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Research Article

The Number of Limit Cycles of a Polynomial System on the Plane

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We perturb the vector field $\dot{x} = -yC(x, y)$, $\dot{y} = xC(x, y)$ with a polynomial perturbation of degree n, where $C(x, y) = (1 - y^2)^m$, and study the number of limit cycles bifurcating from the period annulus surrounding the origin.

1. Introduction and Main Result

The main task in the qualitative theory of real plane differential systems is to determine the number of limit cycles, which is related to Hilbert's 16th problem as well as weakened Hilbert's 16th problem, posed by Arnold in [1].

Consider a planar system of the form

$$\begin{aligned} \dot{x} &= -yC(x, y) + \varepsilon P(x, y), \\ \dot{y} &= xC(x, y) + \varepsilon Q(x, y), \end{aligned} \tag{1}$$

where P, Q, and C are real polynomials, $C(0, 0) \neq 0$, and ε is a small real parameter. It is well known that the number of zeros of the Abelian integral

$$M(\rho) = \int_{\gamma_\rho} \frac{Q(x, y)\,dx - P(x, y)\,dy}{C(x, y)}, \tag{2}$$

where $\gamma_\rho = \{(x, y) : x^2 + y^2 = \rho^2\}$, controls the number of limit cycles of (1) that bifurcate from the periodic orbits of the unperturbed system (1) with $\varepsilon = 0$; see [2].

The problem of finding lower and upper bounds for the number of zeros of $M(\rho)$, when P and Q are arbitrary polynomials of a given degree, say n, and C is a particular polynomial, has been faced in several recent papers. In the case of perturbing the linear center by arbitrary polynomials P and Q of degree n, that is, considering $\dot{x} = -y+\varepsilon p(x, y)$, $\dot{y} = x+\varepsilon q(x, y)$, there are at most $[(n-1)/2]$ limit cycles up to first order in ε, see [3], where $[\cdot]$ denotes the integer part function. Also it is known that perturbing the quadratic center $\dot{x} = -y(1 + x)$, $\dot{y} = x(1 + x)$ inside the polynomial systems of degree n we can obtain at most n limit cycles up to first order in ε (see [4]). The authors of [5] studied the perturbation of the cubic center $\dot{x} = -y(1+x)(2+x)$, $\dot{y} = x(1+x)(2+x)$ inside the polynomial differential systems of degree n, and they obtained that $2n+2-(-1)^n$ is an upped bound for the number of limit cycles up to first order in ε. In [6], the authors studied the perturbations of $\dot{x} = -y(a+x)(b+y)$, $\dot{y} = x(a+x)(b+y)$, and they obtained that $3[(n-1)/2]+4$ if $a \neq b$ and, respectively, $2[(n-1)/2]+2$ if $a = b$, up to first order in ε, are upper bounds for the number of the limit cycles. In [7] the authors studied the maximum number σ of limit cycles which can bifurcate from the periodic orbits of the quartic center $\dot{x} = -yf(x, y)$, $\dot{y} = xf(x, y)$ with $f(x, y) = (x + a)(y + b)(x + c)$ and $abc \neq 0$ by perturbing it inside the class of polynomial vector fields of degree n. They proved that $4[(n - 1)/2] + 4 \leq \sigma \leq 5[(n-1)/2] + 14$. In [8] the authors studied the bifurcation of limit cycles of the system $\dot{x} = -y(x^2 - a^2)(y^2 - b^2) + \varepsilon P(x, y)$, $\dot{y} = x(x^2 - a^2)(y^2 - b^2) + \varepsilon Q(x, y)$ for ε sufficiently small, where $a, b \in R - 0$ and P, Q are polynomials of degree n. They obtained that up to first order in ε the upper bound for the number of limit cycles that bifurcate from the period annulus of the quintic center given by $\varepsilon = 0$ is $(3/2)(n+\sin^2(n\pi/2))+1$ if $a \neq b$ or $n - 1$ if $a = b$. More results can be found in [9, 10].

But few of these algebraic curves have a multiple factor. In [11], the authors took $C(x, y) = (1 - y)^m$ and proved that an upper bound for the number of zeros of $M(\rho)$ on (0,1) is $n + m - 1$ and that this bound is reached when $m = 1$. The approach of [11] is mainly based on the explicit computation of $M(\rho)$. In [12], the authors obtained the maximum of zeros of $M(\rho)$, taking into account their multiplicities, is $[(m+n)/2]-1$ when $n<m-1$ and n when $n\geq m-1$. In [12], the authors improve the upper bound given in [11] and provide the optimal upper bound for the zeros of $M(\rho)$. In this paper, motivated by [12] we take $C(x,y)=(1-y^2)^m$ and obtain the following theorem.

