(t) \geqslant \gamma_j$.

From the above assumptions, it is obvious that the null solution is a trivial solution for system (1).

Definition 2. System (1) is said to be globally exponentially stable if for any initial condition $\phi(\theta) \in C[[-\tau, 0], R^n]$ there exist positive constants λ and μ such that

$$\|x(t; \theta, \phi)\| \leqslant \lambda e^{-\mu t}, \quad \forall t \geqslant 0. \tag{6}$$

Lemma 3 (see [22]). *Let π be a contraction operator on a complete metric space X; then there exists a unique point $\xi \in X$ for which $\pi(\xi) = \xi$.*

2. Main Result

Before giving the main result, we may firstly definite some matrices as follows:

$$\begin{gathered} C = (|c_{jk}|)_{n\times n}, \\ D = (|d_{jk}|)_{n\times n}, \\ \widetilde{A} = \operatorname{diag}(\tilde{a}_1, \ldots, \tilde{a}_n), \\ \overline{A} = \operatorname{diag}(\overline{a}_1, \ldots, \overline{a}_n), \\ \overline{F} = \operatorname{diag}(\overline{F}_1, \ldots, \overline{F}_n), \\ \overline{G} = \operatorname{diag}(\overline{G}_1, \ldots, \overline{G}_n), \\ F = \operatorname{diag}(F_1, \ldots, F_n), \\ G = \operatorname{diag}(G_1, \ldots, G_n), \\ \Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n). \end{gathered} \tag{7}$$

Assume, in addition, there exists the constant h_j with $h_{ji} \leqslant h_j \mu$ for each $j \in \mathcal{N}$, and μ is a positive constant, satisfying $\mu \leqslant \inf_{i=1,2,\ldots}\{t_i - t_{i-1}\}$. Denote $H = \operatorname{diag}(h_1, \ldots, h_n)$.

Theorem 4. *If there exists a positive constant $\alpha < 1$ such that the following LMI condition holds:*

$$C\overline{F}\widetilde{A} + D\overline{G}\widetilde{A} + H + \mu\Gamma + CF\overline{A} + DG\overline{A} \leqslant \alpha\Gamma\overline{A}, \tag{8}$$

then system (1) is globally exponentially stable.

Proof. Our proof is based on the contraction mapping principle (Lemma 3), which may be divided into four steps.

Step 1. We may set up a space frame.

Let $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$, and let Θ_j $(j \in \mathcal{N})$ be the space consisting of functions $q_j(t) : [-\tau, \infty) \to R$ such that, for any $j \in \mathcal{N}$,

(a) $q_j(t)$ is continuous on $t \in [0, \infty)$ with $t \neq t_i$ $(i = 1, 2, \ldots)$;

(b) $q_j(\theta) = \phi_j(\theta)$, $-\tau \leqslant \theta \leqslant 0$;

(c) $e^{\beta t} q_j(t) \to 0$ as $t \to +\infty$, where $\beta > 0$ is the constant with $\beta < \min\{\liminf_{t\to+\infty}(\int_0^t \Upsilon_j(s)ds/t), \gamma_j\}$;

(d) both $\lim_{t\to t_i^-} q_j(t)$ and $\lim_{t\to t_i^+} q_j(t)$ exist; besides, $\lim_{t\to t_i^-} q_j(t) = q_j(t_i)$ for $i = 1, 2, \ldots$.

Moreover, Θ is a complete metric space if it is equipped with a metric defined by

$$\operatorname{dist}(q, \breve{q}) = \sum_{j=1}^{n} \left(\sup_{t \geqslant -\tau} \left| q_j(t) - \breve{q}_j(t) \right| \right), \tag{9}$$

where $q = q(t) = (q_1(t), q_2(t), \ldots, q_n(t))^T \in \Theta$ and $\breve{q} = \breve{q}(t) = (\breve{q}_1(t), \breve{q}_2(t), \ldots, \breve{q}_n(t))^T \in \Theta$.

Remark 5. It is followed by $\beta < \liminf_{t\to+\infty}(\int_0^t \Upsilon_j(s)ds/t)$ that

$$\begin{aligned}
-\infty &\leqslant \liminf_{t\to+\infty} \left(\beta - \frac{\int_0^t \Upsilon_j(s)\,ds}{t} \right) t \\
&\leqslant \limsup_{t\to+\infty} \left(\beta - \frac{\int_0^t \Upsilon_j(s)\,ds}{t} \right) t \\
&\leqslant \left(\beta - \liminf_{t\to+\infty} \frac{\int_0^t \Upsilon_j(s)\,ds}{t} \right) \limsup_{t\to+\infty} t \leqslant -\infty,
\end{aligned} \tag{10}$$

and hence

$$\lim_{t\to+\infty} e^{(\beta t - \int_0^t \Upsilon_j(s)ds)} = 0. \tag{11}$$

Step 2. Frame a mapping on the space Θ.

Multiplying both sides of system (1) with $e^{\int_0^t \Upsilon_j(s)ds}$ yields, for $t > 0$ and $t \neq t_i$ $(i = 1, 2, \ldots)$,

$$\begin{aligned}
&\frac{d\left(e^{\int_0^t \Upsilon_j(s)ds} x_j(t)\right)}{dt} = \Upsilon_j(t) e^{\int_0^t \Upsilon_j(s)ds} x_j(t) + e^{\int_0^t \Upsilon_j(s)ds} \\
&\quad \cdot \frac{dx_j(t)}{dt} = \Upsilon_j(t) e^{\int_0^t \Upsilon_j(s)ds} x_j(t) \\
&\quad + e^{\int_0^t \Upsilon_j(s)ds} \left\{ -a_j(x_j(t)) \left[b_j(x_j(t)) \right.\right. \\
&\quad \left.\left. - \sum_{k=1}^{n} \left(c_{jk} f_k(x_k(t)) + d_{jk} g_k\left(x_k\left(t - \tau_j(t)\right)\right)\right) \right] \right\} \\
&\quad = e^{\int_0^t \Upsilon_j(s)ds} a_j(x_j(t)) \sum_{k=1}^{n} \left(c_{jk} f_k(x_k(t)) \right. \\
&\quad \left. + d_{jk} g_k\left(x_k\left(t - \tau_j(t)\right)\right)\right).
\end{aligned} \tag{12}$$

