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Research Article

Eigenvalues of a Class of Singular Boundary Value Problems of Impulsive Differential Equations in Banach Spaces

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This paper is devoted to investigating the eigenvalue problems of a class of nonlinear impulsive singular boundary value problem in Banach spaces: $\mu x'' + f(t,x) = 0, t \in (0,1), t \neq t_i$; $\Delta x|_{t=t_i} = \alpha_i x(t_i - 0), i = 1,2,\dots,k; ax(0) - bx'(0) = \theta; cx(1) + dx'(1) = \theta$, where θ denotes the zero element of Banach space, $\Delta x|_{t=t_i} = x(t_i+0) - x(t_i-0)$, $\alpha_i > -1, a, b, c, d \in R^+, \gamma = ac + ad + bc > 0$, μ is a parameter, and $f(t,x)$ may be singular at $t = 0, 1$ and $x = \theta$. The arguments are mainly based upon the theory of fixed point index, measure of noncompactness, and the special cone, which is constructed to overcome the singularity.

1. Introduction

Consider the following eigenvalue problems of singular boundary value problem (SBVP) with impulse in Banach space E:

$$\begin{gathered} \mu x'' + f(t,x) = 0, \quad t \in (0,1), \quad t \neq t_i; \\ \Delta x|_{t=t_i} = \alpha_i x(t_i - 0), \quad i = 1,2,\dots,k; \\ ax(0) - bx'(0) = \theta; \\ cx(1) + dx'(1) = \theta, \end{gathered} \tag{1}$$

where θ denotes the zero element of Banach space E, $\Delta x|_{t=t_i} = x(t_i+0) - x(t_i-0)$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} = 1$, $\alpha_i > -1, a, b, c, d \in R^+, \gamma = ac + ad + bc > 0$, μ is a parameter, $f(t,x)$ may be singular at $t = 0, 1$ and $x = \theta$.

During the last few decades, the existence of positive solution of nonlinear singular boundary value problems has gained considerable popularity (see [1–12] and references therein). In recent years, there were also a lot of papers which dealt with eigenvalue problems (see [13–27]). Some of them considered singular case (see, for instance, [13–16, 18, 19], etc.).

On the other hand, as we know, the theory of impulsive differential equations has found its extensive applications in realistic mathematical modeling of a wide variety of practical situations such as physics, chemical technology, population dynamics, biotechnology, and economics. It has emerged as an important area of investigation in recent years (see [28–36] and references therein).

To the best of our knowledge, there is no paper studying the eigenvalue problems of the impulsive singular boundary value problem in Banach spaces. The main purpose of this paper is to fill this gap. By using the theory of fixed point index, measure of noncompactness, and the special cone which is constructed to overcome the singularity, we investigate the existence of eigenvalues of (1).

The main features of the present paper are as follows. By virtue of a special transformation, we first convert (1) into another solvable form such that the associated operator can be used to overcome the influence of impulse and parameter μ. Then a special cone is constructed to deal with the singularity of (1).

This paper is organized as follows. In Section 2, we provide some basic definitions, preliminaries facts, and lemmas. Meanwhile, some transformations are introduced to convert (1) into another solvable form. In Section 3, the main results are presented and proved. Finally, an example is worked out to demonstrate the application of the main result.

2. Preliminaries and Conversion of (1)

Let P be a normal solid cone of real Banach space E. Without loss of generality, suppose the normal constant is 1. Let P^* denote the dual cone of P, $J = [0,1]$, and $P_r = \{x \in P : \|x\| < r\}$, $\overline{P}_r = \{x \in P : \|x\| \le r\}$ $(r > 0)$. Denoted by $C[J,E]$ the Banach space of all continuous functions $x : J \to E$ with norm $\|x\|_c = \max_{t\in J}\|x(t)\|$.

