

Research Article

Hedging Long-Term Exposures of a Well-Diversified Portfolio with Short-Term Stock Index Futures Contracts

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It is difficult for passive portfolio strategy to manage the long-term exposure of a well-diversified portfolio because stock index futures contracts have a finite life limited by their maturity. In this paper, we investigate the problem of the rollover hedge strategy for the long-term exposure of a well-diversified portfolio. First, we consider the rollover hedge strategy for the well-diversified portfolio when the portfolio is not adjusted during the period. In order to obtain the optimal solution of the proposed model, the auxiliary models are constructed using the equivalent transformation technique. Moreover, dynamic programming is employed to derive the optimal positions of stock index futures contracts for the long-term exposure of the well-diversified portfolio. In addition, we extend the result to the case of the rollover hedge strategy with transaction costs and derive the optimal number of stock index futures contracts.

1. Introduction

Stock index futures contracts are often used for the risk management because they have low transaction costs and counterparty risk is small. A key question for stock index futures hedging is how to determine the optimal hedge ratio or the optimal number of stock index futures contracts. The vast majority of the hedging literature has focused on the key question. One of the most widely used methods for calculating the optimal hedge ratio is based on the minimization of the variance of the hedged portfolio. One of the first studies was by Figlewski [1]. He calculated the minimum variance hedge ratio (MVHR) by the ordinary least squares (OLS) method on historical spot and futures returns. On this basis, some studies (e.g., the literature [2–4]) examined the effectiveness of hedges with duration effect and expiration effects. Lee and Chien [5] investigated the impact of stock market liquidity on the hedging performance of stock index futures by the conditional OLS model with stock market liquidity. Ghosh [6] applied an error correction

model (ECM) to capture the lead and lag relationships between stock index and stock index futures and estimated the optimal hedge ratio. He found that ECM incorporated the long-run equilibrium relationship as well as the short-run dynamics and had significant improvement in estimating the optimal hedge ratio and reducing the risk of risk minimizing portfolio. Eftekhari [7] and Lien and Tse [8] minimized downside risk of the hedged portfolio to obtain the optimal hedge ratio. Chen et al. [9] adopted the mean-generalized semivariance approach to determine the hedge ratio. These studies estimated the optimal hedge ratio under the assumption that the hedge ratio was fixed at the optimum level and was not revised during the hedging period.

The fixed hedge ratio might be inappropriate because the variance and covariance matrix of the spot and futures returns are time-varying over time. Under such circumstances, many dynamic models were proposed to estimate the time-varying hedge ratio. Specifically, the GARCH models are widely used for index futures hedging. The bivariate GARCH models are employed to design the dynamic hedging

strategy (e.g., the literature [10–17]). The results showed that the hedging strategy based on the bivariate GARCH estimation had better hedging performance than the constant hedging strategy. Sim and Zurbrugg [18] and Wang and Hsu [19] developed error-corrected GARCH models to investigate the dynamic hedging effectiveness in index futures hedging. Yang and Allen [20] and Floros and Vougas [21] examined the hedging effectiveness of multivariate GARCH hedge ratios in Australian and Greek stock index futures market, respectively. Pok et al. [22] and Aragón and Salvador [23] used asymmetric GARCH models to investigate the performance of dynamic hedging strategy.

In addition to the GARCH hedging model, there are several studies using other methods to generate the optimal hedge ratio, such as Markov regime switching approach [24], the exponential weighted moving average (EMWA) method [25], the dynamic hedging algorithm based on a Reverse Order CUSUM-squared testing procedure [26], time-varying copulas methodology [27, 28], and autoregressive conditional density (ARCD) models [29].

Although the derivation methods for the time-varying hedge ratio capture the time-varying second moments of financial time series [17], they mainly focused on hedging with one index futures contract. As we know, stock index futures contracts have a finite life span and the most liquid contracts tend to be of short maturity. It is difficult for investors to hedge a long-term exposure with one futures contract. In order to solve the problem, a portfolio of nearby futures contracts, each with different maturities, is used to hedge the long-term exposure. This is the so-called “rollover hedge” strategy.

The “rollover hedge” strategy is mainly applied in hedging the risk of commodities (e.g., the literature [30–34]). But it is strange that the “rollover hedge” strategy is seldom used for stock index futures hedging. The only research of which we are aware that considered the rollover hedge with index futures contracts is by Carchano and Pardo [35]. They devised a rollover hedge strategy with five criteria to manage the long-term exposure of single stock index. However, the issue of hedging the long-term exposure of a well-diversified portfolio was not considered in their studies.

This paper is also closely related to the passive portfolio management. Passive portfolio management is an important strategy in financial investments. The idea of the strategy is to make a well-diversified portfolio with low turnover. Nevertheless, the well-diversified portfolio can do nothing for the systematic risk of the portfolio while it diversifies the unsystematic risk of each security out of the portfolio. In addition, passive portfolio management usually holds the well-diversified portfolio for a long time to reduce the transaction costs. Therefore, it is necessary to discuss the problem of hedging the long-term exposure of a well-diversified portfolio.

The main purpose of this paper is to investigate the rollover hedge for the long-term exposure of the well-diversified portfolio. A new theoretical model is proposed to design the rollover hedge strategy. The rest of the paper is organized as follows. Section 2 presents how to construct the rollover hedging model for the well-diversified portfolio.

Section 3 describes how to solve the proposed model by the equivalent transformation technique and dynamic programming. Section 4 discusses the rollover hedging problem with transaction costs. Finally, we draw a conclusion in Section 5.

2. The Rollover Hedging Model for the Well-Diversified Portfolio

In general, passive management funds, such as Barclays Global Investors and State Street Corp., primarily hold a well-diversified portfolio for a long time. To hedge against the risk exposure of the portfolio, they can roll over short-term contracts in stock index futures market. This section lays out the hedge problem of the passive management fund and constructs a new model for hedging the long-term exposure of the well-diversified portfolio without adjustment.

First, we will introduce the rollover hedge strategy with stock index futures. It follows the step below to manage the long-term risk of the portfolio. At the beginning of each period, the passive management fund takes a position in stock index futures contract nearest to expiration. At the end of each period, the former nearby contract is closed out and the subsequent nearby contract is opened. This process is continued over the life of the well-diversified portfolio.

Next, we start to build the new model. The available capital is divided into three separate parts. The first part is invested in the well-diversified portfolio that is not adjusted during the whole investment period. The second part is used for stock index futures contracts. The third part is deposited in the saving account of the bank to deal with the possible liquidity risk.

For the sake of clarity, we use the following notations:

V_0 : the available investment capital for a passive management fund,

Q_j : the number of stock j during the investment period,

$S_{j,t}$: the price of stock j at time t ,

F_t : the price of stock index futures contract at time t ,

Z : the unit of stock index futures contract,

H_{t-1} : the number of stock index futures contracts for hedging the long-term exposure,

α : the ratio of the maintenance margin,

$W_{S,t}$: the capital invested in the well-diversified portfolio at time t ,

$W_{F,t}$: the capital invested in stock index futures contracts at time t ,

$W_{C,t}$: the saving in the bank at time t ,

M_t : the total amount of $W_{F,t}$ and $W_{C,t}$.

At time $t = t_0$, the amount of the well-diversified portfolio is

$$W_{S,t_0} = \sum_{j=1}^n Q_j S_{j,t_0}. \quad (1)$$

After the well-diversified portfolio is constructed, the total amount of W_{F,t_0} and W_{C,t_0} is

$$M_0 = V_0 - W_{S,t_0} = W_{F,t_0} + W_{C,t_0} = H_0 Z F_{t_0} \alpha + W_{C,t_0}, \quad (2)$$

where W_{F,t_0} and W_{C,t_0} are the amount of stock index futures contracts and the saving in the bank, respectively.