Let $\varepsilon > 0$ be small enough. After integrating from $t_{i-1} + \varepsilon$ to $t \in (t_{i-1}, t_i)$ $(i = 1, 2, \ldots)$, we get

$$\begin{aligned}
&e^{\int_0^t \Upsilon_j(s)ds} x_j(t) - e^{\int_0^{t_{i-1}+\varepsilon} \Upsilon_j(s)ds} x_j(t_{i-1} + \varepsilon) \\
&\quad = \int_{t_{i-1}+\varepsilon}^{t} e^{\int_0^s \Upsilon_j(l)dl} a_j(x_j(s)) \\
&\quad \cdot \sum_{k=1}^{n} \left(c_{jk} f_k(x_k(s)) + d_{jk} g_k\left(x_k\left(s - \tau_j(s)\right)\right)\right) ds.
\end{aligned} \tag{13}$$

Letting $\varepsilon \to 0$ in (13), we have

$$\begin{aligned}
&e^{\int_0^t \Upsilon_j(s)ds} x_j(t) - e^{\int_0^{t_{i-1}} \Upsilon_j(s)ds} x_j(t_{i-1}^+) \\
&\quad = \int_{t_{i-1}}^{t} e^{\int_0^s \Upsilon_j(l)dl} a_j(x_j(s)) \\
&\quad \cdot \sum_{k=1}^{n} \left(c_{jk} f_k(x_k(s)) + d_{jk} g_k\left(x_k\left(s - \tau_j(s)\right)\right)\right) ds, \\
&\qquad t \in (t_{i-1}, t_i) \quad (i = 1, 2, \ldots).
\end{aligned} \tag{14}$$

Setting $t = t_i - \varepsilon$ $(\varepsilon > 0)$ in (14), we obtain

$$\begin{aligned}
&e^{\int_0^{t_i-\varepsilon} \Upsilon_j(s)ds} x_j(t_i - \varepsilon) - e^{\int_0^{t_{i-1}} \Upsilon_j(s)ds} x_j(t_{i-1}^+) \\
&\quad = \int_{t_{i-1}}^{t_i-\varepsilon} e^{\int_0^s \Upsilon_j(l)dl} a_j(x_j(s)) \\
&\quad \cdot \sum_{k=1}^{n} \left(c_{jk} f_k(x_k(s)) + d_{jk} g_k\left(x_k\left(s - \tau_j(s)\right)\right)\right) ds,
\end{aligned} \tag{15}$$

which yields by letting $\varepsilon \to 0$

$$\begin{aligned} & e^{\int_0^{t_i} \Upsilon_j(s)ds} x_j(t_i) - e^{\int_0^{t_{i-1}} \Upsilon_j(s)ds} x_j(t_{i-1}^+) \\ & = e^{\int_0^{t_i} \Upsilon_j(s)ds} x_j(t_i^-) - e^{\int_0^{t_{i-1}} \Upsilon_j(s)ds} x_j(t_{i-1}^+) \\ & = \int_{t_{i-1}}^{t_i} e^{\int_0^s \Upsilon_j(l)dl} a_j(x_j(s)) \\ & \cdot \sum_{k=1}^n \left(c_{jk} f_k(x_k(s)) + d_{jk} g_k(x_k(s - \tau_j(s)))\right) ds. \end{aligned} \tag{16}$$

Combining (14) and (16) results in

$$\begin{aligned} & e^{\int_0^{t} \Upsilon_j(s)ds} x_j(t) - e^{\int_0^{t_{i-1}} \Upsilon_j(s)ds} \left(x_j(t_{i-1}) \right. \\ & \left. + \rho_{j(i-1)}(x_j(t_{i-1}))\right) = e^{\int_0^{t} \Upsilon_j(s)ds} x_j(t) \\ & - e^{\int_0^{t_{i-1}} \Upsilon_j(s)ds} x_j(t_{i-1}^+) = \int_{t_{i-1}}^{t} e^{\int_0^s \Upsilon_j(l)dl} a_j(x_j(s)) \\ & \cdot \sum_{k=1}^n \left(c_{jk} f_k(x_k(s)) + d_{jk} g_k(x_k(s - \tau_j(s)))\right) ds, \\ & \qquad t_{i-1} < t \leqslant t_i \quad (i = 1, 2, \ldots). \end{aligned} \tag{17}$$

Hence,

$$\begin{aligned} & e^{\int_0^{t_{i-1}} \Upsilon_j(s)ds} x_j(t_{i-1}) - e^{\int_0^{t_{i-2}} \Upsilon_j(s)ds} \left(x_j(t_{i-2}) \right. \\ & \left. + \rho_{j(i-2)}(x_j(t_{i-2}))\right) = \int_{t_{i-2}}^{t_{i-1}} e^{\int_0^s \Upsilon_j(l)dl} a_j(x_j(s)) \\ & \cdot \sum_{k=1}^n \left(c_{jk} f_k(x_k(s)) + d_{jk} g_k(x_k(s - \tau_j(s)))\right) ds, \\ & \vdots \\ & e^{\int_0^{t_2} \Upsilon_j(s)ds} x_j(t_2) - e^{\int_0^{t_1} \Upsilon_j(s)ds} \left(x_j(t_1) + \rho_{j1}(x_j(t_1))\right) \\ & = \int_{t_1}^{t_2} e^{\int_0^s \Upsilon_j(l)dl} a_j(x_j(s)) \\ & \cdot \sum_{k=1}^n \left(c_{jk} f_k(x_k(s)) + d_{jk} g_k(x_k(s - \tau_j(s)))\right) ds, \\ & e^{\int_0^{t_1} \Upsilon_j(s)ds} x_j(t_1) - \phi_j(0) = \int_0^{t_1} e^{\int_0^s \Upsilon_j(l)dl} a_j(x_j(s)) \\ & \cdot \sum_{k=0}^n \left(c_{jk} f_k(x_k(s)) + d_{jk} g_k(x_k(s - \tau_j(s)))\right) ds, \end{aligned} \tag{18}$$

which produces

$$\begin{aligned} x_j(t) &= e^{-\int_0^t \Upsilon_j(s)ds} \phi_j(0) \\ & + e^{-\int_0^t \Upsilon_j(s)ds} \int_0^t e^{\int_0^s \Upsilon_j(l)dl} \left[a_j(x_j(s)) \right. \\ & \left. \cdot \sum_{k=1}^n \left(c_{jk} f_k(x_k(s)) + d_{jk} g_k(x_k(s - \tau_j(s)))\right) \right] ds \\ & + e^{-\int_0^t \Upsilon_j(s)ds} \sum_{0<t_i<t} \left(\rho_{ji}(x_j(t_i))\right) e^{\int_0^{t_i} \Upsilon_j(s)ds}, \quad t > 0. \end{aligned} \tag{19}$$