We define $PC[J,E] = \{x : x$ is a map from J into E such that $x(t)$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and its right limit at $t = t_i$ (denoted by $x(t_i^+)$) exists for $i = 1,2,\dots,m\}$, and $PC^1[J,E] = \{x \in PC[J,E] : x'(t)$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and the right limit at $t = t_i$ (denoted by $x'(t_i^+)$) exists for $i = 1,2,\dots,m\}$.

Let $Y = \{y \in C^1[J,E] : ay(0) - by'(0) = \theta, cy(1) + dy'(1) = \theta\}$, and $X = \{x \in PC^1[J,E] : \Delta x|_{t=t_i} = \alpha_i x(t_i - 0)$, $\Delta x'|_{t=t_i} = \alpha_i x'(t_i - 0)$, $ax(0) - bx'(0) = \theta$, $cx(1) + dx'(1) = \theta\}$. It is well known that Y is a Banach space with the norm $\|y\|_1 = \max\{\max_{t\in J}\|y(t)\|, \max_{t\in J}\|y'(t)\|\}$.

Evidently, $PC[J,E]$ and $PC^1[J,E]$ are Banach spaces with norm $\|x\|_{PC} = \sup_{t\in J}\|x(t)\|$ and $\|x\|_{PC^1} = \max\{\sup_{t\in J}\|x(t)\|, \sup_{t\in J}\|x'(t)\|\}$, respectively. Furthermore, X is a closed subspace of $PC^1[J,E]$.

For each $y \in Y$, let $x(t) = \prod_{0<t_i<t}(1+\alpha_i)y(t)$; it is easy to see $x \in X$. Conversely, for each $x \in X$, let $y(t) = x(t)/\prod_{0<t_i<t}(1+\alpha_i)$; then $y \in Y$.

We convert (1) into another solvable form first.

Lemma 1. *If $x \in X$ is a solution of impulsive SBVP* (1)*, then $y(t) = x(t)/\prod_{0<t_i<t}(1+\alpha_i)$ satisfies SBVP:*

$$\mu y'' + \frac{1}{\prod_{0<t_i<t}(1+\alpha_i)} f\left(t, \prod_{0<t_i<t}(1+\alpha_i)\, y(t)\right) = 0, \quad t \in (0,1); \qquad (2)$$
$$ay(0) - by'(0) = \theta;$$
$$cy(1) + dy'(1) = \theta.$$

Conversely, if $y \in Y$ is a solution of SBVP (2)*, then $x(t) = \prod_{0<t_i<t}(1+\alpha_i)y(t)$ is a solution of impulsive SBVP* (1)*.*

Lemma 2. *$y \in Y$ is a solution of SBVP* (2) *if and only if y is a solution of the integral equation:*

$$\mu y(t) = \int_J \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} G(t,s) \times f\left(t, \prod_{0<t_i<s}(1+\alpha_i)\, y(s)\right) ds, \qquad (3)$$

where

$$G(t,s) = \begin{cases} \gamma^{-1}(at+b)(c(1-s)+d), & 0 \le t \le s \le 1, \\ \gamma^{-1}(as+b)(c(1-t)+d), & 0 \le s \le t \le 1. \end{cases} \qquad (4)$$

By Lemmas 1 and 2, we can obtain the following.

Lemma 3. *$x(t) \in PC[J,E]\bigcap X$ is a solution of impulsive SBVP* (1) *if and only if $x \in PC[J,E]\bigcap X$ is a solution of the integral equation:*

$$\mu x(t) = \int_J G^*(t,s) f(s, x(s))\, ds, \qquad (5)$$

where

$$G^*(t,s) = \frac{\prod_{0<t_i<t}(1+\alpha_i)}{\prod_{0<t_i<s}(1+\alpha_i)} G(t,s). \qquad (6)$$

Let $\alpha(\cdot)$ and $\alpha_{pc}(\cdot)$ donated the Kuratowski noncompactness measure of bounded sets in E and $PC[J,E]$, respectively. If nontrivial function $x \in PC[J,E]$ satisfies problem (5) for some $\mu \neq 0$, then μ is called an eigenvalue and x is called an eigenfunction of problem (1) corresponding to the eigenvalue μ.