At time $t = t_{i-1}$, that is, the beginning of the period i , the amounts of stock index futures contracts and the saving in the bank are $W_{F,t_{i-1}} = H_{i-1} Z F_{t_{i-1}} \alpha$ and $W_{C,t_{i-1}}$, respectively. Then the total amount of $W_{F,t_{i-1}}$ and $W_{C,t_{i-1}}$ is

$$M_{i-1} = W_{F,t_{i-1}} + W_{C,t_{i-1}} = H_{i-1} Z F_{t_{i-1}} \alpha + W_{C,t_{i-1}}. \quad (3)$$

At time $t = t_i$, the amounts of stock index futures contracts are expressed by

$$\begin{aligned} W_{F,t_i} &= H_{i-1} Z F_{t_{i-1}} \alpha + H_{i-1} Z F_{t_{i-1}} - H_{i-1} Z F_{t_i} \\ &= H_{i-1} Z [F_{t_{i-1}} (1 + \alpha) - F_{t_i}]. \end{aligned} \quad (4)$$

At time $t = t_i$, that is, the end of the period i , the total amount of W_{F,t_i} and W_{C,t_i} needs to be discussed in two cases because liquidity risk might happen; that is, the balance of the futures account falls below the maintenance margin and the added margin is needed.

If the firm does not face liquidity risk, the saving at time $t = t_i$ is

$$W_{C,t_i} = W_{C,t_{i-1}} (1 + r_{C,i}), \quad (5)$$

where $r_{C,i}$ is the interest rate of current deposits from $t = t_{i-1}$ to $t = t_i$.

In the case without liquidity risk, the total amount of W_{F,t_i} and W_{C,t_i} is

$$\begin{aligned} M_i &= W_{F,t_i} + W_{C,t_i} \\ &= H_{i-1} Z [F_{t_{i-1}} (1 + \alpha) - F_{t_i}] + W_{C,t_{i-1}} (1 + r_{C,i}). \end{aligned} \quad (6)$$

If the firm receives a margin call and is expected to pay additional margin into the futures account to maintain its position, the amount of the rest capital at time $t = t_i$ is different from the case without liquidity risk. The proposed method by Fu et al. [36] is used to calculate the additional margin.

From $t = t_{i-1}$ to $t = t_i$, we assume that there are N_i trading days that the balance of the account falls below the deposit required to maintain the position H_{i-1} . Let $F_{t_{i-1}+g_i(k)}$ be the settlement price of stock index futures contracts on the k th trading day ($k = 1, 2, \dots, N_i, N_i \geq 1$) that the manufacturer receives a margin call, and $g_i(k)$ is the number of the trading days between the k th trading day with a margin call and the former with a margin call. In addition, $F_{t_{i-1}+g_i(k)}$ should satisfy

$$\begin{aligned} F_{t_{i-1}} &< F_{t_{i-1}+g_i(1)} < F_{t_{i-1}+g_i(2)} < \dots \\ &< F_{t_{i-1}+g_i(N_i)}, t_{i-1} + g_i(N_i) < t_i. \end{aligned} \quad (7)$$

Let $g_i(0) = 0$; the inequality becomes

$$\begin{aligned} F_{t_{i-1}} &= F_{t_{i-1}+g_i(0)} < F_{t_{i-1}+g_i(1)} \\ &< F_{t_{i-1}+g_i(2)} < \dots < F_{t_{i-1}+g_i(N_i)}. \end{aligned} \quad (8)$$

At time $t = t_{i-1} + g_i(1)$, the balance of the stock index futures account is

$$H_{i-1} F_{t_{i-1}+g_i(0)} \alpha + H_{i-1} F_{t_{i-1}+g_i(0)} - H_{i-1} F_{t_{i-1}+g_i(1)}. \quad (9)$$

Then the maintenance margin of stock index futures contracts is $H_{i-1} F_{t_{i-1}+g_i(1)} \alpha$. Obviously, the balance of the account is less than the maintenance margin of the position H_{i-1} . In order to maintain the position H_{i-1} , the amount of the additional margin at time $t_{i-1} + g_i(1)$ is

$$\begin{aligned} \text{AM}_{t_{i-1}+g_i(1)} &= H_{i-1} F_{t_{i-1}+g_i(1)} \alpha \\ &\quad - H_{i-1} [F_{t_{i-1}+g_i(0)} \alpha + F_{t_{i-1}+g_i(0)} - F_{t_{i-1}+g_i(1)}] \\ &= H_{i-1} [F_{t_{i-1}+g_i(1)} - F_{t_{i-1}+g_i(0)}] (1 + \alpha). \end{aligned} \quad (10)$$

After depositing the additional margin $\text{AM}_{t_{i-1}+g_i(1)}$, the saving in the bank is

$$\begin{aligned} W_{C,t_{i-1}+g_i(1)} &= W_{C,t_{i-1}} [1 + r_i(1)] \\ &\quad - H_{i-1} [F_{t_{i-1}+g_i(1)} - F_{t_{i-1}+g_i(0)}] (1 + \alpha), \end{aligned} \quad (11)$$

where $r_i(1) = r \cdot g_i(1)/360$.

At time $t = t_{i-1} + g_i(2)$, the balance of the account $H_{i-1} [F_{t_{i-1}+g_i(1)} \alpha + F_{t_{i-1}+g_i(1)} - F_{t_{i-1}+g_i(2)}]$ is also less than the maintenance margin $H_{i-1} F_{t_{i-1}+g_i(2)} \alpha$. The amount of the additional margin should be

$$\begin{aligned} \text{AM}_{t_{i-1}+g_i(2)} &= H_{i-1} F_{t_{i-1}+g_i(2)} \alpha \\ &\quad - H_{i-1} [F_{t_{i-1}+g_i(1)} \alpha + F_{t_{i-1}+g_i(1)} - F_{t_{i-1}+g_i(2)}] \\ &= H_{i-1} [F_{t_{i-1}+g_i(2)} - F_{t_{i-1}+g_i(1)}] (1 + \alpha). \end{aligned} \quad (12)$$

Then the saving in the bank is

$$\begin{aligned} W_{C,t_{i-1}+g_i(2)} &= W_{C,t_{i-1}+g_i(1)} [1 + r_i(2)] \\ &\quad - H_{i-1} [F_{t_{i-1}+g_i(2)} - F_{t_{i-1}+g_i(1)}] (1 + \alpha) \\ &= \{W_{C,t_{i-1}} [1 + r_i(1)] - H_{i-1} (F_{t_{i-1}+g_i(1)} - F_{t_{i-1}+g_i(0)}) (1 + \alpha)\} \\ &\quad \times [1 + r_i(2)] - H_{i-1} [F_{t_{i-1}+g_i(2)} - F_{t_{i-1}+g_i(1)}] (1 + \alpha) \\ &= W_{C,t_{i-1}} \prod_{m=1}^2 [1 + r_i(m)] - H_{i-1} (F_{t_{i-1}+g_i(1)} - F_{t_{i-1}+g_i(0)}) \\ &\quad \times (1 + \alpha) [1 + r_i(2)] \\ &\quad - H_{i-1} [F_{t_{i-1}+g_i(2)} - F_{t_{i-1}+g_i(1)}] (1 + \alpha), \end{aligned} \quad (13)$$

where $r_i(2) = r \cdot g_i(2)/360$.