Thereby, we may define the mapping $\Phi : x(t) \to \Phi(x)(t)$, where $x(t) = (x_1(t), \ldots, x_j(t), \ldots, x_n(t))^T \in \Theta$, and $\Phi(x)(t) = (\Phi(x_1)(t), \ldots, \Phi(x_j)(t), \ldots, \Phi(x_n)(t))^T$. In addition, for any $j \in \mathcal{N}$, the mapping $\Phi(x_j)(t) : [-\tau, \infty) \to R$ satisfies

$$\begin{aligned} \Phi(x_j)(t) &= e^{-\int_0^t \Upsilon_j(s)ds} \phi_j(0) \\ & + \int_0^t e^{-\int_s^t \Upsilon_j(l)dl} \left[a_j(x_j(s)) \right. \\ & \left. \cdot \sum_{k=1}^n \left(c_{jk} f_k(x_k(s)) + d_{jk} g_k(x_k(s - \tau_j(s)))\right) \right] ds \\ & + e^{-\int_0^t \Upsilon_j(s)ds} \sum_{0<t_i<t} \left(\rho_{ji}(x_j(t_i)) e^{\int_0^{t_i} \Upsilon_j(s)ds}\right), \quad t \geqslant 0, \end{aligned} \tag{20}$$

and $\Phi(x_j)(\theta) = \phi_j(\theta)$ as $-\tau \leqslant \theta \leqslant 0$.

Next, we will prove that $\Phi(\Theta_j) \subset \Theta_j, \ \forall j \in \mathcal{N}$.

It is obvious that conditions (a) and (b) hold in $\Phi(\Theta_j)$. So we only need to prove that, for any $x_j(t) \in \Theta_j$,

$$\left| e^{\beta t} \Phi(x_j) \right| \longrightarrow 0, \quad t \longrightarrow +\infty. \tag{21}$$

Indeed, it is obvious that

$$\left| e^{\beta t} \Phi(x_j) \right| \leqslant Q_1 + Q_2 + Q_3 + Q_4, \tag{22}$$

where

$$\begin{aligned} Q_1 &= \left| e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} \phi_j(0) \right|, \\ Q_2 &= \left| e^{\beta t} \int_0^t e^{-\int_s^t \Upsilon_j(l)dl} a_j(x_j(s)) \sum_{k=1}^n c_{jk} f_k(x_k(s)) ds \right|, \\ Q_3 &= \left| e^{\beta t} \int_0^t e^{-\int_s^t \Upsilon_j(l)dl} a_j(x_j(s)) \right. \\ & \left. \cdot \sum_{k=1}^n d_{jk} g_k(x_k(s - \tau_j(s))) ds \right|, \\ Q_4 &= \left| e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} \sum_{0<t_i<t} \left(\rho_{ji}(x_j(t_i))\right) e^{\int_0^{t_i} \Upsilon_j(s)ds} \right|. \end{aligned} \tag{23}$$

It follows by $\beta < \liminf_{t\to+\infty}(\int_0^t \Upsilon_j(s)ds/t)$ that $e^{\beta t}e^{-\int_0^t \Upsilon_j(s)ds} = e^{(\beta-\int_0^t \Upsilon_j(s)ds/t)t} \to 0$ as $t \to +\infty$. And hence $Q_1 = |e^{\beta t}e^{-\int_0^t \Upsilon_j(s)ds}\phi_j(0)| \to 0$ as $t \to \infty$.

Note that $e^{\beta t}x_j(t) \to 0$ if $x_j(t) \in \Theta_j$, $\forall j \in \mathcal{N}$. So, for any $\varepsilon > 0$, there exists a positive constant N such that

$$\left|e^{\beta t}x_j(t)\right| < \varepsilon, \quad \forall t \in [N, +\infty). \tag{24}$$

Thereby,

$$\begin{aligned}\left|e^{\int_0^t \Upsilon_j(s)ds}x_j(t)\right| &= e^{(\int_0^t \Upsilon_j(s)ds-\beta t)}\left|e^{\beta t}x_j(t)\right| \\ &\leqslant e^{(\int_0^t \Upsilon_j(s)ds-\beta t)}\varepsilon, \quad \forall t \in [N, +\infty).\end{aligned} \tag{25}$$

The continuity of $x_j(t)$ derives that there exists a constant $M > 0$ such that

$$\left|e^{\beta t}x_j(t)\right| < M, \quad \forall t \in [0, N], \tag{26}$$

$$\begin{aligned}\left|e^{\int_0^t \Upsilon_j(s)ds}x_j(t)\right| &= e^{(\int_0^t \Upsilon_j(s)ds-\beta t)}\left|e^{\beta t}x_j(t)\right| \\ &\leqslant M\max_{t\in[0,N]}\left(e^{(\int_0^t \Upsilon_j(s)ds-\beta t)}\right), \\ &\qquad \forall t \in [0, N].\end{aligned} \tag{27}$$

In addition, condition (H1) yields that $|f_k(x_k(s))| = |f_k(x_k(s)) - f_k(0)| \leqslant F_k|x_k(s) - 0|$. Let $t > N$, and we can derive from (25), (27), and the change of variable formula for integrals that

$$\begin{aligned}Q_2 &\leqslant e^{-(\int_0^t \Upsilon_j(s)ds-\beta t)}\left[\int_0^N \overline{a}_j\sum_{k=1}^n\left|c_{jk}\right| \right. \\ &\quad\cdot F_k\left|e^{\int_0^s \Upsilon_j(l)dl}x_k(s)\right|ds + \int_N^t \overline{a}_j\sum_{k=1}^n\left|c_{jk}\right| \\ &\quad\left.\cdot F_k\left|e^{\int_0^s \Upsilon_j(l)dl}x_k(s)\right|ds\right] \leqslant e^{-(\int_0^t \Upsilon_j(s)ds-\beta t)}\left[M\right. \\ &\quad\cdot\max_{t\in[0,N]}\left(e^{(\int_0^t \Upsilon_j(s)ds-\beta t)}\right)\int_0^N \overline{a}_j\sum_{k=1}^n\left|c_{jk}\right|F_k ds \\ &\quad\left.+\overline{a}_j\sum_{k=1}^n\left|c_{jk}\right|F_k\varepsilon\int_N^t e^{(\int_0^s \Upsilon_j(l)dl-\beta s)}ds\right] \\ &\leqslant e^{-(\int_0^t \Upsilon_j(s)ds-\beta t)}\left[M\max_{t\in[0,N]}\left(e^{(\int_0^t \Upsilon_j(s)ds-\beta t)}\right)\right. \\ &\quad\left.\cdot N\overline{a}_j\sum_{k=1}^n\left|c_{jk}\right|F_k\right] + \frac{\overline{a}_j\sum_{k=1}^n\left|c_{jk}\right|F_k}{\gamma_j-\beta}\varepsilon.\end{aligned} \tag{28}$$

Now we can conclude from the arbitrariness of ε that $Q_2 \to 0$, $t \to \infty$.