Theorem 1. *Consider* (1) *with* $C(x,y)=(1-y^2)^m$. *Let*

$$M(\rho)=\int_{\gamma_\rho}\frac{Q(x,y)\,dx-P(x,y)\,dy}{(1-y^2)^m}, \tag{3}$$

where $\gamma_\rho=\{(x,y):x^2+y^2=\rho^2\}$, $\rho\in(0,1)$, *and* P *and* Q *are polynomials of degree* n. *Then the maximum number of zeros of* $M(\rho)$, *taking into account their multiplicities, is* $m+[(n-1)/2]-1$ *when* $n<2m-1$ *and* $2[(n+1)/2]-1$ *when* $n\geq 2m-1$. *Moreover, when* $m=1,2$ *and* $n=2$, *the corresponding maximum number, which is 1, can be reached by taking suitable* P *and* Q.

2. Preliminary Results

To study the property of $M(\rho)$, we need to make some preliminaries. First we introduce a function of the form

$$I_j(\rho)=\int_0^{2\pi}\frac{1}{(1-\rho^2\sin^2\theta)^j}d\theta, \tag{4}$$

where $\rho\in[0,1)$ and j is an integer.

This section contains some preliminary computations to express the Abelian integral $M(\rho)$ given in (3) in terms of polynomials.

Lemma 2. *Let* $R_l(x)$ *be a polynomial of degree* l *in* x. *Then for* $\rho\in[0,1)$ *and* $m\geq 0$ *it holds that*

$$\int_0^{2\pi}\frac{R_l(\rho^2\sin^2\theta)}{(1-\rho^2\sin^2\theta)^m}d\theta=\sum_{j=0}^{l}\alpha_{m-j}I_{m-j}(\rho), \tag{5}$$

for some $\alpha_j\in R$.

Proof. Note that, for $j\geq 0$,

$$\begin{aligned}\int_0^{2\pi}\frac{\rho^{2j}\sin^{2j}\theta}{(1-\rho^2\sin^2\theta)^m}d\theta &=\int_0^{2\pi}\frac{(1-(1-\rho^2\sin^2\theta))^j}{(1-\rho^2\sin^2\theta)^m}d\theta\\ &=\sum_{k=0}^{j}\binom{j}{k}(-1)^k\int_0^{2\pi}\frac{(1-\rho^2\sin^2\theta)^k}{(1-\rho^2\sin^2\theta)^m}d\theta\\ &=\sum_{k=0}^{j}\binom{j}{k}(-1)^k I_{m-k}(\rho).\end{aligned} \tag{6}$$

The result follows by applying the above formula to each term of

$$R_l(\rho^2\sin^2\theta)=\sum_{j=0}^{l}r_j(\rho^2\sin^2\theta)^j. \tag{7}$$

□

Lemma 3. *Let* I_j *be the functions introduced in* (4). *Then, for* $1\neq j\in Z$,

$$(\rho^2-1)I_j(\rho)=\frac{3-2j}{2j-2}(2-\rho^2)I_{j-1}(\rho)+\frac{j-2}{j-1}I_{j-2}(\rho). \tag{8}$$

Proof. Note that

$$\begin{aligned}I_{j-1}(\rho)&=\int_0^{2\pi}\frac{1-\rho^2\sin^2\theta}{(1-\rho^2\sin^2\theta)^j}d\theta\\ &=I_j(\rho)-\rho^2\int_0^{2\pi}\frac{\sin^2\theta}{(1-\rho^2\sin^2\theta)^j}d\theta.\end{aligned} \tag{9}$$

Using integration by parts in the last integral, we obtain

$$\begin{aligned}&\rho^2\int_0^{2\pi}\frac{\sin^2\theta}{(1-\rho^2\sin^2\theta)^j}d\theta\\ &=\rho^2\int_0^{2\pi}\frac{\sin\theta}{(1-\rho^2\sin^2\theta)^j}d(-\cos\theta)\\ &=\left.\frac{-\rho^2\sin\theta\cos\theta}{(1-\rho^2\sin^2\theta)^j}\right|_0^{2\pi}\\ &\quad+\int_0^{2\pi}\Big(\big(\rho^2\cos^2\theta(1-\rho^2\sin^2\theta)^j\\ &\qquad+2j\rho^4\sin^2\theta\cos^2\theta(1-\rho^2\sin^2\theta)^{j-1}\big)\\ &\qquad\times\left(\left(1-\rho^2\sin^2\theta\right)^{2j}\right)^{-1}\Big)d\theta\\ &=\int_0^{2\pi}\frac{\rho^2\cos^2\theta}{(1-\rho^2\sin^2\theta)^j}d\theta+2j\int_0^{2\pi}\frac{\rho^4\sin^2\theta\cos^2\theta}{(1-\rho^2\sin^2\theta)^{j+1}}d\theta\\ &=\rho^2I_j+\int_0^{2\pi}\frac{1-\rho^2\sin^2\theta-1}{(1-\rho^2\sin^2\theta)^j}d\theta\\ &\quad+2j\int_0^{2\pi}\frac{\rho^4\sin^2\theta(1-\sin^2\theta)}{(1-\rho^2\sin^2\theta)^{j+1}}d\theta.\end{aligned} \tag{10}$$