By analogy, the amounts of the additional margin and the saving at time $t_{i-1} + g_i(N_i)$ should be

$$\begin{aligned}
AM_{t_{i-1}+g_i(N_i)} &= H_{i-1} \left[F_{t_{i-1}+g_i(N_i)} - F_{t_{i-1}+g_i(N_{i-1})} \right] (1 + \alpha), \\
W_{C,t_{i-1}+g_i(N_i)} &= W_{C,t_{i-1}+g_i(N_{i-1})} \left[1 + r_i(N_i) \right] \\
&\quad - H_{i-1} \left[F_{t_{i-1}+g_i(N_i)} - F_{t_{i-1}+g_i(N_{i-1})} \right] (1 + \alpha) \\
&= \left\{ W_{C,t_{i-1}+g_i(N_{i-2})} \left[1 + r_i(N_i - 1) \right] \right. \\
&\quad \left. - H_{i-1} \left[F_{t_{i-1}+g_i(N_{i-1})} - F_{t_{i-1}+g_i(N_{i-2})} \right] (1 + \alpha) \right\} \\
&\quad \times \left[1 + r_i(N_i) \right] \\
&\quad - H_{i-1} \left[F_{t_{i-1}+g_i(N_i)} - F_{t_{i-1}+g_i(N_{i-1})} \right] (1 + \alpha) \\
&= W_{C,t_{i-1}+g_i(N_{i-2})} \prod_{m=N_{i-1}}^{N_i} \left[1 + r_i(j) \right] \\
&\quad - H_{i-1} \left[F_{t_{i-1}+g_i(N_{i-1})} - F_{t_{i-1}+g_i(N_{i-2})} \right] \\
&\quad \times (1 + \alpha) \left[1 + r_i(N_i) \right] \\
&\quad - H_{i-1} \left[F_{t_{i-1}+g_i(N_i)} - F_{t_{i-1}+g_i(N_{i-1})} \right] (1 + \alpha) \\
&\quad \vdots \\
&= W_{C,t_{i-1}} \prod_{m=1}^{N_i} \left[1 + r_i(j) \right] - H_{i-1} A_{i-1}, \tag{14}
\end{aligned}$$

where

$$\begin{aligned}
A_{i-1} &= \sum_{k=1}^{N_i} \left\{ \left[F_{t_{i-1}+g_i(k)} - F_{t_{i-1}+g_i(k-1)} \right] (1 + \alpha) \prod_{m=k+1}^{N_i} \left[1 + r_i(m) \right] \right\} \\
&\quad + \left[F_{t_{i-1}+g_i(N_i)} - F_{t_{i-1}+g_i(N_{i-1})} \right] (1 + \alpha). \tag{15}
\end{aligned}$$

At time $t = t_i$, the saving in the bank is given by

$$\begin{aligned}
W_{C,t_i} &= W_{C,t_{i-1}+g_i(N_i)} \left[1 + r_i(N_i + k_i) \right] \\
&= \left\{ W_{C,t_{i-1}} \prod_{m=1}^{N_i} \left[1 + r_i(j) \right] - H_{i-1} A_{i-1} \right\} \\
&\quad \times \left[1 + r_i(N_i + k_i) \right], \tag{16}
\end{aligned}$$

where $r_i(N_i + k_i) = r \cdot g_i(N_i + k_i)/360$ and $g_i(N_i + k_i)$ is the number of the trading days from time $t_{i-1} + g_i(N_i)$ to time t_i .

At the end of period i , the total amount of W_{F,t_i} and W_{C,t_i} with liquidity risk is

$$\begin{aligned}
M_i &= W_{F,t_i} + W_{C,t_i} \\
&= H_{i-1} Z \left[F_{t_{i-1}} (1 + \alpha) - F_{t_i} \right] \\
&\quad + \left[W_{C,t_{i-1}} \prod_{m=1}^{N_i} \left[1 + r_i(j) \right] - H_{i-1} A_{i-1} \right] \\
&\quad \times \left[1 + r_i(N_i + k_i) \right] \\
&= H_{i-1} Z \left[F_{t_{i-1}} (1 + \alpha) - F_{t_i} \right] \\
&\quad + W_{C,t_{i-1}} \left[1 + r_i(N_i + k_i) \right] \prod_{m=1}^{N_i} \left[1 + r_i(j) \right] \\
&\quad - H_{i-1} A_{i-1} \left[1 + r_i(N_i + k_i) \right] \\
&= H_{i-1} \left\{ Z \left[F_{t_{i-1}} (1 + \alpha) - F_{t_i} \right] - A_{i-1} \left[1 + r_i(N_i + k_i) \right] \right\} \\
&\quad + W_{C,t_{i-1}} \left[1 + r_i(N_i + k_i) \right] \prod_{m=1}^{N_i} \left[1 + r_i(j) \right]. \tag{17}
\end{aligned}$$

Let

$$\begin{aligned}
e_{F,i-1} &= \begin{cases} Z \left[F_{t_{i-1}} (1 + \alpha) - F_{t_i} \right], & \text{without liquidity risk,} \\ Z \left[F_{t_{i-1}} (1 + \alpha) - F_{t_i} \right] - A_{i-1} \left[1 + r_i(N_i + k_i) \right], & \text{with liquidity risk,} \end{cases} \\
e_{C,i-1} &= \begin{cases} 1 + r_{C,i}, & \text{without liquidity risk,} \\ \left[1 + r_i(N_i + k_i) \right] \prod_{m=1}^{N_i} \left[1 + r_i(j) \right], & \text{with liquidity risk.} \end{cases} \tag{18}
\end{aligned}$$

According to (6) and (17), the total amount of W_{F,t_i} and W_{C,t_i} at the end of the period i can be given by

$$M_i = H_{i-1} e_{F,i-1} + W_{C,t_{i-1}} e_{C,i-1}. \tag{19}$$

Substituting (3) into (19), the total amount of W_{F,t_i} and W_{C,t_i} is expressed by

$$\begin{aligned}
M_i &= H_{i-1} e_{F,i-1} + W_{C,t_{i-1}} e_{C,i-1} \\
&= H_{i-1} e_{F,i-1} + \left(M_{i-1} - H_{i-1} Z F_{t_{i-1}} \alpha \right) e_{C,i-1} \\
&= M_{i-1} e_{C,i-1} + H_{i-1} \left(e_{F,i-1} - Z F_{t_{i-1}} \alpha e_{C,i-1} \right). \tag{20}
\end{aligned}$$

Let $P_{i-1} = e_{F,i} - ZF_{t_{i-1}} \alpha e_{C,i-1}$. Then (20) can be simplified as

$$M_i = M_{i-1}e_{C,i-1} + H_{i-1}P_{i-1}. \quad (21)$$

During the whole period, the terminal profit of the investment is

$$\Delta R = (M_T + W_{S,t_T}) - (M_0 + W_{S,t_0}), \quad (22)$$

where $W_{S,t_T} = \sum_{j=1}^n Q_j S_{j,t_T}$.

The variance of terminal profit of the investment is

$$\begin{aligned} \text{Var}(\Delta R) &= \text{Var}(M_T + W_{S,t_T}) \\ &= \text{Var}(M_T) + 2\text{Cov}(M_T, W_{S,t_T}) + \text{Var}(W_{S,t_T}) \\ &= E(M_T^2) - E^2(M_T) + 2E(M_T W_{S,t_T}) \\ &\quad - 2E(W_{S,t_T})E(M_T) + \text{Var}(W_{S,t_T}). \end{aligned} \quad (23)$$

The firm chooses the number of index futures contracts by minimizing the variance of terminal profit of the investment:

$$\min \text{Var}(\Delta R) = \text{Var}(M_T + W_{S,t_T}). \quad (24)$$

Combining (24) with (21), we obtain the following model for hedging the long-term exposure of the well-diversified portfolio:

$$\begin{aligned} \min \quad & \text{Var}(M_T + W_{S,t_T}) \\ \text{s.t.} \quad & M_i = M_{i-1}e_{C,i-1} + H_{i-1}P_{i-1}. \end{aligned} \quad (25)$$

3. The Analytical Solution for the Proposed Model

This section derives the minimum-risk hedge positions for the rollover hedge strategies. The method by Li and Ng [37] is employed. Model (25) is nonseparable in the sense of dynamic programming. It is necessary for the optimal solutions of the proposed model to transform model (25) into a separable model.