Similarly, condition (H2) yields

$$\begin{aligned}Q_3 &\leqslant e^{-(\int_0^t \Upsilon_j(s)ds-\beta t)}\int_0^t e^{\int_0^s \Upsilon_j(l)dl}\overline{a}_j\sum_{k=1}^n\left|d_{jk}\right| \\ &\quad\cdot G_k\left|x_k\left(s-\tau_j(s)\right)\right|ds \\ &\leqslant e^{\beta\tau}e^{-(\int_0^t \Upsilon_j(s)ds-\beta t)}\overline{a}_j\sum_{k=1}^n\left|d_{jk}\right|G_k \\ &\quad\cdot\int_0^t e^{(\int_0^s \Upsilon_j(l)dl-\beta s)}\left|e^{\beta(s-\tau_j(s))}x_k\left(s-\tau_j(s)\right)\right|ds.\end{aligned} \tag{29}$$

Let $t > N + \tau$; we can get by (24)

$$\begin{aligned}&\int_0^t e^{(\int_0^s \Upsilon_j(l)dl-\beta s)}\left|e^{\beta(s-\tau_j(s))}x_k\left(s-\tau_j(s)\right)\right|ds \\ &\quad= \int_0^{N+\tau} e^{(\int_0^s \Upsilon_j(l)dl-\beta s)}\left|e^{\beta(s-\tau_j(s))}x_k\left(s-\tau_j(s)\right)\right|ds \\ &\quad+ \int_{N+\tau}^t e^{(\int_0^s \Upsilon_j(l)dl-\beta s)}\left|e^{\beta(s-\tau_j(s))}x_k\left(s-\tau_j(s)\right)\right|ds \\ &\quad\leqslant \sup_{s\in[0,N+\tau]}\left|e^{\beta(s-\tau_j(s))}x_k\left(s-\tau_j(s)\right)\right| \\ &\quad\cdot\int_0^{N+\tau} e^{(\int_0^s \Upsilon_j(l)dl-\beta s)}ds + \int_{N+\tau}^t e^{(\int_0^s \Upsilon_j(l)dl-\beta s)}\varepsilon\, ds \\ &\quad\leqslant \sup_{s\in[-\tau,N+\tau]}\left|e^{\beta s}x_k(s)\right|(N+\tau) \\ &\quad\cdot\max_{s\in[-\tau,N+\tau]}e^{(\int_0^s \Upsilon_j(l)dl-\beta s)} + \varepsilon\frac{e^{(\int_0^t \Upsilon_j(s)ds-\beta t)}}{\gamma_j-\beta}.\end{aligned} \tag{30}$$

Combining (29) and (30) results in

$$\begin{aligned}Q_3 &\leqslant e^{\beta\tau}e^{-(\int_0^t \Upsilon_j(s)ds-\beta t)}\overline{a}_j\sum_{k=1}^n\left|d_{jk}\right|G_k \\ &\quad\cdot\int_0^t e^{(\int_0^s \Upsilon_j(l)dl-\beta s)}\left|e^{\beta(s-\tau_j(s))}x_k\left(s-\tau_j(s)\right)\right|ds \\ &\leqslant\left(e^{\beta\tau}\overline{a}_j\sum_{k=1}^n\left|d_{jk}\right|G_k\sup_{s\in[-\tau,N+\tau]}\left|e^{\beta s}x_k(s)\right|(N+\tau)\right. \\ &\quad\left.\cdot\max_{s\in[-\tau,N+\tau]}e^{(\int_0^s \Upsilon_j(l)dl-\beta s)}\right)e^{-(\int_0^t \Upsilon_j(s)ds-\beta t)} \\ &\quad+\frac{e^{\beta\tau}\overline{a}_j\sum_{k=1}^n\left|d_{jk}\right|G_k}{\gamma_j-\beta}\varepsilon.\end{aligned} \tag{31}$$

Then the arbitrariness of ε yields that $Q_3 \to 0$, $t \to \infty$.

It follows by (H3) and (25) that

$$\begin{aligned}
Q_4 &= \left| e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} \sum_{0<t_i<t} \left(\rho_{ji}\left(x_j\left(t_i\right)\right)\right) e^{\int_0^{t_i} \Upsilon_j(s)ds} \right| \\
&\leqslant e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} \sum_{0<t_i<t} \left|\rho_{ji}\left(x_j\left(t_i\right)\right)\right| e^{\int_0^{t_i} \Upsilon_j(s)ds} \\
&\leqslant e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} \sum_{0<t_i<t} h_{ji} \left|x_j\left(t_i\right)\right| e^{\int_0^{t_i} \Upsilon_j(s)ds} \\
&= e^{-(\int_0^t \Upsilon_j(s)ds - \beta t)} \sum_{0<t_i\leqslant N} h_{ji} \left|x_j\left(t_i\right)\right| e^{\int_0^{t_i} \Upsilon_j(s)ds} \\
&\quad + e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} \sum_{N<t_i<t} h_{ji} \left|x_j\left(t_i\right)\right| e^{\int_0^{t_i} \Upsilon_j(s)ds} \\
&= e^{-(\int_0^t \Upsilon_j(s)ds - \beta t)} \sum_{0<t_i\leqslant N} h_{ji} \left|x_j\left(t_i\right)\right| e^{\int_0^{t_i} \Upsilon_j(s)ds} \\
&\quad + e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} \sum_{N<t_i<t} h_{ji} \left|x_j\left(t_i\right)\right| e^{\int_0^{t_i} \Upsilon_j(s)ds}.
\end{aligned} \tag{32}$$