It is well known that the following conclusions hold.

Lemma 4 (see [17]). *Let P be a cone in Banach space E, $P_r = \{x \in P : \|x\| < r\}$ $(r > 0)$, and $\overline{P}_r = \{x \in P : \|x\| \le r\}$. Let operator $A : \overline{P}_r \to P$ be a strict set contraction. If $\|Ax\| \ge \|x\|$ and $Ax \neq x$ for $x \in \partial P_r$, then $i(A, P_r, P) = 0$.*

Lemma 5 (see [17]). *$H \subset C[J,E]$ is relatively compact if and only if H is equicontinuous and for any $t \in J$, $H(t)$ is a relatively compact set in E.*

3. Main Results

For convenience, let

$$\sigma_* =: \min_{1\le j\le k}\prod_{1\le i\le j}(1+\alpha_i), \qquad \sigma^* =: \max_{1\le j\le k}\prod_{1\le i\le j}(1+\alpha_i),$$
$$\Omega(t) = \frac{\varrho(t)\prod_{0<t_i<t}(1+\alpha_i)}{\sigma^*}, \qquad (7)$$
$$\varrho(t) = \min\left\{\frac{b+at}{b+a}, \frac{d+c(1-t)}{d+c}\right\}, \quad \forall t \in (0,1).$$

For the forthcoming analysis, we list the following assumptions:

(H_1) $f \in C[(0,1)\times P\setminus\{\theta\}, P]$,

$$\|f(t,x)\| \le g(t)\|l(x)\|, \quad t \in (0,1), \ x \in P\setminus\{\theta\}, \qquad (8)$$

where $g : (0,1) \to (0,+\infty)$, $l \in C[P\setminus\{\theta\}, P]$ satisfy

$$\int_0^1 G(s,s)\, g(s)\, l[\Omega(s)r, R]\, ds < +\infty, \quad \forall R > r > 0, \qquad (9)$$

where $l[r,R] = \sup_{x\in\overline{P}_R\setminus P_r}\|l(x)\| < +\infty$.

(H_2) There exists a constant $L \geq 0$ such that

$$\alpha(f(t,D)) \leq L\alpha(D), \quad \forall t \in (0,1),$$
$$D \subset \overline{P}_R \setminus P_r, \quad \forall R > r > 0,$$
$$L < \frac{\sigma_*}{2M_0\sigma^*}, \quad M_0 = \max_{s\in J} \gamma^{-1}(as+b)(c(1-s)+d) > 0. \tag{10}$$

(H_3) There exist $u_0 \in P \setminus \{\theta\}$, $[a_0, b_0] \subset (0,1)$ such that

$$f(t,x) \geq h(\|x\|) u_0, \quad t \in J_0 = [a_0, b_0],$$
$$\frac{2\sigma^*}{\|u_0\| \sigma_* \max_{t\in J} \int_{a_0}^{b_0} G(t,s)\Omega(s)\, ds} < \liminf_{t\to 0+} \frac{h(t)}{t} \leq +\infty, \tag{11}$$

where $h \in C[R^+, R^+]$.

(H_4) There exist $\phi \in P^*$ and $[a_1, b_1] \subset (0,1)$ such that $\phi(x) > 0$ for $x > \theta$ and

$$\liminf_{\|x\|\to\infty, x\in P} \frac{\phi(f(t,x))}{\phi(x)} = \xi(t) \tag{12}$$

uniformly for $t \in J_1 = [a_1, b_1]$, and $\int_{J_1} G(t,s)\xi(s)ds > \sigma^*/\sigma_*$.

(H_5) There exist $\phi \in P^*$ and $[a_1, b_1] \subset (0,1)$ such that $\phi(x) > 0$ for $x > \theta$ and

$$\liminf_{\|x\|\to 0, x\in P} \frac{\phi(f(t,x))}{\phi(x)} = \xi(t) \tag{13}$$

uniformly for $t \in J_1 = [a_1, b_1]$, and $\int_{J_1} G(t,s)\xi(s)ds > \sigma^*/\sigma_*$.