By equivalent transformation of the objective function, model (25) is rewritten as

$$\begin{aligned} \max \quad & -\text{Var}(M_T + W_{S,t_T}) \\ \text{s.t.} \quad & M_i = M_{i-1}e_{C,i-1} + H_{i-1}P_{i-1}. \end{aligned} \quad (26)$$

Let

$$\begin{aligned} & U \{E(M_T^2), E(M_T), E(M_T W_{S,t_T})\} \\ &= -\text{Var}(M_T + W_{S,t_T}) \\ &= -E(M_T^2) + E^2(M_T) - 2E(M_T W_{S,t_T}) \\ &\quad + 2E(W_{S,t_T})E(M_T) - \text{Var}(W_{S,t_T}). \end{aligned} \quad (27)$$

Then model (26) is expressed by

$$\begin{aligned} \max \quad & U \{E(M_T^2), E(M_T), E(M_T W_{S,t_T})\} \\ \text{s.t.} \quad & M_i = M_{i-1}e_{C,i-1} + H_{i-1}P_{i-1}. \end{aligned} \quad (28)$$

In order to obtain the analytical solution for model (28), we can construct the following auxiliary model:

$$\begin{aligned} \text{(AU)} \quad \max \quad & E \{-M_T^2 + \lambda M_T - 2M_T W_{S,t_T}\} \\ \text{s.t.} \quad & M_i = M_{i-1}e_{C,i-1} + H_{i-1}P_{i-1}. \end{aligned} \quad (29)$$

Under certain conditions, the auxiliary model (29) is equivalent to model (28). In other words, the set of the optimal solutions of the auxiliary model (29) is also the one of model (28). In the following, we will give the condition for equivalent transformation of the two models. Define Z_M and Z_A to be the set of models (28) and (29), respectively.

Theorem 1. For any $\pi^* \in Z_M$, $\pi^* \in Z_A$.

Proof. By contradiction, assume that $\pi^* \notin Z_A$. Then, there exists a π such that

$$\begin{aligned} & (-1, 2E(M_T) + 2E(W_{S,t_T}), -2) \left[\begin{array}{c} E(M_T^2) \\ E(M_T) \\ E(M_T W_{S,t_T}) \end{array} \right] \Big|_{\pi} \\ & > (-1, 2E(M_T) + 2E(W_{S,t_T}), -2) \left[\begin{array}{c} E(M_T^2) \\ E(M_T) \\ E(M_T W_{S,t_T}) \end{array} \right] \Big|_{\pi^*}. \end{aligned} \quad (30)$$

It is obvious that $U\{E(M_T^2), E(M_T), E(M_T W_{S,t_T})\}$ is a convex function of $E(M_T^2)$, $E(M_T)$, and $E(M_T W_{S,t_T})$. So the following property is satisfied:

$$\begin{aligned} & U \{E(M_T^2), E(M_T), E(M_T W_{S,t_T})\} \Big|_{\pi} \\ & \geq U \{E(M_T^2), E(M_T), E(M_T W_{S,t_T})\} \Big|_{\pi^*} \\ & \quad + (-1, 2E(M_T) + 2E(W_{S,t_T}), -2) \\ & \quad \times \left\{ \left[\begin{array}{c} E(M_T^2) \\ E(M_T) \\ \sum_{j=1}^n Q_j S_{j,t_T} \end{array} \right] \Big|_{\pi} - \left[\begin{array}{c} E(M_T^2) \\ E(M_T) \\ \sum_{j=1}^n Q_j S_{j,t_T} \end{array} \right] \Big|_{\pi^*} \right\}. \end{aligned} \quad (31)$$

According to the analysis above, we have

$$\begin{aligned} & U \{E(M_T^2), E(M_T), E(M_T W_{S,t_T})\} \Big|_{\pi} \\ & \geq U \{E(M_T^2), E(M_T), E(M_T W_{S,t_T})\} \Big|_{\pi^*}. \end{aligned} \quad (32)$$

It shows that the above inequation contradicts the assumption of $\pi^* \notin Z_A$. Thus, the proof of Theorem 1 is ended. \square

Theorem 2. Assume $\pi^* \in Z_A$. A necessary condition for $\pi^* \in Z_M$ is

$$\lambda = 2E(M_T) + 2E(W_{S,t_T}). \quad (33)$$

Proof. Obviously, the solution set of the auxiliary model (29) can be parameterized by λ . In other words, $E(M_T^2)$, $E(M_T)$, and $E(M_T W_{S,t_T})$ can be expressed in terms of $E(M_T^2(\lambda))$, $E(M_T(\lambda))$, and $E(W_{S,t_T} M_T(\lambda))$. Then we can obtain

$$\begin{aligned} & \max_{\lambda} U \{E(M_T^2(\lambda)), E(M_T(\lambda)), E(W_{S,t_T} M_T(\lambda))\} \\ &= \max_{\lambda} -E(M_T^2(\lambda)) + E^2(M_T(\lambda)) - 2E(W_{S,t_T} M_T(\lambda)) \\ & \quad + 2E(W_{S,t_T}) E(M_T(\lambda)) - \text{Var}(W_{S,t_T}). \end{aligned} \quad (34)$$

A first-order necessary condition for optimal λ^* is

$$\begin{aligned} & -\frac{\partial E(M_T^2(\lambda^*))}{\partial \lambda} + 2[E(M_T) + E(W_{S,t_T})] \frac{\partial E(M_T(\lambda^*))}{\partial \lambda} \\ & - 2\frac{\partial E(W_{S,t_T} M_T(\lambda^*))}{\partial \lambda} = 0. \end{aligned} \quad (35)$$

According to Reid and Citron [38], when $\pi^* \in Z_A$, we get the following property:

$$\begin{aligned} & -\frac{\partial E(M_T^2(\lambda^*))}{\partial \lambda} + \lambda \frac{\partial E(M_T(\lambda^*))}{\partial \lambda} \\ & - 2\frac{\partial E(W_{S,t_T} M_T(\lambda^*))}{\partial \lambda} = 0. \end{aligned} \quad (36)$$

Since the vector $[-1, 2E(M_T) + 2E(W_{S,t_T}), -2]$ is proportional to $[-1, \lambda, -2]$, the condition necessary for $\pi^* \in Z_M$ is $\lambda = 2E(M_T) + 2E(W_{S,t_T})$ under the assumption of $\pi^* \in Z_A$. \square

It should become clear that the optimal solution of model (28) is generated by solving the auxiliary model (29). The optimal solution of the auxiliary model (29) can be derived analytically using dynamic programming.

At first, we consider the case of period $T - 1$.