In addition,

$$\frac{d}{dt}\left(\int_0^t \Upsilon_j(s)\,ds - \beta t\right) = \Upsilon_j(t) - \beta \geqslant \gamma - \beta > 0, \tag{33}$$

which implies the function $(\int_0^t \Upsilon_j(s)ds - \beta t)$ is an increasing function on $t \geqslant 0$. This yields

$$\begin{aligned}
& e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} \sum_{N<t_i<t} h_{ji} \left|x_j\left(t_i\right)\right| e^{\int_0^{t_i} \Upsilon_j(s)ds} \\
&= e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} \sum_{N<t_i\leqslant t_k} h_{ji} \left(\left|x_j\left(t_i\right)\right| e^{\beta t_i}\right) e^{(\int_0^{t_i} \Upsilon_j(s)ds - \beta t_i)} \\
&\leqslant e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} h_j \varepsilon \sum_{N<t_i\leqslant t_k} \mu e^{(\int_0^{t_i} \Upsilon_j(s)ds - \beta t_i)} \\
&\leqslant e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} h_j \varepsilon \left[\sum_{N<t_i<t_k} \left(e^{(\int_0^{t_i} \Upsilon_j(s)ds - \beta t_i)} \left(t_{i+1} - t_i\right)\right) \right. \\
&\left. + \mu e^{(\int_0^{t_k} \Upsilon_j(s)ds - \beta t_k)} \right] \\
&\leqslant e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} h_j \varepsilon \left[\int_N^t e^{(\int_0^s \Upsilon_j(l)dl - \beta s)} ds \right. \\
&\left. + \mu e^{(\int_0^t \Upsilon_j(s)ds - \beta t)} \right] \leqslant e^{\beta t} e^{-\int_0^t \Upsilon_j(s)ds} h_j \frac{\varepsilon}{\gamma - \beta} \\
&\cdot \int_N^t e^{(\int_0^s \Upsilon_j(l)dl - \beta s)} d\left(\int_0^s \Upsilon_j(l)\,dl - \beta s\right) + \mu h_j \varepsilon \\
&\leqslant h_j \frac{\varepsilon}{\gamma - \beta} + \mu h_j \varepsilon.
\end{aligned} \tag{34}$$

From (32) and (34), the arbitrariness of ε yields that $Q_4 \to 0$, $t \to \infty$.

So we can deduce from the above analysis that $|e^{\beta t}\Phi(x_j)| \to 0$, $t \to +\infty$, and hence the condition (c) is satisfied.

Next we are to prove that condition (d) holds in $\Phi(\Theta_j)$.

Indeed, for a given t_i, it follows from (20) that

$$\Phi(x)(t_i + r) - \Phi(x)(t_i) = P_1 + P_2, \tag{35}$$

where

$$\begin{aligned}
P_1 &= \left\{ e^{-\int_0^{t_i+r} \Upsilon_j(s)ds} \phi_j(0) \right. \\
&\quad + \int_0^{t_i+r} e^{-\int_s^{t_i+r} \Upsilon_j(l)dl} \left[a_j\left(x_j(s)\right) \right. \\
&\quad \left.\left. \cdot \sum_{k=1}^n \left(c_{jk} f_k\left(x_k(s)\right) + d_{jk} g_k\left(x_k\left(s - \tau_j(s)\right)\right)\right) \right] ds \right\} \\
&\quad - \left\{ e^{-\int_0^{t_i} \Upsilon_j(s)ds} \phi_j(0) + \int_0^{t_i} e^{-\int_s^{t_i} \Upsilon_j(l)dl} \left[a_j\left(x_j(s)\right) \right.\right. \\
&\quad \left.\left. \cdot \sum_{k=1}^n \left(c_{jk} f_k\left(x_k(s)\right) + d_{jk} g_k\left(x_k\left(s - \tau_j(s)\right)\right)\right) \right] ds \right\}, \\
P_2 &= e^{-\int_0^{t_i+r} \Upsilon_j(s)ds} \sum_{0<t_k<t_i+r} \left(\rho_{jk}\left(x_j\left(t_k\right)\right)\right) e^{\int_0^{t_k} \Upsilon_j(s)ds} \\
&\quad - e^{-\int_0^{t_i} \Upsilon_j(s)ds} \sum_{0<t_k<t_i} \left(\rho_{jk}\left(x_j\left(t_k\right)\right)\right) e^{\int_0^{t_k} \Upsilon_j(s)ds}.
\end{aligned} \tag{36}$$

It is obvious that $P_1 \to 0$ as $r \to 0$. From (36), we can get by letting $r < 0$ be small enough

$$\begin{aligned}
P_2 &= \left(e^{-\int_0^{t_i+r} \Upsilon_j(s)ds} - e^{-\int_0^{t_i} \Upsilon_j(s)ds} \right) \\
&\quad \cdot \sum_{0<t_k<t_i} \left(\rho_{jk}\left(x_j\left(t_k\right)\right)\right) e^{\int_0^{t_k} \Upsilon_j(s)ds},
\end{aligned} \tag{37}$$

which implies that $\lim_{r\to 0^-} P_2 = 0$. On the other hand, letting $r > 0$ be small enough, we have

$$\begin{aligned}
P_2 &= e^{-\int_0^{t_i+r} \Upsilon_j(s)ds} \left[\sum_{0<t_k<t_i} \left(\rho_{jk}\left(x_j\left(t_k\right)\right) e^{\int_0^{t_k} \Upsilon_j(s)ds} \right) \right. \\
&\quad \left. + \rho_{ji}\left(x_j\left(t_i\right)\right) e^{\int_0^{t_i} \Upsilon_j(s)ds} \right] \\
&\quad - e^{-\int_0^{t_i} \Upsilon_j(s)ds} \sum_{0<t_k<t_i} \left(\rho_{jk}\left(x_j\left(t_k\right)\right)\right) e^{\int_0^{t_k} \Upsilon_j(s)ds} \\
&= \left(e^{-\int_0^{t_i+r} \Upsilon_j(s)ds} - e^{-\int_0^{t_i} \Upsilon_j(s)ds} \right) \\
&\quad \cdot \sum_{0<t_k<t_i} \left(\rho_{jk}\left(x_j\left(t_k\right)\right) e^{\int_0^{t_k} \Upsilon_j(s)ds} \right) \\
&\quad + e^{-\int_0^{t_i+r} \Upsilon_j(s)ds} \rho_{ji}\left(x_j\left(t_i\right)\right) e^{\int_0^{t_i} \Upsilon_j(s)ds},
\end{aligned} \tag{38}$$

which leads to $\lim_{r\to 0^+} P_2 = \rho_{ji}(x_j(t_i))$. Hence, condition (d) holds in $\Phi(\Theta_j)$.