Note that

$$\begin{aligned}
&\int_0^{2\pi} \frac{\rho^4\sin^2\theta\left(1-\sin^2\theta\right)}{\left(1-\rho^2\sin^2\theta\right)^{j+1}}d\theta \\
&= \rho^2\int_0^{2\pi} \frac{\rho^2\sin^2\theta}{\left(1-\rho^2\sin^2\theta\right)^{j+1}}d\theta \\
&\quad - \int_0^{2\pi} \frac{\rho^4\sin^4\theta}{\left(1-\rho^2\sin^2\theta\right)^{j+1}}d\theta \\
&= \rho^2\int_0^{2\pi} \frac{1-\left(1-\rho^2\sin^2\theta\right)}{\left(1-\rho^2\sin^2\theta\right)^{j+1}}d\theta \\
&\quad - \int_0^{2\pi} \frac{\left[1-\left(1-\rho^2\sin^2\theta\right)\right]^2}{\left(1-\rho^2\sin^2\theta\right)^{j+1}}d\theta \\
&= \rho^2\left(I_{j+1}-I_j\right)-\left(I_{j+1}-2I_j+I_{j-1}\right).
\end{aligned} \tag{11}$$

Then

$$\begin{aligned}
&\rho^2\int_0^{2\pi} \frac{\sin^2\theta}{\left(1-\rho^2\sin^2\theta\right)^{j}}d\theta \\
&= \rho^2 I_j+\left(I_{j-1}-I_j\right) \\
&\quad + 2j\left[\rho^2\left(I_{j+1}-I_j\right)-I_{j+1}+2I_j-I_{j-1}\right].
\end{aligned} \tag{12}$$

Substituting the formula above into (9), we find

$$\begin{aligned}
I_{j-1}(\rho) &= (1-2j)\left(2-\rho^2\right)I_j(\rho)+(2j-1)I_{j-1}(\rho) \\
&\quad + 2j\left(1-\rho^2\right)I_{j+1}(\rho).
\end{aligned} \tag{13}$$

Replacing j by $j-1$, we can obtain the conclusion. □

Lemma 4. *The functions in* (4) *satisfy*

$$I_j(\rho) = \frac{1}{\left(1-\rho^2\right)^{j-1/2}}I_{1-j}(\rho), \quad j\in Z. \tag{14}$$

Moreover,

$$\begin{aligned}
I_j(\rho) &= \overline{R}_{(-j)}\left(\rho^2\right), \quad j\le 0, \\
I_j(\rho) &= \frac{\overline{R}_{j-1}\left(\rho^2\right)}{\left(1-\rho^2\right)^{j-1/2}}, \quad j\ge 1,
\end{aligned} \tag{15}$$

where $\overline{R}_l$ denotes a polynomial of (exact) degree l.

Proof. It is easy to check that equality (14) is true for $j=0,1$. Now we prove that it is true for any $j\ge 2$ by induction.

Suppose that it is true for j and $j+1$; that is,

$$\begin{aligned}
\left(1-\rho^2\right)^{j-1/2}I_j(\rho) &= I_{1-j}(\rho), \\
\left(1-\rho^2\right)^{j+1/2}I_{j+1}(\rho) &= I_{-j}(\rho).
\end{aligned} \tag{16}$$

We need to prove that

$$\left(1-\rho^2\right)^{j+3/2}I_{j+2}(\rho) = I_{-(j+1)}(\rho). \tag{17}$$

By Lemma 3, we have

$$\left(1-\rho^2\right)I_{j+2}(\rho) = -\frac{2j+1}{2j+2}\left(\rho^2-2\right)I_{j+1}(\rho)-\frac{j}{j+1}I_j(\rho). \tag{18}$$

Hence,

$$\begin{aligned}
&\left(1-\rho^2\right)^{j+3/2}I_{j+2}(\rho) \\
&= -\frac{2j+1}{2j+2}\left(\rho^2-2\right)\left(1-\rho^2\right)^{j+1/2}I_{j+1}(\rho) \\
&\quad - \frac{j}{j+1}\left(1-\rho^2\right)^{j+1/2}I_j(\rho).
\end{aligned} \tag{19}$$

Then it follows from assumption (16) that

$$\begin{aligned}
&\left(1-\rho^2\right)^{j+3/2}I_{j+2}(\rho) \\
&= -\frac{2j+1}{2j+2}\left(\rho^2-2\right)I_{-j}(\rho) \\
&\quad - \frac{j}{j+1}\left(1-\rho^2\right)I_{1-j}(\rho).
\end{aligned} \tag{20}$$

By Lemma 3 again we have

$$\begin{aligned}
&\left(1-\rho^2\right)I_{1-j}(\rho) \\
&= -\frac{2j+1}{2j}\left(\rho^2-2\right)I_{-j}(\rho)-\frac{j+1}{j}I_{-(j+1)}(\rho).
\end{aligned} \tag{21}$$