Substituting (21) into $E\{-M_T^2 + \lambda M_T - 2M_T W_{S,t_T}\}$, the optimization problem is given as follows:

$$\begin{aligned} & \max J_{T-1}(H_{T-1} | M_{T-1}) \\ &= \max E\{-M_T^2 + \lambda M_T - 2M_T W_{S,t_T}\} \\ &= \max E\{-(M_{T-1} e_{C,T-1} + P_{T-1} H_{T-1})^2 \\ & \quad + (\lambda - 2W_{S,t_T})(M_{T-1} e_{C,T-1} + P_{T-1} H_{T-1})\} \end{aligned}$$

$$\begin{aligned} &= \max \{-H_{T-1}^2 E(P_{T-1}^2) \\ & \quad + [-2M_{T-1} E(e_{C,T-1} P_{T-1}) + \lambda E(P_{T-1}) \\ & \quad - 2E(W_{S,t_T} P_{T-1})] H_{T-1} \\ & \quad - M_{T-1}^2 E(e_{C,T-1}^2) + \lambda M_{T-1} E(e_{C,T-1}) \\ & \quad - 2M_{T-1} E(W_{S,t_T} e_{C,T-1})\}. \end{aligned} \quad (37)$$

Solving the first-order necessary optimality condition of (37), we get

$$H_{T-1}^* = \frac{(\lambda/2) E(P_{T-1}) - M_{T-1} E(e_{C,T-1} P_{T-1}) - E(W_{S,t_T} P_{T-1})}{E(P_{T-1}^2)}. \quad (38)$$

Substituting (38) into (37), the optimal $J_{T-1}(H_{T-1} | M_{T-1})$ is

$$\begin{aligned} & J_{T-1}^*(M_{T-1}) \\ &= -\omega_{T-1} M_{T-1}^2 + (\lambda \delta_{T-1} + K) M_{T-1} + \frac{\lambda^2}{4} \beta_{T-1} + \gamma, \end{aligned} \quad (39)$$

where

$$\begin{aligned} \omega_{T-1} &= E(e_{C,T-1}^2) - \frac{E^2(e_{C,T-1} P_{T-1})}{E(P_{T-1}^2)}, \\ \delta_{T-1} &= E(e_{C,T-1}) - \frac{E(P_{T-1}) E(e_{C,T-1} P_{T-1})}{E(P_{T-1}^2)}, \end{aligned}$$

$$\beta_{T-1} = \frac{E^2(P_{T-1})}{E(P_{T-1}^2)},$$

$$\begin{aligned} K &= 2 \left[\frac{E(W_{S,t_T} P_{T-1}) E(e_{C,T-1} P_{T-1})}{E^2(P_{T-1}^2)} - E(W_{S,t_T} e_{C,T-1}) \right], \\ \gamma &= \frac{[E(W_{S,t_T} P_{T-1}) - \lambda E(P_{T-1})] E(W_{S,t_T} P_{T-1})}{E(P_{T-1}^2)}. \end{aligned} \quad (40)$$

Next, we discuss the case of period $T - 1$. Substituting (21) into (39), (39) can be rewritten as

$$\begin{aligned} & J_{T-1}^*(M_{T-1}) \\ &= \max J_{T-2}(H_{T-2} | M_{T-2}) \\ &= \max E \left\{ -\omega_{T-1} M_{T-1}^2 + (\lambda \delta_{T-1} + K) M_{T-1} + \frac{\lambda^2}{4} \beta_{T-1} + \gamma \right\} \\ &= \max E \left\{ -\omega_{T-1} (M_{T-2} e_{C,T-2} + P_{T-2} H_{T-2})^2 \right. \\ & \quad + (\lambda \delta_{T-1} + K) (M_{T-2} e_{C,T-2} + P_{T-2} H_{T-2}) \\ & \quad \left. + \frac{\lambda^2}{4} \beta_{T-1} + \gamma \right\} \end{aligned}$$

$$\begin{aligned}
 = \max & \left\{ -\omega_{T-1} H_{T-2}^2 E(P_{T-2}^2) \right. \\
 & + [(\lambda \delta_{T-1} + K) E(P_{T-2}) \\
 & \quad - 2\omega_{T-1} M_{T-2} E(e_{C,T-2} P_{T-2})] H_{T-2} \\
 & - \omega_{T-1} M_{T-2}^2 E(e_{C,T-2}^2) \\
 & + (\lambda \delta_{T-1} + K) M_{T-2} E(e_{C,T-2}) \\
 & \left. + \frac{\lambda^2}{4} \beta_{T-1} + \gamma \right\}.
 \end{aligned} \tag{41}$$

The optimal solution of (41) is given by

$$H_{T-2}^* = \frac{((\lambda \delta_{T-1} + K) / 2\omega_{T-1}) E(P_{T-2}) - M_{T-2} E(e_{C,T-2} P_{T-2})}{E(P_{T-2}^2)}. \tag{42}$$

Then (41) can be rewritten as

$$\begin{aligned}
 J_{T-2}^*(M_{T-2}) = & -\omega_{T-1} \omega_{T-2} M_{T-2}^2 + (\lambda \delta_{T-1} + K) \delta_{T-2} M_{T-2} \\
 & + \frac{(\lambda \delta_{T-1} + K)^2}{4\omega_{T-1}} \beta_{T-2} + \frac{\lambda^2}{4} \beta_{T-1} + \gamma,
 \end{aligned} \tag{43}$$

where

$$\begin{aligned}
 \omega_{T-2} &= E(e_{C,T-2}^2) - \frac{E^2(e_{C,T-2} P_{T-2})}{E(P_{T-2}^2)}, \\
 \delta_{T-2} &= E(e_{C,T-2}) - \frac{E(P_{T-2}) E(e_{C,T-2} P_{T-2})}{E(P_{T-2}^2)}, \\
 \beta_{T-2} &= \frac{E^2(P_{T-2})}{E(P_{T-2}^2)}.
 \end{aligned} \tag{44}$$

Similarly, we can derive the optimal H_i and $J_i(M_i)$ ($0 \leq i \leq T-3$)

$$\begin{aligned}
 H_i^* &= \left(\frac{(\lambda \delta_{T-1} + K)}{2\omega_{T-1}} \prod_{k=i+1}^{T-2} \frac{\delta_k}{\omega_k} E(P_i) \right. \\
 & \quad \left. - M_i E(e_{C,i} P_i) \right) \\
 & \quad \times (E(P_i^2))^{-1},
 \end{aligned} \tag{45}$$

$$\begin{aligned}
 J_i^*(M_i) = & -M_i^2 \prod_{k=i}^{T-1} \omega_k + (\lambda \delta_{T-1} + K) M_i \prod_{k=i}^{T-2} \delta_k \\
 & + \frac{(\lambda \delta_{T-1} + K)^2}{4\omega_{T-1}} \left[\sum_{k=i}^{T-3} \left(\beta_k \prod_{s=k+1}^{T-2} \frac{\delta_s^2}{\omega_s} \right) \right] \\
 & + \frac{(\lambda \delta_{T-1} + K)^2}{4\omega_{T-1}} \beta_{T-2} + \frac{\lambda^2}{4} \beta_{T-1} + \gamma,
 \end{aligned} \tag{46}$$

where

$$\begin{aligned}
 \omega_k &= E(e_{C,k}^2) - \frac{E^2(e_{C,k} P_k)}{E(P_k^2)}, \\
 \delta_k &= E(e_{C,k}) - \frac{E(P_k) E(e_{C,k} P_k)}{E(P_k^2)}, \\
 \beta_k &= \frac{E^2(P_k)}{E(P_k^2)}.
 \end{aligned} \tag{47}$$

With the optimal solution of the auxiliary model (29), we can solve model (28).