It is derived by the above analysis that $\Phi(\Theta_j) \subset \Theta_j$ for all j.

Step 3. We claim that Φ is a contraction mapping on Θ.

Indeed, for $x(t) = (x_1(t),\dots,x_j(t),\dots,x_n(t))^T \in \Theta$ and $y(t) = (y_1(t),\dots,y_j(t),\dots,y_n(t))^T \in \Theta$, we have

$$\left|\Phi\left(x_j\right)(t) - \Phi\left(y_j\right)(t)\right| \leqslant J_{j1} + J_{j2} + J_{j3}, \tag{39}$$

where

$$\begin{aligned}
J_{j1} &= \int_0^t e^{-\int_s^t \Upsilon_j(l)dl}\left|a_j\left(x_j(s)\right)\sum_{k=1}^n c_{jk} f_k\left(x_k(s)\right) - a_j\left(y_j(s)\right)\sum_{k=1}^n c_{jk} f_k\left(y_k(s)\right)\right| ds, \\
J_{j2} &= \int_0^t e^{-\int_s^t \Upsilon_j(l)dl}\left|a_j\left(x_j(s)\right)\cdot\sum_{k=1}^n d_{jk} g_k\left(x_k\left(s-\tau_j(s)\right)\right) - a_j\left(y_j(s)\right)\cdot\sum_{k=1}^n d_{jk} g_k\left(y_k\left(s-\tau_j(s)\right)\right)\right| ds, \\
J_{j3} &= e^{-\int_0^t \Upsilon_j(s)ds}\sum_{0<t_i<t}\left(\left|\rho_{ji}\left(x_j\left(t_i\right)\right) - \rho_{ji}\left(y_j\left(t_i\right)\right)\right|\cdot e^{\int_0^{t_i}\Upsilon_j(s)ds}\right).
\end{aligned} \tag{40}$$

It follows by the triangle inequality, (H1), (H2), and the differential mean value theorem that

$$\begin{aligned}
&\left|a_j\left(x_j(s)\right)\sum_{k=1}^n c_{jk} f_k\left(x_k(s)\right) - a_j\left(y_j(s)\right)\sum_{k=1}^n c_{jk} f_k\left(y_k(s)\right)\right| \leqslant \sum_{k=1}^n\left[\left|c_{jk}\right|\left(\left|a_j\left(x_j(s)\right) f_k\left(x_k(s)\right) - a_j\left(x_j(s)\right) f_k\left(y_k(s)\right)\right|\right.\right. \\
&\quad \left.\left.+ \left|a_j\left(x_j(s)\right) f_k\left(y_k(s)\right) - a_j\left(y_j(s)\right) f_k\left(y_k(s)\right)\right|\right)\right] \leqslant \sum_{k=1}^n\left(\left|c_{jk}\right|\overline{a}_j F_k\left|x_k(s) - y_k(s)\right|\right) + \left(\sum_{k=1}^n\left|c_{jk}\right|\overline{F}_k\right)\widetilde{a}_j\left|x_j(s)\right. \\
&\quad \left.- y_j(s)\right|, \\
&\left|a_j\left(x_j(s)\right)\sum_{k=1}^n d_{jk} g_k\left(x_k\left(s-\tau_j(s)\right)\right) - a_j\left(y_j(s)\right)\sum_{k=1}^n d_{jk} g_k\left(y_k\left(s-\tau_j(s)\right)\right)\right| \leqslant \sum_{k=1}^n\left[\left|d_{jk}\right|\right. \\
&\quad \cdot\left(\left|a_j\left(x_j(s)\right) g_k\left(x_k\left(s-\tau_j(s)\right)\right) - a_j\left(x_j(s)\right) g_k\left(y_k\left(s-\tau_j(s)\right)\right)\right|\right. \\
&\quad \left.\left.+ \left|a_j\left(x_j(s)\right) g_k\left(y_k\left(s-\tau_j(s)\right)\right) - a_j\left(y_j(s)\right) g_k\left(y_k\left(s-\tau_j(s)\right)\right)\right|\right)\right] \leqslant \sum_{k=1}^n\left(\left|d_{jk}\right|\overline{a}_j G_k\left|x_k\left(s-\tau_j(s)\right)\right.\right. \\
&\quad \left.\left.- y_k\left(s-\tau_j(s)\right)\right|\right) + \left(\sum_{k=1}^n\left|d_{jk}\right|\overline{G}_k\right)\widetilde{a}_j\left|x_j(s) - y_j(s)\right|.
\end{aligned} \tag{41}$$

On the other hand,

$$\begin{aligned}
\int_0^t e^{\int_0^s \Upsilon_j(l)dl} ds &= \int_0^t \frac{e^{\int_0^s \Upsilon_j(l)dl}}{\Upsilon_j(s)} d\left(\int_0^s \Upsilon_j(l)\, dl\right) \\
&\leqslant \int_0^t \frac{e^{\int_0^s \Upsilon_j(l)dl}}{\gamma_j} d\left(\int_0^s \Upsilon_j(l)\, dl\right) \\
&\leqslant \frac{e^{\int_0^t \Upsilon_j(s)ds}}{\gamma_j}.
\end{aligned} \tag{42}$$