Substituting it into (20) we obtain

$$\begin{aligned}
&\left(1-\rho^2\right)^{j+3/2}I_{j+2}(\rho) \\
&= -\frac{2j+1}{2j+2}\left(\rho^2-2\right)I_{-j}(\rho) \\
&\quad - \frac{j}{j+1}\left[-\frac{2j+1}{2j}\left(\rho^2-2\right)I_{-j}(\rho)-\frac{j+1}{j}I_{-(j+1)}(\rho)\right] \\
&= I_{-(j+1)}(\rho),
\end{aligned} \tag{22}$$

which gives (17). Hence equality (14) holds for $j\ge 0$.

If $j\le -1$, then $\tilde{j}=1-j\ge 2$. By applying (14) for $\tilde{j}\ge 2$, it is easy to see that (14) holds for all $j\le -1$.

The first formula in (15) for the case of $j\le 0$ follows directly from (4). The second one follows from the first one together with (14). This completes the proof. □

Some explicit expressions of $I_j(\rho)$ are

$$\begin{aligned} I_1(\rho) &= \frac{2\pi}{(1-\rho^2)^{1/2}}, \\ I_2(\rho) &= \frac{(2-\rho^2)\pi}{(1-\rho^2)^{3/2}}, \\ I_3(\rho) &= \frac{(3\rho^4-8\rho^2+8)\pi}{4(1-\rho^2)^{5/2}}, \\ I_0(\rho) &= 2\pi, \\ I_{-1}(\rho) &= (2-\rho^2)\pi, \\ I_{-2}(\rho) &= \frac{(3\rho^4-8\rho^2+8)\pi}{4}. \end{aligned} \tag{23}$$

Lemma 5. *For any nonnegative integer numbers p, q, and m, one has*

$$\int_0^{2\pi} \frac{\sin^{2p+1}\theta\cos^q\theta}{(1-\rho^2\sin^2\theta)^m}d\theta = 0, \tag{24}$$

$$\int_0^{2\pi} \frac{\sin^{p}\theta\cos^{2q+1}\theta}{(1-\rho^2\sin^2\theta)^m}d\theta = 0. \tag{25}$$

Proof. Since the integrand is an odd function of θ, (24) follows. Further

$$\int_0^{2\pi} \frac{\sin^{p}\theta\cos^{2q+1}\theta}{(1-\rho^2\sin^2\theta)^j}d\theta = \int_0^{2\pi} \frac{\sin^{p}\theta(1-\sin^2\theta)^q}{(1-\rho^2\sin^2\theta)^j}d\sin\theta = 0. \tag{26}$$

Then (25) follows, and the proof is ended. □

Lemma 6. *Let $M(\rho)$ be the Abelian integral given in* (3). *Then, there exist polynomials $\widetilde{R}_l$ of degree l, $l = 0, 1, \ldots, [(n+1)/2]$, such that, for $\rho \in [0, 1)$,*

$$M(\rho) = \sum_{j=1}^{[(n+1)/2]+1} \widetilde{R}_{[(n+1)/2]+1-j}(\rho^2) I_{m+1-j}(\rho). \tag{27}$$

Proof. In polar coordinates, $x = \rho\cos\theta$ and $y = \rho\sin\theta$, the integral $M(\rho)$ writes as

$$\begin{aligned} M(\rho) = -\int_0^{2\pi} \Big(&(Q(\rho\cos\theta, \rho\sin\theta)\rho\sin\theta \\ &+P(\rho\cos\theta, \rho\sin\theta)\rho\cos\theta) \\ &\times \left(\left(1-\rho^2\sin^2\theta\right)^m\right)^{-1}\Big) d\theta. \end{aligned} \tag{28}$$

Note that

$$\int_0^{2\pi} \frac{\sin^{p}\theta\cos^{2q}\theta}{(1-\rho^2\sin^2\theta)^m}d\theta = \int_0^{2\pi} \frac{\sin^{p}\theta(1-\sin^2\theta)^q}{(1-\rho^2\sin^2\theta)^m}d\theta. \tag{29}$$