Substituting (38) and (45) into (21), the total amount of $W_{F,T}$ and $W_{C,T}$ can be expressed by the following analytical form:

$$M_i = \begin{cases} M_0 \left[e_{C,0} - \frac{P_0 E(e_{C,0} P_0)}{E(P_0^2)} \right] + \frac{((\lambda \delta_{T-1} + K) / 2\omega_{T-1}) \prod_{k=1}^{T-1} (\delta_k / 2\omega_k) E(P_0)}{E(P_0^2)}, & i = 1; \\ M_0 \prod_{k=0}^{i-1} \left[e_{C,k} - \frac{P_k E(e_{C,k} P_k)}{E(P_k^2)} \right] + \frac{(\lambda \delta_{T-1} + K)}{2\omega_{T-1}} \\ \quad \times \sum_{g=0}^{i-2} \left\{ \frac{P_g E(P_g)}{E(P_g^2)} \prod_{k=g+1}^{T-2} \frac{\delta_k}{\omega_k} \prod_{k=g+1}^{i-1} \left[e_{C,k} - \frac{P_k E(e_{C,k} P_k)}{E(P_k^2)} \right] \right\} \\ \quad + \frac{(\lambda \delta_{T-1} + K)}{2\omega_{T-1}} \prod_{k=i}^{T-2} \frac{\delta_k}{\omega_k} \frac{P_{i-1} E(P_{i-1})}{E(P_{i-1}^2)}, & 2 \leq i \leq T-2; \\ M_{T-2} \left[e_{C,T-2} - \frac{P_{T-2} E(e_{C,T-2} P_{T-2})}{E(P_{T-2}^2)} \right] + \frac{\lambda \delta_{T-1} + K}{2\omega_{T-1}} \frac{P_{T-2} E(P_{T-2})}{E(P_{T-2}^2)}, & i = T-1; \\ M_{T-1} \left[e_{C,T-1} - \frac{P_{T-1} E(e_{C,T-1} P_{T-1})}{E(P_{T-1}^2)} \right] + \frac{(\lambda/2) P_{T-1} E(P_{T-1}) - P_{T-1} E(W_{S,t_T} P_{T-1})}{E(P_{T-1}^2)} & i = T. \end{cases} \tag{48}$$

Combining (48) with the necessary condition $\lambda = 2E(M_T) + 2E(W_{S,t_T})$, we have

$$\begin{aligned} \lambda &= 2E(M_T) + 2E(W_{S,t_T}) \\ &= 2M_0 \prod_{k=0}^{T-1} \delta_k + \frac{(\lambda \delta_{T-1} + K)}{\omega_{T-1}} \sum_{g=0}^{T-4} \left\{ \beta_g \prod_{k=g+1}^{T-2} \frac{\delta_k}{\omega_k} \prod_{k=g+1}^{T-1} \omega_k \right\} \\ &\quad + \frac{(\lambda \delta_{T-1} + K)}{\omega_{T-1}} \frac{\delta_{T-2}}{\omega_{T-2}} \beta_{T-3} \prod_{k=T-2}^{T-1} \delta_k \\ &\quad + \frac{\lambda \delta_{T-1} + K}{\omega_{T-1}} \beta_{T-2} \delta_{T-1} + \lambda \beta_{T-1} \\ &\quad - \frac{2E(P_{T-1}) E(W_{S,t_T} P_{T-1})}{E(P_{T-1}^2)} + 2E(W_{S,t_T}). \end{aligned} \tag{49}$$

By solving the above equation, we can obtain

$$\lambda = \frac{2\omega_{T-1} [M_0 \prod_{k=0}^{T-1} \delta_k + E(W_{S,t_T})] + K\eta}{\omega_{T-1} (1 - \beta_{T-1}) - \delta_{T-1}\eta}, \tag{50}$$

where

$$\begin{aligned} \eta &= \sum_{g=0}^{T-4} \left\{ \beta_g \prod_{k=g+1}^{T-2} \frac{\delta_k}{\omega_k} \prod_{k=g+1}^{T-1} \omega_k \right\} \\ &\quad + \frac{\delta_{T-2} \beta_{T-3}}{\omega_{T-2}} \prod_{k=T-2}^{T-1} \delta_k + \beta_{T-2} \delta_{T-1}. \end{aligned} \tag{51}$$

Then the optimal solution of the proposed model is

$$H_i^* = \begin{cases} \frac{((\lambda \delta_{T-1} + K) / 2\omega_{T-1}) \prod_{k=i+1}^{T-2} (\delta_k / \omega_k) E(P_i) - M_i E(e_{C,i} P_i)}{E(P_i^2)}, & 0 \leq i \leq T-3, \\ \frac{((\lambda \delta_{T-1} + K) / 2\omega_{T-1}) E(P_{T-2}) - M_{T-2} E(e_{C,T-2} P_{T-2})}{E(P_{T-2}^2)}, & i = T-2, \\ \frac{(\lambda/2) E(P_{T-1}) - M_{T-1} E(e_{C,T-1} P_{T-1}) - E(W_{S,t_T} P_{T-1})}{E(P_{T-1}^2)}, & i = T-1, \end{cases} \tag{52}$$

where M_i and λ satisfy (48) and (50), respectively.

4. The Rollover Hedging Model with Transaction Costs

The change of stock index futures contracts is essential in the rollover hedge strategy and causes lots of transaction costs, which has a great impact on the hedging performance. In this section, we will focus on the rollover hedging model with transaction costs. Suppose that θ is the transaction cost rate of stock index futures contracts.

At time $t = t_0$, the capital invested in stock index futures contracts is the sum of the maintenance margin and the transaction cost of stock index futures contracts, that is,

$$W_{F,t_0} = H_0 Z F_{t_0} \alpha + H_0 Z F_{t_0} \theta = H_0 Z F_{t_0} (\alpha + \theta). \tag{53}$$

So the total amount of W_{F,t_0} and W_{C,t_0} is

$$Y_0 = V_0 - W_{S,t_0} = W_{F,t_0} + W_{C,t_0} = H_0 Z F_{t_0} (\alpha + \theta) + W_{C,t_0}. \tag{54}$$

At time $t = t_{i-1}$, the capital invested in stock index futures contracts is

$$W_{F,t_{i-1}} = H_{i-1} Z F_{t_{i-1}} \alpha + H_{i-1} Z F_{t_{i-1}} \theta = H_{i-1} Z F_{t_{i-1}} (\alpha + \theta). \tag{55}$$

Then the total amount of $W_{F,t_{i-1}}$ and $W_{C,t_{i-1}}$ is

$$Y_{i-1} = W_{F,t_{i-1}} + W_{C,t_{i-1}} = H_{i-1} Z F_{t_{i-1}} (\alpha + \theta) + W_{C,t_{i-1}}. \tag{56}$$

Similarly, the total amount of W_{F,t_i} and W_{C,t_i} also needs to be discussed in two cases.

If the firm does not face liquidity risk, the amounts of stock index futures contracts and the saving at time $t = t_i$ are

$$\begin{aligned} W_{F,t_i} &= H_{i-1} Z F_{t_{i-1}} \alpha + H_{i-1} Z F_{t_{i-1}} - H_{i-1} Z F_{t_i} - H_{i-1} Z F_{t_i} \theta \\ &= H_{i-1} Z [F_{t_{i-1}} (1 + \alpha) - F_{t_i} (1 + \theta)], \\ W_{C,t_i} &= W_{C,t_{i-1}} (1 + r_{C,i}). \end{aligned} \tag{57}$$

Then the total amount of W_{F,t_i} and W_{C,t_i} is

$$\begin{aligned} Y_i &= W_{F,t_i} + W_{C,t_i} \\ &= H_{i-1}Z \left[F_{t_{i-1}}(1+\alpha) - F_{t_i}(1+\theta) \right] \\ &\quad + W_{C,t_{i-1}}(1+r_{C,i}). \end{aligned} \quad (58)$$

If there is liquidity risk during period i , the saving at time $t = t_i$ is given by

$$\begin{aligned} W_{C,t_i} &= W_{C,t_{i-1}+\theta_i(N_i)} [1+r_i(N_i+k_i)] \\ &= \left\{ W_{C,t_{i-1}} \prod_{m=1}^{N_i} [1+r_i(j)] - H_{i-1}A_{i-1} \right\} \\ &\quad \times [1+r_i(N_i+k_i)] \end{aligned} \quad (59)$$

$$e_{K,i-1} = \begin{cases} Z [F_{t_{i-1}}(1+\alpha) - F_{t_i}(1+\theta)], & \text{without liquidity risk,} \\ Z [F_{t_{i-1}}(1+\alpha) - F_{t_i}(1+\theta)] - A_{i-1} [1+r_i(N_i+k_i)], & \text{with liquidity risk.} \end{cases} \quad (61)$$