So we can get by the above analysis

$$\begin{aligned}
J_{j1} &\leqslant \int_0^t e^{-\int_s^t \Upsilon_j(l)dl}\left[\sum_{k=1}^n\left(\left|c_{jk}\right|\overline{a}_j F_k\left|x_k(s) - y_k(s)\right|\right) + \left(\sum_{k=1}^n\left|c_{jk}\right|\overline{F}_k\right)\widetilde{a}_j\left|x_j(s) - y_j(s)\right|\right] ds \\
&\leqslant e^{-\int_0^t \Upsilon_j(s)ds}\left[\sum_{k=1}^n\left(\left|c_{jk}\right|\overline{a}_j F_k \sup_{s\geqslant -\tau}\left|x_k(s) - y_k(s)\right|\right) + \left(\sum_{k=1}^n\left|c_{jk}\right|\overline{F}_k\right)\widetilde{a}_j \sup_{s\geqslant -\tau}\left|x_j(s) - y_j(s)\right|\right]\int_0^t e^{\int_0^s \Upsilon_j(l)dl} ds
\end{aligned}$$

$$\leqslant \frac{\left(\max_{k\in\mathcal{N}}\left|c_{jk}\right|F_k\right)\overline{a}_j}{\gamma_j}\operatorname{dist}(x,y)+\frac{\left(\sum_{k=1}^{n}\left|c_{jk}\right|\overline{F}_k\right)\widetilde{a}_j}{\gamma_j}\sup_{t\geqslant-\tau}\left|x_j(t)-y_j(t)\right|,$$

$$J_{j2}\leqslant\int_0^t e^{-\int_s^t\Upsilon_j(l)dl}\left[\sum_{k=1}^{n}\left(\left|d_{jk}\right|\overline{a}_jG_k\left|x_k\left(s-\tau_j(s)\right)-y_k\left(s-\tau_j(s)\right)\right|\right)+\left(\sum_{k=1}^{n}\left|d_{jk}\right|\overline{G}_k\right)\widetilde{a}_j\left|x_j(s)-y_j(s)\right|\right]ds$$

$$\leqslant e^{-\int_0^t\Upsilon_j(s)ds}\left[\sum_{k=1}^{n}\left(\left|d_{jk}\right|\overline{a}_jG_k\sup_{s\geqslant-\tau}\left|x_k(s)-y_k(s)\right|\right)+\left(\sum_{k=1}^{n}\left|d_{jk}\right|\overline{G}_k\right)\widetilde{a}_j\sup_{s\geqslant-\tau}\left|x_j(s)-y_j(s)\right|\right]\int_0^t e^{\int_0^s\Upsilon_j(l)dl}ds$$

$$\leqslant\frac{\max_{k\in\mathcal{N}}\left(\left|d_{jk}\right|G_k\right)\overline{a}_j}{\gamma_j}\operatorname{dist}(x,y)+\frac{\left(\sum_{k=1}^{n}\left|d_{jk}\right|\overline{G}_k\right)\widetilde{a}_j}{\gamma_j}\sup_{t\geqslant-\tau}\left|x_j(t)-y_j(t)\right|,$$

$$J_{j3}\leqslant e^{-\int_0^t\Upsilon_j(s)ds}\sum_{0<t_i<t}\left(h_{ji}\left|x_j(t_i)-y_j(t_i)\right|e^{\int_0^{t_i}\Upsilon_j(s)ds}\right)\leqslant h_je^{-\int_0^t\Upsilon_j(s)ds}\sup_{t\geqslant-\tau}\left|x_j(t_i)-y_j(t_i)\right|\sum_{0<t_i<t}\left(\mu e^{\int_0^{t_i}\Upsilon_j(s)ds}\right)$$

$$\leqslant h_je^{-\int_0^t\Upsilon_j(s)ds}\sup_{t\geqslant-\tau}\left|x_j(t_i)-y_j(t_i)\right|\left(\sum_{0<t_i<t_k}e^{\int_0^{t_i}\Upsilon_j(s)ds}\left(t_{r+1}-t_r\right)+\mu e^{\int_0^{t_k}\Upsilon_j(s)ds}\right)\leqslant h_je^{-\int_0^t\Upsilon_j(s)ds}\sup_{t\geqslant-\tau}\left|x_j(t_i)-y_j(t_i)\right|$$

$$\cdot\left(\int_0^t e^{\int_0^s\Upsilon_j(l)dl}ds+\mu e^{\int_0^t\Upsilon_j(s)ds}\right)\leqslant h_j\left(\frac{1}{\gamma_j}+\mu\right)\sup_{t\geqslant-\tau}\left|x_j(t_i)-y_j(t_i)\right|. \tag{43}$$

So we have

$$\begin{aligned}\sup_{t\geqslant-\tau}&\left|\Phi(x_j)(t)-\Phi(y_j)(t)\right|\leqslant\sup_{t\geqslant-\tau}\left(J_{j1}+J_{j2}+J_{j3}\right)\\
&\leqslant\frac{\left(\max_{k\in\mathcal{N}}\left|c_{jk}\right|F_k\right)\overline{a}_j}{\gamma_j}\operatorname{dist}(x,y)\\
&+\frac{\left(\sum_{k=1}^{n}\left|c_{jk}\right|\overline{F}_k\right)\widetilde{a}_j}{\gamma_j}\sup_{t\geqslant-\tau}\left|x_j(t)-y_j(t)\right|\\
&+\frac{\max_{k\in\mathcal{N}}\left(\left|d_{jk}\right|G_k\right)\overline{a}_j}{\gamma_j}\operatorname{dist}(x,y)\\
&+\frac{\left(\sum_{k=1}^{n}\left|d_{jk}\right|\overline{G}_k\right)\widetilde{a}_j}{\gamma_j}\sup_{t\geqslant-\tau}\left|x_j(t)-y_j(t)\right|\\
&+h_j\left(\frac{1}{\gamma_j}+\mu\right)\sup_{t\geqslant-\tau}\left|x_j(t_i)-y_j(t_i)\right|\\
&\leqslant\frac{\left(\max_{k\in\mathcal{N}}\left|c_{jk}\right|F_k\right)\overline{a}_j+\max_{k\in\mathcal{N}}\left(\left|d_{jk}\right|G_k\right)\overline{a}_j}{\gamma_j}\\
&\cdot\operatorname{dist}(x,y)\\
&+\frac{\left(\sum_{k=1}^{n}\left|c_{jk}\right|\overline{F}_k\right)\widetilde{a}_j+\left(\sum_{k=1}^{n}\left|d_{jk}\right|\overline{G}_k\right)\widetilde{a}_j+h_j+\mu\gamma_j}{\gamma_j}\\
&\cdot\sup_{t\geqslant-\tau}\left|x_j(t)-y_j(t)\right|.\end{aligned}\tag{44}$$