Then by Lemma 5, we have

$$\begin{aligned} M(\rho) &= \sum_{j=1}^{n} \frac{R_j(\cos\theta, \sin\theta)\rho^j}{(1-\rho^2\sin^2\theta)^m}d\theta \\ &= \sum_{j=1}^{[(n+1)/2]} \frac{S_j(\sin^2\theta)\rho^{2j}}{(1-\rho^2\sin^2\theta)^m}d\theta, \end{aligned} \tag{30}$$

where $R_j(x, y)$ denotes a homogeneous polynomial of degree j in (x, y), and $S_j(x)$ denotes a polynomial of degree j in x having the form

$$S_j(\sin^2\theta) = \sum_{l=0}^{j} s_{2j,2l}(\sin^2\theta)^l. \tag{31}$$

By the above formula, we get

$$\begin{aligned} &\sum_{j=1}^{[(n+1)/2]} S_j(\sin^2\theta)\rho^{2j} \\ &= s_{2,2}\sin^2\theta\rho^2 + s_{4,4}\sin^4\theta\rho^4 \\ &\quad + \cdots + s_{2[(n+1)/2],2[(n+1)/2]}\sin^{2[(n+1)/2]}\theta\rho^{2[(n+1)/2]} \\ &\quad + \rho^2\big(s_{2,0} + s_{4,2}\sin^2\theta\rho^2 + \cdots + s_{2[(n+1)/2],2[(n+1)/2]-2} \\ &\qquad \times \sin^{2[(n+1)/2]-2}\theta\rho^{2[(n+1)/2]-2}\big) \\ &\quad + \cdots + \rho^{2[(n+1)/2]-2}\left(s_{2[(n+1)/2]-2,0} + s_{2[(n+1)/2],2}\sin^2\theta\rho^2\right) \\ &\quad + \rho^{2[(n+1)/2]}\left(s_{2[(n+1)/2],0}\right) \\ &= T_{[(n+1)/2]}(\rho^2\sin^2\theta) + \rho^2 T_{[(n+1)/2]-1}(\rho^2\sin^2\theta) \\ &\quad + \rho^4 T_{[(n+1)/2]-2}(\rho^2\sin^2\theta) \\ &\quad + \cdots + \rho^{2[(n+1)/2]}T_0(\rho^2\sin^2\theta) \\ &= \sum_{j=0}^{[(n+1)/2]} \rho^{2j} T_{[(n+1)/2]-j}(\rho^2\sin^2\theta), \end{aligned} \tag{32}$$

where T_i is a polynomial of degree i in $\rho^2\sin^2\theta$, $i = 0, 1, \ldots, [(n+1)/2]$.

Hence by (30) and Lemma 2,

$$\begin{aligned} &M(\rho) \\ &= \sum_{j=1}^{[(n+1)/2]} \frac{S_j(\sin^2\theta)\rho^{2j}}{(1-\rho^2\sin^2\theta)^m}d\theta \\ &= \sum_{j=0}^{[(n+1)/2]} \rho^{2j}\int_0^{2\pi} \frac{T_{[(n+1)/2]-j}(\rho^2\sin^2\theta)}{(1-\rho^2\sin^2\theta)^m}d\theta \end{aligned}$$

$$= \sum_{j=0}^{[(n+1)/2]} \rho^{2j} \left(\alpha_m I_m(\rho) + \alpha_{m-1} I_{m-1}(\rho) + \cdots + \alpha_{m-[(n+1)/2]+j} I_{m-[(n+1)/2]+j}(\rho)\right) = \sum_{j=1}^{[(n+1)/2]+1} \widetilde{R}_{([(n+1)/2]+1-j)}\left(\rho^2\right) I_{m+1-j}(\rho). \tag{33}$$

This ends the proof. □

Assume $\delta = \sqrt{1-\rho^2}$, $\rho \in [0,1)$. By Lemma 4, we have

$$I_j^*(\delta) = R_{-j}^*\left(\delta^2\right), \quad j \le 0, \qquad I_j^*(\delta) = \frac{R_{j-1}^*\left(\delta^2\right)}{\delta^{2j-1}}, \quad j \ge 1, \tag{34}$$

where

$$R_{-j}^*\left(\delta^2\right) = R_{-j}\left(1-\delta^2\right), \qquad I_j^*(\delta) = I_j\left(\sqrt{1-\delta^2}\right), \tag{35}$$

where R_l^* denotes a polynomial of degree l.

Lemma 7. *Let $M(\rho)$ be the Abelian integral given in* (3). *Then for $\delta \in (0,1]$*

$$\begin{gathered} M(\rho) = \frac{R_{m+[(n-1)/2]}^*\left(\delta^2\right)}{\delta^{2m-1}}, \quad n < 2m-1, \\ M(\rho) = \frac{R_{2[(n+1)/2]}^*(\delta)}{\delta}, \quad m = 1 \ (n \ge 2m-1), \\ M(\rho) = \frac{R_{m+[(n+1)/2]-1}^*\left(\delta^2\right)}{\delta^{2m-1}} + R_{[(n+1)/2]-m}^*\left(\delta^2\right), \\ m \ne 1 \ (n \ge 2m-1). \end{gathered} \tag{36}$$

Proof. We have the following cases.
Case 1 ($m+1-([(n+1)/2]+1) > 0$). In this case we have $[(n+1)/2] < m$, or $n < 2m-1$.