Define

$$Q = \{x \in PC[J,P] : x(t) \geq \Omega(t) x(s), \forall t, s \in J\}. \tag{14}$$

It is easy to check that Q is a cone in space $PC[J,E]$ and $Q \subset PC[J,P]$. Let $Q_r = \{x \in Q : \|x\| < r\}$. As in [2], we can prove that

$$G(t,\tau) \geq \varrho(t) G(s,\tau), \quad \forall t, s, \tau \in [0,1], \tag{15}$$

where $G(t,s)$ is defined in Lemma 2. So for all $t, s, \tau \in [0,1]$, we have

$$\begin{aligned} G^*(t,s) &\geq \frac{\varrho(t)\prod_{0<t_i<t}(1+\alpha_i)}{\prod_{0<t_i<\tau}(1+\alpha_i)} \cdot \frac{\prod_{0<t_i<\tau}(1+\alpha_i)}{\prod_{0<t_i<s}(1+\alpha_i)} G(\tau,s) \\ &\geq \Omega(t) G^*(\tau,s). \end{aligned} \tag{16}$$

To solve eigenvalues of impulsive SBVP (1), we first consider operator A associated with (5) and defined by

$$(Ax)(t) = \int_J G^*(t,s) f(s,x(s))\, ds, \quad \forall x \in Q \setminus \{\theta\}. \tag{17}$$

For $\forall x \in Q \setminus \{\theta\}$, from the definition of Q, we have

$$\Omega(t)\|x\|_{pc} \leq \|x(t)\| \leq \|x\|_{pc}, \quad t \in J. \tag{18}$$

By (H_1), we know

$$\begin{aligned} &\|(Ax)(t)\| \\ &\leq \prod_{0<t_i<t}(1+\alpha_i) \int_0^1 \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} G(s,s)\|f(s,x(s))\|\, ds \\ &\leq \prod_{0<t_i<t}(1+\alpha_i) \int_0^1 \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} \\ &\qquad \times G(s,s) g(s) \|l(x(s))\|\, ds \\ &\leq \frac{\sigma^*}{\sigma_*} \int_0^1 G(s,s) g(s) l\left[\Omega(s)\|x\|_{pc}, \|x\|_{pc}\right] ds < +\infty. \end{aligned} \tag{19}$$

So the operator A is well defined in $Q \setminus \{\theta\}$.

For the sake of overcoming the singularity, choose $e \in \operatorname{int} P$ with $\|e\| = 1$ and consider the approximate problem of (17):

$$(A_m x)(t) = \int_0^1 G^*(t,s) f\left(s, x(s) + \frac{e}{m}\right) ds. \tag{20}$$

For any $x \in Q, t, \tau \in J$,

$$\begin{aligned} (A_m x)(t) &= \int_0^1 G^*(t,s) f\left(s, x(s) + \frac{e}{m}\right) ds \\ &\geq \Omega(t) \int_0^1 G^*(\tau,s) f\left(s, x(s) + \frac{e}{m}\right) ds \\ &= \Omega(t)(A_m x)(\tau). \end{aligned} \tag{21}$$

Hence $A_m Q \subset Q$.

Lemma 6. *Let conditions (H_1) and (H_2) be satisfied; then for any $r > 0$, the operator A_m is a strict set contraction from Q into Q.*

Proof. Obviously $A_m Q \subset Q$. Now we prove that A_m is continuous. Let $\|x_n - x\|_{pc} \to 0$ as $n \to \infty$ ($x_n, x \in \overline{Q}_r$); we have