Therefore, we can derive by the above analysis and the LMI condition

$$\begin{aligned}\operatorname{dist}(\Phi(x),\Phi(y))&=\sum_{j=1}^{n}\sup_{t\geqslant-\tau}\left|\Phi(x_j)(t)-\Phi(y_j)(t)\right|\\
&\leqslant\sum_{j=1}^{n}\left\{\frac{\left(\max_{k\in\mathcal{N}}\left|c_{jk}\right|F_k\right)\overline{a}_j+\max_{k\in\mathcal{N}}\left(\left|d_{jk}\right|G_k\right)\overline{a}_j}{\gamma_j}\right.\\
&\cdot\operatorname{dist}(x,y)\\
&+\frac{\left(\sum_{k=1}^{n}\left|c_{jk}\right|\overline{F}_k\right)\widetilde{a}_j+\left(\sum_{k=1}^{n}\left|d_{jk}\right|\overline{G}_k\right)\widetilde{a}_j+h_j+\mu\gamma_j}{\gamma_j}\\
&\left.\cdot\sup_{t\geqslant-\tau}\left|x_j(t)-y_j(t)\right|\right\}\leqslant\alpha\operatorname{dist}(x,y),\end{aligned}\tag{45}$$

which implies that Φ is a contraction mapping on Θ. And then the contraction mapping theorem yields the idea that Φ has the fixed point $x(t)$ on Θ, which implies that $x(t)$ is the solution for system (1), satisfying $e^{\beta t}\|x(t)\| \to 0$ as $t \to \infty$. And then the proof is completed. □

Remark 6. As far as we can know, there is not any previous literature related to the fixed point theory where LMI-based stability criteria were presented, except for [15]. In this paper, the LMI-based stability criterion is the first time to be proposed for impulsive CGNNs via fixed point theorems. This is another main contribution of this paper.

3. Numerical Example

Example 1. Consider

$$\frac{dx_1(t)}{dt} = -a_1(x_1(t))\Big\{b_1(x_1(t)) - \sum_{k=1}^{2}[c_{1k}f_k(x_k(t)) + d_{1k}g_k(x_k(t-\tau_1(t)))]\Big\}, \quad t \geqslant 0,\ t \neq t_i,$$
$$\frac{dx_2(t)}{dt} = -a_2(x_2(t))\Big\{b_2(x_2(t)) - \sum_{k=1}^{2}[c_{2k}f_k(x_k(t)) + d_{2k}g_k(x_k(t-\tau_2(t)))]\Big\}, \quad t \geqslant 0,\ t \neq t_i,$$
$$\Delta x_j(t_i) = x_j(t_i^+) - x_j(t_i) = \rho_{ji}(x_j(t_i)), \quad i = 1, 2, \ldots,$$
$$x_j(\theta) = \phi_j(\theta), \quad -\tau \leqslant \theta \leqslant 0,\ j \in \mathcal{N}, \tag{46}$$

where $j \in \mathcal{N} \triangleq \{1, 2\}$, $b_1(x_1(t)) = 2x_1(t)$, $b_2(x_2(t)) = 2x_2(t)$, $a_1(x_1(t)) = a_2(x_2(t)) = 2 + \cos t$, and then $a_j(r)b_j(r) = 2(2 + \cos t)r = \Upsilon_j(t)r$; hence $\Upsilon_j(t) = 2(2 + \cos t)$. Let $\gamma_j = 2$, and then $\Upsilon_j(t) = 2(2 + \cos t) \geqslant \gamma_j > 0$, let $\overline{a}_j = 3$; then $0 < a_j(r) = (2 + \cos t) \cdot r^0 \leqslant \overline{a}_j$, and $a_j'(r) = 0$. So we let $\tilde{a}_j = 0$, and $|a_j'(r)| \leqslant \tilde{a}_j$. Let $\int_0^t \Upsilon_j(s)ds = 4t + 2\sin t$. Let $\beta = 1 > 0$, and then $\liminf_{t\to+\infty}(\int_0^t \Upsilon_j(s)ds/t) = 4 > \beta$ and $\gamma_j > \beta$. Let $f_i(r) = \sin^2(r/10) = g_i(r)$, $i = 1, 2$; then $\overline{F}_i = \overline{G}_i = 1$, $b_i(0) = f_i(0) = g_i(0) = 0$, and $F_i = G_i = 0.2$, $i = 1, 2$. Define $\rho_{ji} = \arctan(1.7x_j(t_i))$ and $t_i - t_{i-1} = 0.5i$ for $j = 1, 2$, $i = 1, 2, \ldots$. Let $h_{ji} = 1.7$, $h_j = 3.4$, and $\mu = 0.5$, and then $h_{ji} \leqslant h_j\mu$ holds.

In addition, let $c_{jk} = d_{jk} = 0.05$ for $j, k \in \mathcal{N}$; then we can use Matlab LMI toolbox to solve the LMI condition in Theorem 4 and obtain

$$\alpha = 0.8351 < 1. \tag{47}$$

According to Theorem 4, system (46) is globally exponentially stable.

Remark 7. To compare the upper bounds of time delay and impulse in various related literature, we need to compute and compare the ratios between the maximum of allowable impulse and maximum of parameters in various related literature, because the maximum of allowable impulse may rise as parameters of numerical examples become bigger. From Table 1, the ratios of Example 1 (ours) are bigger than those of [1, 4, 10, 11] to some extent. In addition, impulse value of [9, Example 4.1] is less than 1 while ours of Example 1 is $1.7 > 1$. It is well known that bigger impulse gives bigger influence on the stability. Thereby, the large allowable variation range of impulse illustrates the effectiveness of our new stability criterion.

Table 1: Ratios between allowable upper bounds of impulse $\mathcal{C}$ and maximum of parameters $\mathcal{P}$.

	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{C}/\mathcal{P}$
Example 1 (ours)	1.7	2	0.850
[1, Example 1]	1	3	0.333
[4, Example 1]	0.8	3.5	0.2286
[9, Example 4.1]	0.55	0.2	2.75
[10, Example Case 1]	0.81	4.8	0.1688
[11, Example]	0.7	6	0.1167

Remark 8. The super limit τ of time delays is actually infinite.

Remark 9. As far as we know, it is the first time to use the contraction mapping theorem to prove the stability for CGNNs. The method employed in this paper is different from those of related literature [20, 21].

Remark 10. Of course, [1, 4, 9–11] involved more complicated systems than ours, such as reaction-diffusion phenomena, stochastic Markovian jumping, and parameters uncertainty. Moreover, fixed point theories may not be employed to such complicated systems. Thereby, we can not claim that the new stability criterion is better than the criteria of [1, 4, 9–11] in all aspects, for those results obtained in [1, 4, 9–11] solved what Theorem 4 (ours) cannot involve. However, a further research on applications of fixed point theories may have the widest appeal of all. This is another purpose of writing this paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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