By Lemmas 4 and 6, we have

$$M(\rho) = \widetilde{R}_{[(n+1)/2]}\left(\rho^2\right) I_m(\rho) + \widetilde{R}_{[(n+1)/2]-1}\left(\rho^2\right) I_{m-1}(\rho) + \cdots + \widetilde{R}_0\left(\rho^2\right) I_{m-[(n+1)/2]}(\rho). \tag{37}$$

Let $M(\rho) = M^*(\delta)$. Then

$$\begin{aligned} M^*(\delta) &= R_{[(n+1)/2]}^*\left(\delta^2\right) I_m^*(\delta) + R_{[(n+1)/2]-1}^*\left(\delta^2\right) I_{m-1}^*(\delta) + \cdots + R_0^*\left(\delta^2\right) I_{m-[(n+1)/2]}^*(\delta) \\ &= R_{[(n+1)/2]}^*\left(\delta^2\right) \frac{R_{m-1}^*\left(\delta^2\right)}{\delta^{2m-1}} + R_{[(n+1)/2]-1}^*\left(\delta^2\right) \frac{R_{m-2}^*\left(\delta^2\right)}{\delta^{2m-3}} + \cdots + R_0^*\left(\delta^2\right) \frac{R_{m-[(n+1)/2]-1}^*\left(\delta^2\right)}{\delta^{2m-2[(n+1)/2]-1}} \\ &= \frac{R_{m+[(n-1)/2]}^*\left(\delta^2\right)}{\delta^{2m-1}}, \end{aligned} \tag{38}$$

where

$$R_{[(n+1)/2]+m-1}^*\left(\delta^2\right) = R_{[(n+1)/2]}^*\left(\delta^2\right) R_{m-1}^*\left(\delta^2\right) + \delta^2 R_{[(n+1)/2]-1}^*\left(\delta^2\right) R_{m-2}^*\left(\delta^2\right) + \cdots + \delta^{(2[(n+1)/2])} R_0^*\left(\delta^2\right) R_{m-[(n+1)/2]-1}^*\left(\delta^2\right), \tag{39}$$

which is a polynomial of degree $[(n+1)/2]+m-1$ (= $[(n-1)/2]+m$) in δ^2.

Hence,

$$M(\rho) = M^*(\delta) = \frac{R_{[(n-1)/2]+m}^*\left(\delta^2\right)}{\delta^{2m-1}}. \tag{40}$$

Case 2 ($m+1-([(n+1)/2]+1) \le 0$). In this case we have $[(n+1)/2] \ge m$ or $n \ge 2m-1$.

(i) When $m = 1$, we have

$$\begin{aligned} M(\rho) &= \sum_{j=1}^{[(n+1)/2]+1} \widetilde{R}_{[(n+1)/2]+1-j}\left(\rho^2\right) I_{2-j}(\rho) \\ &= \widetilde{R}_{[(n+1)/2])}\left(\rho^2\right) I_1(\rho) + \widetilde{R}_{[(n+1)/2]-1}\left(\rho^2\right) I_0(\rho) + \widetilde{R}_{[(n+1)/2]-2}\left(\rho^2\right) I_{-1}(\rho) + \cdots + \widetilde{R}_0\left(\rho^2\right) I_{1-[(n+1)/2]}(\rho). \end{aligned} \tag{41}$$

Thus,

$$\begin{aligned} M^*(\delta) &= R_{[(n+1)/2]}^*\left(\delta^2\right) \frac{R_0^*\left(\delta^2\right)}{\delta} + R_{[(n+1)/2]-1}^*\left(\delta^2\right) \cdot 2\pi + R_{[(n+1)/2]-2}^*\left(\delta^2\right) R_1^*\left(\delta^2\right) + \cdots + R_0^*\left(\delta^2\right) R_{[(n+1)/2]-1}^*\left(\delta^2\right) \\ &= \frac{R_{2[(n+1)/2]}^*(\delta)}{\delta}, \end{aligned} \tag{42}$$

where

$$\begin{aligned}
&R^*_{2[(n+1)/2]}(\delta) \\
&\quad= R^*_{[(n+1)/2]}(\delta^2) R^*_0(\delta^2) \\
&\qquad+ 2\pi\delta R^*_{[(n+1)/2]-1}(\delta^2) + \delta R^*_{[(n+1)/2]-2}(\delta^2) R^*_1(\delta^2) \\
&\qquad+ \cdots + \delta R^*_0(\delta^2) R^*_{[(n+1)/2]-1}(\delta^2).
\end{aligned} \tag{43}$$