$$\begin{aligned}
&\left\|(A_m x_n)(t) - (A_m x)(t)\right\| \\
&\quad \le \prod_{0<t_i<t} (1+\alpha_i) \int_0^1 \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} G(t,s) \\
&\qquad \times \left\| f\left(s, x_n(s) + \frac{e}{m}\right) - f\left(s, x(s) + \frac{e}{m}\right)\right\| ds \\
&\quad \le \prod_{0<t_i<t} (1+\alpha_i) \int_0^1 \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} G(s,s)\, g(s) \\
&\qquad \times \left(\left\| l\left(x_n(s) + \frac{e}{m}\right)\right\| + \left\| l\left(x(s) + \frac{e}{m}\right)\right\|\right) \\
&\quad \le 2 \prod_{0<t_i<t} (1+\alpha_i) \int_0^1 \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} G(s,s)\, g(s) \\
&\qquad \times l\left[\Omega(s) r, r+1\right] ds \\
&\quad \le 2 \frac{\sigma^*}{\sigma_*} \int_0^1 G(s,s)\, g(s)\, l\left[\Omega(s) r, r+1\right] ds.
\end{aligned} \tag{22}$$

So (H_1) and the dominated convergence theorem imply that

$$\lim_{n\to\infty} (A_m x_n)(t) = (A_m x)(t). \tag{23}$$

We now show that

$$\lim_{n\to\infty} \left\|A_m x_n - A_m x\right\|_{pc} = 0. \tag{24}$$

In fact, if (24) is not true, then there exist a positive number ε_0 and a sequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\left\|A_m x_{n_i} - A_m x\right\|_{pc} \ge \varepsilon_0 \qquad (i = 1, 2, \ldots). \tag{25}$$

Since $\{A_m x_n\}$ is relatively compact, there is a subsequence of $\{A_m x_{n_i}\}$ which converges to some $x_0 \in Q$. Without loss of generality, we may assume that $\{A_m x_{n_i}\}$ itself converges to x_0; that is,

$$\lim_{i\to\infty} \left\|A_m x_{n_i} - x_0\right\|_{pc} = 0. \tag{26}$$

By virtue of (23) and (26), we have $x_0 = A_m x$, and so (26) contradicts with (25). Hence (24) holds, and the continuity of A_m is proved.

By (H_1), it is easy to see that A_m is bounded from Q into Q.

Now we will prove that the operator A_m is a strict set contraction from Q into Q. Let

$$\left(A_m^{(n)} x\right)(t) = \int_{1/n}^{1-(1/n)} G^*(t,s)\, f\left(s, x(s) + \frac{e}{m}\right) ds. \tag{27}$$

By (H_1), as $n \to \infty$, we have

$$\begin{aligned}
&\left\|(A_m x)(t) - \left(A_m^{(n)} x\right)(t)\right\| \\
&\quad \le \int_0^{1/n} G^*(t,s) \left\| f\left(s, x(s) + \frac{e}{m}\right)\right\| ds \\
&\qquad + \int_{1-(1/n)}^1 G^*(t,s) \left\| f\left(s, x(s) + \frac{e}{m}\right)\right\| ds \longrightarrow 0.
\end{aligned} \tag{28}$$

So for any bounded set $U \subset Q$, we have

$$\lim_{n\to\infty} \left(A_m^{(n)} x\right)(t) = (A_m x)(t), \quad \forall x \in U,\ t \in J. \tag{29}$$

Hence

$$d_h\left(\left(A_m^{(n)} U\right)(t), (A_m U)(t)\right) \longrightarrow 0, \quad n \longrightarrow \infty, \tag{30}$$

where $d_h(\cdot,\cdot)$ denotes the Hausdorff metric, which implies

$$\alpha\left(\left(A_m^{(n)} U\right)(t)\right) \longrightarrow \alpha\left((A_m U)(t)\right), \quad n \longrightarrow \infty,\ t \in J. \tag{31}$$