Then the total amount of W_{F,t_i} and W_{C,t_i} can be rewritten as

$$Y_i = H_{i-1}e_{K,i-1} + W_{C,t_{i-1}}e_{C,i-1}. \quad (62)$$

Substituting (56) into (60), the total amount of W_{F,t_i} and W_{C,t_i} is expressed by

$$\begin{aligned} Y_i &= H_{i-1}e_{K,i-1} + W_{C,t_{i-1}}e_{C,i-1} \\ &= H_{i-1}e_{K,i-1} + [Y_{i-1} - H_{i-1}ZF_{t_{i-1}}(\alpha+\theta)]e_{C,i-1} \\ &= H_{i-1} [e_{K,i-1} - ZF_{t_{i-1}}(\alpha+\theta)]e_{C,i-1} + Y_{i-1}e_{C,i-1}. \end{aligned} \quad (63)$$

Let $L_{i-1} = e_{K,i-1} - ZF_{t_{i-1}}(\alpha+\theta)e_{C,i-1}$. Then the total amount of W_{F,t_i} and W_{C,t_i} can be simplified as

$$Y_i = H_{i-1}L_{i-1} + Y_{i-1}e_{C,i-1}. \quad (64)$$

and the total amount of W_{F,t_i} and W_{C,t_i} is

$$\begin{aligned} Y_i &= W_{C,t_i} + W_{F,t_i} \\ &= \left\{ W_{C,t_{i-1}} \prod_{m=1}^{N_i} [1+r_i(j)] - H_{i-1}A_{i-1} \right\} \\ &\quad \times [1+r_i(N_i+k_i)] \\ &\quad + H_{i-1}Z [F_{t_{i-1}}(1+\alpha) - F_{t_i}(1+\theta)] \\ &= H_{i-1} \left\{ Z [F_{t_{i-1}}(1+\alpha) - F_{t_i}(1+\theta)] \right. \\ &\quad \left. - A_{i-1} [1+r_i(N_i+k_i)] \right\} \\ &\quad + W_{C,t_{i-1}} [1+r_i(N_i+k_i)] \prod_{m=1}^{N_i} [1+r_i(j)]. \end{aligned} \quad (60)$$

Let

During the whole period, the terminal profit of the investment is

$$\Delta V = V_T - V_0 = Y_T + W_{S,t_T} - (Y_0 + W_{S,t_0}), \quad (65)$$

where V_T is the terminal wealth at the end of the investment period.

The optimal number of stock index futures contracts is determined by minimizing the variance of terminal profit of the investment:

$$\min \text{Var}(\Delta V) = \text{Var}(Y_T + W_{S,t_T}). \quad (66)$$

Combining (66) with (64), we can propose the rollover hedging model with transaction costs

$$\min \text{Var}(Y_T + W_{S,t_T}) \quad (67)$$

$$\text{s.t. } Y_i = H_{i-1}L_{i-1} + Y_{i-1}e_{C,i-1}.$$

Model (67) is also nonseparable in the sense of dynamic programming and needs to be transformed into a separable model.

Let

$$\begin{aligned}
 U \{ & E(Y_T^2), E(Y_T), E(Y_T W_{S,t_T}) \} \\
 &= -\text{Var}(Y_T + W_{S,t_T}) \\
 &= -E(Y_T^2) + E^2(Y_T) - 2E(Y_T W_{S,t_T}) \\
 &\quad + 2E(W_{S,t_T})E(Y_T) - \text{Var}(W_{S,t_T}).
 \end{aligned} \tag{68}$$

Then model (67) is expressed by

$$\begin{aligned}
 \max \quad & U \{ E(Y_T^2), E(Y_T), E(Y_T W_{S,t_T}) \} \\
 \text{s.t.} \quad & Y_i = H_{i-1} L_{i-1} + Y_{i-1} e_{C,i-1}.
 \end{aligned} \tag{69}$$

In order to obtain the analytical solution for model (69), we use the following auxiliary model:

$$\begin{aligned}
 \text{(AU2):} \quad \max \quad & E \{ -Y_T^2 + \lambda Y_T - 2Y_T W_{S,t_T} \} \\
 \text{s.t.} \quad & Y_i = H_{i-1} L_{i-1} + Y_{i-1} e_{C,i-1}.
 \end{aligned} \tag{70}$$

Define Z_{MU} and Z_{AU} to be the set of models (69) and (70), respectively. We can obtain the condition for equivalent transformation of the two models.

Theorem 3. For any $\pi^* \in Z_{MU}$, $\pi^* \in Z_{AU}$.

Theorem 4. Assume $\pi^* \in Z_{AU}$. A necessary condition for $\pi^* \in Z_{MU}$ is

$$\lambda = 2E(Y_T) + 2E(W_{S,t_T}). \tag{71}$$

The proofs of Theorems 3 and 4 are similar to those of Theorems 1 and 2 and are omitted.

Dynamic programming is employed to obtain the optimal solution of the following auxiliary model. The dynamic programming starts from the period $T-1$. The optimization problem is given as follows:

$$\begin{aligned}
 \max \quad & J_{T-1}(H_{T-1} | Y_{T-1}) \\
 &= \max E \{ -Y_T^2 + \lambda Y_T - 2Y_T W_{S,t_T} \}
 \end{aligned}$$

$$\begin{aligned}
 &= \max E \{ -(Y_{T-1} e_{C,T-1} + L_{T-1} H_{T-1})^2 \\
 &\quad + (\lambda - 2W_{S,t_T})(Y_{T-1} e_{C,T-1} + L_{T-1} H_{T-1}) \} \\
 &= \max \{ -H_{T-1}^2 E(L_{T-1}^2) \\
 &\quad + [-2Y_{T-1} E(e_{C,T-1} L_{T-1}) \\
 &\quad + \lambda E(L_{T-1}) - 2E(W_{S,t_T} L_{T-1})] H_{T-1} \\
 &\quad - Y_{T-1}^2 E(e_{C,T-1}^2) + \lambda Y_{T-1} E(e_{C,T-1}) \\
 &\quad - 2Y_{T-1} E(W_{S,t_T} e_{C,T-1}) \}.
 \end{aligned} \tag{72}$$

From (72), we have

$$\begin{aligned}
 H_{T-1}^* &= \frac{(\lambda/2) E(L_{T-1}) - Y_{T-1} E(e_{C,T-1} L_{T-1}) - E(W_{S,t_T} L_{T-1})}{E(L_{T-1}^2)}, \\
 J_{T-1}^*(Y_{T-1}) &= -\omega_{T-1} Y_{T-1}^2 + (\lambda \delta_{T-1} + K) Y_{T-1} + \frac{\lambda^2}{4} \beta_{T-1} + \gamma,
 \end{aligned} \tag{73}$$

where

$$\begin{aligned}
 \varphi_{T-1} &= E(e_{C,T-1}^2) - \frac{E^2(e_{C,T-1} L_{T-1})}{E(L_{T-1}^2)}, \\
 \gamma_{T-1} &= E(e_{C,T-1}) - \frac{E(L_{T-1}) E(e_{C,T-1} L_{T-1})}{E(L_{T-1}^2)}, \\
 \rho_{T-1} &= \frac{E^2(L_{T-1})}{E(L_{T-1}^2)}, \\
 K &= 2 \left[\frac{E(W_{S,t_T} L_{T-1}) E(e_{C,T-1} L_{T-1})}{E^2(L_{T-1}^2)} - E(W_{S,t_T} e_{C,T-1}) \right], \\
 \gamma &= \frac{[E(W_{S,t_T} L_{T-1}) - \lambda E(L_{T-1})] E(W_{S,t_T} L_{T-1})}{E(L_{T-1}^2)}.
 \end{aligned} \tag{74}$$