Then

$$M^*(\delta) = \frac{R^*_{2[(n+1)/2]}(\delta)}{\delta}. \tag{44}$$

(ii) When $m \neq 1$, by Lemmas 4 and 6, we have

$$\begin{aligned}
M(\rho) &= \frac{\widetilde{R}_{[(n+1)/2]+m-1}(\rho^2)}{(1-\rho^2)^{m-1/2}} \\
&\quad+ \sum_{j=0}^{[(n+1)/2]-m} \widetilde{R}_{[(n+1)/2]-m-j}(\rho^2) I_{-j}(\rho) \\
&= \frac{\widetilde{R}_{[(n+1)/2]+m-1}(\rho^2)}{(1-\rho^2)^{m-1/2}} + \widetilde{R}_{[(n+1)/2]-m}(\rho^2) I_0(\rho) \\
&\quad+ \widetilde{R}_{[(n+1)/2]-m-1}(\rho^2) I_{-1}(\rho) \\
&\quad+ \cdots + \widetilde{R}_0(\rho^2) I_{-([(n+1)/2]-m)}(\rho).
\end{aligned} \tag{45}$$

Thus,

$$\begin{aligned}
&M^*(\delta) \\
&\quad= \frac{R^*_{[(n+1)/2]+m-1}(\delta^2)}{\delta^{2m-1}} + R^*_{[(n+1)/2]-m}(\delta^2) R^*_0(\delta^2) \\
&\qquad+ R^*_{[(n+1)/2]-m-1}(\delta^2) R^*_1(\delta^2) \\
&\qquad+ \cdots + R^*_0(\delta^2) R^*_{[(n+1)/2]-m}(\delta^2) \\
&\quad= \frac{R^*_{[(n+1)/2]+m-1}(\delta^2)}{\delta^{2m-1}} + R^*_{[(n+1)/2]-m}(\delta^2).
\end{aligned} \tag{46}$$

This ends the proof. □

The following lemma can be found in [5].

Lemma 8. *Consider the family of functions*

$$F(x) = A_i(x) + B_j(x)(a-x)^{\alpha}, \tag{47}$$

defined on $(-\infty, a)$*, where* A_i *and* B_j *are polynomials of degrees* i *and* j*, respectively, and* $\alpha \notin Z$*. Then each nontrivial function of the form has at most* $i+j+1$ *real zeros, taking into account their multiplicities. Moreover, there exist polynomials* A_i *and* B_j *such that the corresponding function has exactly this number of zeros on* $(-\infty, a)$*.*

3. Proof of Theorem 1

Proof. By Lemma 7, when $n < 2m-1$ we have

$$M(\rho) = \frac{R^*_{m+[(n-1)/2]}(\delta^2)}{\delta^{2m-1}}. \tag{48}$$

As the numerator of the above expression is a polynomial in δ^2 of degree $m + [(n-1)/2]$, the maximum number of zeros of $M(\rho)$ in [0,1) is $m + [(n-1)/2]$.

When $m = 1$ $(n \geq 2m-1)$, then

$$M(\rho) = \frac{R^*_{2[(n+1)/2]}(\delta)}{\delta}. \tag{49}$$

As before, the maximum number of zeros of $M(\rho)$ in [0,1) is $2[(n+1)/2]$.

When $m \neq 1$ $(n \geq 2m-1)$, we have

$$M(\rho) = \widetilde{M}(\delta^2) = \frac{R^*_{m+[(n+1)/2]-1}(\delta^2)}{\delta^{2m-1}} + R^*_{[(n+1)/2]-m}(\delta^2). \tag{50}$$

Then, using Lemma 8, $\widetilde{M}(\delta^2)$ has at most $(m + [(n+1)/2] - 1) + ([(n+1)/2] - m) + 1 = 2[(n+1)/2]$ zeros in $\delta^2 \in (0,1]$.

Finally, for all n and m we know that $M(0) = \widetilde{M}(1) = 0$. Then the maximum number of zeros of $M(\rho)$, taking into account their multiplicities, is $m + [(n-1)/2] - 1$ when $n < 2m-1$ and $2[(n+1)/2] - 1$ when $n \geq 2m-1$. The proof is completed. □

4. Two Illustration Examples on the Maximum Number

Consider the system

$$\begin{aligned}
\dot{x} &= -y(1-y^2) + \varepsilon P(x,y), \\
\dot{y} &= x(1-y^2) + \varepsilon Q(x,y),
\end{aligned} \tag{51}$$

where

$$P(x,y) = \sum_{i+j\leq 2} p_{ij} x^i y^j, \qquad Q(x,y) = \sum_{i+j\leq 2} q_{ij} x^i y^j \tag{52}$$

and ε is a small real parameter.

Let

$$M(\rho) = \int_{\gamma_\rho} \frac{Q(x,y)\,dx - P(x,y)\,dy}{1-y^2}, \tag{53}$$

where $\gamma_\rho = \{(x,y) : x^2 + y^2 = \rho^2\}$, $\rho \in (0,1)$. Assuming $x = \rho\cos\theta$, $y = \rho\sin\theta$, we have

$$\begin{aligned}
M(\rho) &= -\int_0^{2\pi} \frac{p_{10}\rho^2 + (q_{01} - p_{10})\rho^2\sin^2\theta}{1-\rho^2\sin^2\theta} d\theta \\
&= \left[-p_{10}\rho^2 - (q_{01} - p_{10})\right] I_1 + (q_{01} - p_{10}) I_0,
\end{aligned} \tag{54}$$

where

$$I_j(\rho) = \int_0^{2\pi} \frac{1}{(1-\rho^2\sin^2\theta)^j} d\theta, \quad \rho \in [0,1), \tag{55}$$