For any $t \in J$, by (H_2), we have

$$\begin{aligned}
&\alpha\left(\left(A_m^{(n)} U\right)(t)\right) \\
&\quad = \alpha\Bigg(\Bigg\{\prod_{0<t_i<t} (1+\alpha_i) \\
&\qquad \times \int_{1/n}^{1-(1/n)} \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} G(t,s) \\
&\qquad \times f\left(s, x(s) + \frac{e}{m}\right) ds, x \in U\Bigg\}\Bigg) \\
&\quad \le \left(1 - \frac{1}{2n}\right) \prod_{0<t_i<t} (1+\alpha_i) \\
&\qquad \times \alpha\Bigg(\overline{co}\Bigg\{\frac{1}{\prod_{0<t_i<s}(1+\alpha_i)}\ G(t,s) \\
&\qquad \times f\left(s, x(s) + \frac{e}{m}\right) : s \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right], x \in U\Bigg\}\Bigg)
\end{aligned}$$

$$\begin{aligned}
&\leq \prod_{0<t_i<t}(1+\alpha_i)\\
&\quad\times\alpha\left(\left\{\frac{1}{\prod_{0<t_i<s}(1+\alpha_i)}G(s,s)\right.\right.\\
&\qquad\left.\left.\times f\left(s,x(s)+\frac{e}{m}\right):s\in\left[\frac{1}{n},1-\frac{1}{n}\right],x\in U\right\}\right)\\
&\leq\frac{\sigma^*}{\sigma_*}M_0\alpha\left(f\left(s,U\left(I_n\right)\right)\right),\qquad I_n=\left[\frac{1}{n},1-\frac{1}{n}\right],\ s\in I_n,\\
&\leq LM_0\frac{\sigma^*}{\sigma_*}\alpha\left(U\left(I_n\right)\right)\\
&\leq 2LM_0\frac{\sigma^*}{\sigma_*}\alpha_{pc}(U)\\
&<\alpha_{pc}(U).
\end{aligned}\tag{32}$$

It is easy to show that $\alpha_{pc}(A_m(U)) = \sup_{t\in J}\alpha((A_m(U))(t))$. Therefore, we can have $\alpha_{pc}(A_mU) \leq 2LM_0(\sigma^*/\sigma_*)\alpha_{pc}(U) < \alpha_{pc}(U)$. Consequently, the operator A_m is a strict set contraction from Q into Q. The proof is thus completed. □

Theorem 7. *Let conditions* $(H_1), (H_2)$, *and* (H_3) *be satisfied; then there exists a positive number* δ *such that for any given* $r \in (0,\delta)$, *the impulsive SBVP* (1) *has a nontrivial eigenfunction* $x_r \in PC[J,P]$ *with* $\|x_r\|_{pc} = r$ *corresponding to eigenvalue* $\mu_r \geq 1$.

Proof. By Lemma 6, the operator A_m is a strict set contraction from $\overline{Q}_r$ into Q. Observing (H_3), if $\liminf_{t\to 0+}(h(t)/t) \neq +\infty$, we can choose a v_0 with

$$\frac{1}{2}\liminf_{t\to 0+}\frac{h(t)}{t}\leq v_0<\liminf_{t\to 0+}\frac{h(t)}{t},\tag{33}$$

and there is a $\delta > 0$ such that

$$h(t)\geq v_0t,\quad \forall 0<t<\delta.\tag{34}$$

Choose $r \in (0,\delta)$ and m sufficiently large such that $(1/m) < \delta - r$; we have

$$\begin{aligned}
&\left(A_mx\right)(t)\\
&\quad=\prod_{0<t_i<t}(1+\alpha_i)\int_0^1\frac{1}{\prod_{0<t_i<s}(1+\alpha_i)}\\
&\qquad\times G(t,s)f\left(s,x(s)+\frac{e}{m}\right)ds\\
&\quad\geq\prod_{0<t_i<t}(1+\alpha_i)\int_{a_0}^{b_0}\frac{1}{\prod_{0<t_i<s}(1+\alpha_i)}G(t,s)\\
&\qquad\times h\left(\left\|x(s)+\frac{e}{m}\right\|\right)u_0ds\\
&\quad\geq\prod_{0<t_i<t}(1+\alpha_i)\int_{a_0}^{b_0}\frac{1}{\prod_{0<t_i<s}(1+\alpha_i)}\\
&\qquad\times G(t,s)v_0\|x(s)\|u_0ds\\
&\quad\geq\frac{\sigma_*}{\sigma^*}\int_{a_0}^{b_0}G(t,s)v_0\Omega(s)\|x\|_{PC}u_0ds.
\end{aligned}\tag{35}$$