By analogy, we can derive the optimal H_i ($0 \leq i \leq T-2$)

$$H_i^* = \begin{cases} \frac{((\lambda\delta_{T-1} + K)/2\omega_{T-1}) \prod_{k=i+1}^{T-2} (\delta_k/\omega_k) E(P_i) - M_i E(e_{C,i}P_i)}{E(P_i^2)}, & 0 \leq i \leq T-3, \\ \frac{((\lambda\delta_{T-1} + K)/2\omega_{T-1}) E(P_{T-2}) - M_{T-2} E(e_{C,T-2}P_{T-2})}{E(P_{T-2}^2)}, & i = T-2, \end{cases}$$

$$J_i^*(M_i) = \begin{cases} -M_i^2 \prod_{k=i}^{T-1} \omega_k + (\lambda\delta_{T-1} + K) M_i \prod_{k=i}^{T-2} \delta_k + \frac{(\lambda\delta_{T-1} + K)^2}{4\omega_{T-1}} \left[\sum_{k=i}^{T-3} \left(\beta_k \prod_{s=k+1}^{T-2} \frac{\delta_s^2}{\omega_s} \right) \right] \\ + \frac{(\lambda\delta_{T-1} + K)^2}{4\omega_{T-1}} \beta_{T-2} + \frac{\lambda^2}{4} \beta_{T-1} + \gamma, & 0 \leq i \leq T-3, \\ -\omega_{T-1} \omega_{T-2} M_{T-2}^2 + (\lambda\delta_{T-1} + K) \delta_{T-2} M_{T-2} + \frac{(\lambda\delta_{T-1} + K)^2}{4\omega_{T-1}} \beta_{T-2} + \frac{\lambda^2}{4} \beta_{T-1} + \gamma, & i = T-2, \end{cases} \quad (75)$$

where

$$\varphi_k = E(e_{C,k}^2) - \frac{E^2(e_{C,k}L_k)}{E(L_k^2)}, \quad \rho_k = \frac{E^2(L_k)}{E(L_k^2)}. \quad (76)$$

$$\gamma_k = E(e_{C,k}) - \frac{E(L_k)E(e_{C,k}L_k)}{E(L_k^2)},$$

Substituting (73) and (75) into (64), the terminal capital of the firm can be expressed by

$$Y_i = \begin{cases} Y_0 \left[e_{C,0} - \frac{L_0 E(e_{C,0}L_0)}{E(L_0^2)} \right] + \frac{((\lambda\delta_{T-1} + K)/2\omega_{T-1}) \prod_{k=1}^{T-1} (\delta_k/2\omega_k) E(L_0)}{E(L_0^2)}, & i = 1; \\ Y_0 \prod_{k=0}^{i-1} \left[e_{C,k} - \frac{L_k E(e_{C,k}L_k)}{E(L_k^2)} \right] + \frac{(\lambda\delta_{T-1} + K)}{2\omega_{T-1}} \sum_{g=0}^{i-2} \left\{ \frac{L_g E(L_g)}{E(L_g^2)} \prod_{k=g+1}^{T-2} \frac{\delta_k}{\omega_k} \prod_{k=g+1}^{i-1} \left[e_{C,k} - \frac{L_k E(e_{C,k}L_k)}{E(L_k^2)} \right] \right\} \\ + \frac{(\lambda\delta_{T-1} + K)}{2\omega_{T-1}} \prod_{k=i}^{T-2} \frac{\delta_k}{\omega_k} \frac{L_{i-1} E(L_{i-1})}{E(L_{i-1}^2)}, & 2 \leq i \leq T-2; \\ Y_{T-2} \left[e_{C,T-2} - \frac{L_{T-2} E(e_{C,T-2}L_{T-2})}{E(L_{T-2}^2)} \right] + \frac{\lambda\delta_{T-1} + K}{2\omega_{T-1}} \frac{L_{T-2} E(L_{T-2})}{E(L_{T-2}^2)}, & i = T-1; \\ Y_{T-1} \left[e_{C,T-1} - \frac{L_{T-1} E(e_{C,T-1}L_{T-1})}{E(L_{T-1}^2)} \right] + \frac{(\lambda/2) L_{T-1} E(L_{T-1}) - L_{T-1} E(W_{S,t_T} L_{T-1})}{E(L_{T-1}^2)} & i = T. \end{cases} \quad (77)$$

Combining (77) with the necessary condition $\lambda = 2E(Y_T) + 2E(W_{S,t_T})$, we have

$$\lambda = 2E(Y_T) + 2E(W_{S,t_T})$$

$$= 2\omega_{T-1} \left[M_0 \prod_{k=0}^{T-1} \delta_k + E(W_{S,t_T}) \right] + K \left\{ \sum_{g=0}^{T-4} \left\{ \beta_g \prod_{k=g+1}^{T-2} \frac{\delta_k}{\omega_k} \prod_{k=g+1}^{T-1} \omega_k \right\} + \frac{\delta_{T-2} \beta_{T-3}}{\omega_{T-2}} \prod_{k=T-2}^{T-1} \delta_k + \beta_{T-2} \delta_{T-1} \right\}. \quad (78)$$

The solution of (78) is given by

$$\lambda = \frac{2\varphi_{T-1} [M_0 \prod_{k=0}^{T-1} \gamma_k + E(W_{S,t_T})] + K\nu}{\varphi_{T-1} (1 - \rho_{T-1}) - \gamma_{T-1}\nu}, \quad (79)$$

where

$$\nu = \sum_{g=0}^{T-4} \left\{ \beta_g \prod_{k=g+1}^{T-2} \frac{\delta_k}{\omega_k} \prod_{k=g+1}^{T-1} \omega_k \right\} + \frac{\delta_{T-2}\beta_{T-3}}{\omega_{T-2}} \prod_{k=T-2}^{T-1} \delta_k + \beta_{T-2}\delta_{T-1}. \quad (80)$$

Therefore, the optimal solution of the proposed model (67) is

$$H_i^* = \begin{cases} \frac{((\lambda\delta_{T-1} + K)/2\omega_{T-1}) \prod_{k=i+1}^{T-2} (\delta_k/\omega_k) E(L_i) - Y_i E(e_{C,i}L_i)}{E(L_i^2)}, & 0 \leq i \leq T-2, \\ \frac{((\lambda\delta_{T-1} + K)/2\omega_{T-1}) E(L_{T-2}) - Y_{T-2} E(e_{C,T-2}L_{T-2})}{E(P_{T-2}^2)}, & i = T-1, \\ \frac{(\lambda/2) E(L_{T-1}) - Y_{T-1} E(e_{C,T-1}L_{T-1}) - E(W_{S,t_T}L_{T-1})}{E(L_{T-1}^2)}, & i = T-1, \end{cases} \quad (81)$$

where Y_i and λ satisfy (77) and (79), respectively.

5. Conclusion

Stock index futures contracts provide an excellent tool for the risk management. However, the expiration date of the hedge is later than the delivery dates of all the index futures contracts that can be used. It is difficult for the hedger to directly use short index futures contracts for managing the long-term exposure in the stock market. The hedger has to choose the rollover hedge strategy. In this paper, we consider the problem of the rollover hedge strategy for the long-term exposure of a well-diversified portfolio. Two new models are presented to solve the problem. The proposed models are nonseparable in the sense of dynamic programming. Therefore, it is necessary to transform the proposed models into separable models. Using equivalent transformation technique, auxiliary models are constructed for the solution of the proposed models. Finally, the optimal numbers of index futures contracts have been derived by dynamic programming.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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