Let $\delta = \sqrt{1-\rho^2}$, $\rho \in [0,1)$. Then

$$\widetilde{M}(\delta) = M(\rho) = \frac{(\delta-1)(p_{10}\delta + q_{01})}{\delta}. \tag{56}$$

Obviously $\widetilde{M}(\delta) = 0$ for $\delta \in (0,1)$ if and only if $\delta = -q_{01}/p_{10}$. Thus for system (51) the function $M(\rho)$ can have 1 simple zero in $\rho \in (0,1)$.

Now we consider the system

$$\begin{aligned} \dot{x} &= -y(1-y^2)^2 + \varepsilon\overline{P}(x,y), \\ \dot{y} &= x(1-y^2)^2 + \varepsilon\overline{Q}(x,y), \end{aligned} \tag{57}$$

where

$$\overline{P}(x,y) = \sum_{i+j\le 2} \overline{p}_{ij} x^i y^j, \tag{58}$$

$$\overline{Q}(x,y) = \sum_{i+j\le 2} \overline{q}_{ij} x^i y^j, \tag{59}$$

and ε is a small real parameter.

Let

$$\begin{aligned} M(\rho) &= \int_{\gamma_\rho} \frac{\overline{Q}(x,y)\,dx - \overline{P}(x,y)\,dy}{(1-y^2)^2} \\ &= \left[-\overline{p}_{10}\rho^2 - (\overline{q}_{01} - \overline{p}_{10})\right] I_2 + (\overline{q}_{01} - \overline{p}_{10}) I_1, \end{aligned} \tag{60}$$

where $\gamma_\rho = \{(x,y) : x^2 + y^2 = \rho^2\}$, $\rho \in (0,1)$.

As before, let $\delta = \sqrt{1-\rho^2}$, $\rho \in [0,1)$. Then

$$\overline{M}(\delta) = M(\rho) = \frac{(1-\delta^2)(-\overline{p}_{10}\delta^2 - \overline{q}_{01})}{\delta^3}. \tag{61}$$

It follows that $\overline{M}(\delta) = 0$ for $\in (0,1)$ if and only if $\delta = \sqrt{-\overline{q}_{01}/\overline{p}_{10}}$. Thus for system (57) the function $M(\rho)$ can have 1 simple zero in $\rho \in (0,1)$.

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References

[1] V. I. Arnold, "Some unsolved problems in the theory of differential equations and mathematical physics," *Russian Mathematical Surveys*, vol. 44, pp. 157–171, 1989.

[2] J. Li, "Hilbert's 16th problem and bifurcations of planar polynomial vector fields," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 13, no. 1, pp. 47–106, 2003.

[3] H. Giacomini, J. Llibre, and M. Viano, "On the nonexistence, existence and uniqueness of limit cycles," *Nonlinearity*, vol. 9, no. 2, pp. 501–516, 1996.

[4] J. Llibre, J. S. Pérez del Río, and J. A. Rodríguez, "Averaging analysis of a perturbated quadratic center," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 46, no. 1, pp. 45–51, 2001.

[5] G. Xiang and M. Han, "Global bifurcation of limit cycles in a family of polynomial systems," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 2, pp. 633–644, 2004.

[6] A. Buică and J. Llibre, "Limit cycles of a perturbed cubic polynomial differential center," *Chaos, Solitons and Fractals*, vol. 32, no. 3, pp. 1059–1069, 2007.

[7] B. Coll, J. Llibre, and R. Prohens, "Limit cycles bifurcating from a perturbed quartic center," *Chaos, Solitons & Fractals*, vol. 44, no. 4-5, pp. 317–334, 2011.

[8] A. Atabaigi, N. Nyamoradi, and H. R. Z. Zangeneh, "The number of limit cycles of a quintic polynomial system," *Computers & Mathematics with Applications*, vol. 57, no. 4, pp. 677–684, 2009.

[9] A. Gasull, J. T. Lázaro, and J. Torregrosa, "Upper bounds for the number of zeroes for some Abelian integrals," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 13, pp. 5169–5179, 2012.

[10] A. Gasull, R. Prohens, and J. Torregrosa, "Bifurcation of limit cycles from a polynomial non-global center," *Journal of Dynamics and Differential Equations*, vol. 20, no. 4, pp. 945–960, 2008.

[11] G. Xiang and M. Han, "Global bifurcation of limit cycles in a family of multiparameter system," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 14, no. 9, pp. 3325–3335, 2004.

[12] A. Gasull, C. Li, and J. Torregrosa, "Limit cycles appearing from the perturbation of a system with a multiple line of critical points," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 75, no. 1, pp. 278–285, 2012.