So

$$\begin{aligned}
&\left\|A_mx\right\|_{PC}\\
&\quad\geq\frac{\sigma_*}{\sigma^*}v_0\left\|u_0\right\|\max_{t\in J}\int_{a_0}^{b_0}G(t,s)\Omega(s)ds\|x\|_{PC}>\|x\|_{pc}.
\end{aligned}\tag{36}$$

By virtue of Lemma 4, we have $i(A_m,Q_r,Q) = 0$. Since $i(\theta,Q_r,Q) = 1$, it follows from the homotopy invariance of fixed point index for strict set contraction that there exist $x_m \in \partial\overline{Q}_r$ and $\lambda_m \in (0,1)$ such that

$$x_m=\lambda_mA_mx_m,\quad \text{i.e. } A_mx_m=\lambda_m^{-1}x_m,\tag{37}$$

where $m \geq m_0 = [1/(\delta - r)] + 1$.

Let $B =: \{x_m : m \geq m_0\}$. Obviously, B is uniformly bounded. We shall show that B is equicontinuous. By virtue of boundary condition, we need to consider only the following eight cases.

Case 1. $b = d = 0$, the boundary value condition is $x(0) = x(1) = 0$. So we need only to show that $\{x_m(t)\}$ uniformly converge to θ with respect to $m \geq m_0$ as $t \to 0^+, t \to 1-0$, and B is equicontinuous on any closed subinterval of $(0,1)$.

By (H_1), let

$$\eta=:\int_0^1G(s,s)g(s)l\left[\Omega(s)r,r+\frac{1}{m_0}\right]ds<+\infty.\tag{38}$$

Using the absolute continuity of integration, for any $\varepsilon > 0$, there exists a $\delta_1 > 0$, such that

$$\int_{t_1}^{t_2}G(s,s)g(s)l\left[\Omega(s)r,r+\frac{1}{m_0}\right]ds<\varepsilon\tag{39}$$

as $|t_1 - t_2| < \delta_1, t_1, t_2 \in J$. Since

$$\begin{aligned}
x_m(t)&=\lambda_m\int_0^1G^*(t,s)f\left(s,x_m(s)+\frac{e}{m}\right)ds\\
&\leq\lambda_m\frac{\sigma^*}{\sigma_*}\left[(1-t)\int_0^tsf\left(s,x_m(s)+\frac{e}{m}\right)ds\right.\\
&\qquad\left.+t\int_t^1(1-s)f\left(s,x_m(s)+\frac{e}{m}\right)ds\right],
\end{aligned}\tag{40}$$

we have

$$\begin{aligned}
&\|x_m(t)\| \\
&\leq \frac{\sigma^*}{\sigma_*}\left\{(1-t)\int_0^t sg(s)\left\|l\left(x_m(s)+\frac{e}{m}\right)\right\| ds\right. \\
&\qquad \left. +t\int_t^1 (1-s)\,g(s)\left\|l\left(x_m(s)+\frac{e}{m}\right)\right\| ds\right\} \\
&\leq \frac{\sigma^*}{\sigma_*}\left\{(1-t)\int_0^t sg(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds\right. \\
&\qquad \left. +t\int_t^1 (1-s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds\right\}.
\end{aligned} \tag{41}$$

To prove that $\{x_m(t)\}$ uniformly converge to θ with respect to $m \geq m_0$ as $t \to 0^+$, we need only to show

$$\lim_{t\to 0^+}(1-t)\int_0^t sg(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds = 0, \tag{42}$$

$$\lim_{t\to 0^+} t\int_t^1 (1-s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds = 0. \tag{43}$$

Notice that

$$\begin{aligned}
&(1-t)\int_0^t sg(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds \\
&\quad \leq \int_0^t (1-s)\,sg(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds \\
&\quad = \int_0^t G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds.
\end{aligned} \tag{44}$$

This together with (39) implies that (42) holds. For ε, δ_1 in (39), choose

$$\delta_2 = \min\left\{\delta_1, \frac{\varepsilon\delta_1}{\eta}\right\}. \tag{45}$$

Then for $t \in (0, \delta_2)$, (39) implies that

$$\begin{aligned}
&t\int_t^1 (1-s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds \\
&\quad \leq t\int_t^{\delta_1} (1-s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds \\
&\quad + \int_{\delta_1}^1 \frac{t}{s} s(1-s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds \\
&\leq \int_t^{\delta_1} s(1-s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds \\
&\quad + \frac{t}{\delta_1}\int_{\delta_1}^1 s(1-s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds \\
&= \int_t^{\delta_1} G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds \\
&\quad + \frac{t}{\delta_1}\int_{\delta_1}^1 G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds \\
&\leq \varepsilon + \frac{t}{\delta_1}\int_0^1 G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right] ds \\
&\leq 2\varepsilon.
\end{aligned} \tag{46}$$

Hence (43) holds. Very similarly, we can obtain that $\{x_m(t)\}$ uniformly converge to θ with respect to $m \geq m_0$ as $t \to 1-0$. Now, we show that B is equicontinuous on $[\zeta, 1-\zeta]$ for any $[\zeta, 1-\zeta] \subset (0,1), \zeta \in (0, 1/2)$. Notice that

$$\begin{aligned}
&x_m(t_2) - x_m(t_1) \\
&\quad = \lambda_m \prod_{0<t_i<t}(1+\alpha_i) \\
&\quad\quad \times \left[(1-t_2)\int_0^{t_2} s\frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} f\left(s, x_m(s)+\frac{e}{m}\right) ds\right. \\
&\quad\quad + t_2\int_{t_2}^1 (1-s)\frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} f\left(s, x_m(s)+\frac{e}{m}\right) ds \\
&\quad\quad - (1-t_1)\int_0^{t_1} s\frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} f\left(s, x_m(s)+\frac{e}{m}\right) ds \\
&\quad\quad \left. - t_1\int_{t_1}^1 (1-s)\frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} f\left(s, x_m(s)+\frac{e}{m}\right) ds\right] \\
&\quad = \lambda_m \prod_{0<t_i<t}(1+\alpha_i) \\
&\quad\quad \times \left[(t_2-t_1)\right. \\
&\quad\quad\quad \times \left(-\int_0^{t_2} s\frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} f\left(s, x_m(s)+\frac{e}{m}\right) ds\right. \\
&\quad\quad\quad + \int_{t_2}^1 (1-s)\frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} \\
&\quad\quad\quad\quad \left. \times f\left(s, x_m(s)+\frac{e}{m}\right) ds\right)
\end{aligned}$$

$$+\left(1-t_1\right)\int_{t_1}^{t_2} s\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times f\left(s,x_m\left(s\right)+\frac{e}{m}\right)ds$$
$$-t_1\int_{t_1}^{t_2}\left(1-s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times f\left(s,x_m\left(s\right)+\frac{e}{m}\right)ds\Bigg].\tag{47}$$

So for any $t_1, t_2 \in [\zeta, 1-\zeta]$, $t_2 > t_1$, $m \geq m_0$, we have

$$\begin{aligned}
&\left\|x_m\left(t_2\right)-x_m\left(t_1\right)\right\| \\
&\leq \prod_{0<t_i<t}\left(1+\alpha_i\right) \\
&\quad\times\Bigg\{\left(t_2-t_1\right)\Bigg[\int_0^{t_2} s\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)\left\|l\left(x_m\left(s\right)+\frac{e}{m}\right)\right\|ds \\
&\quad+\int_{t_2}^{1}\left(1-s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)\left\|l\left(x_m\left(s\right)+\frac{e}{m}\right)\right\|ds\Bigg] \\
&\quad+\left(1-t_1\right)\int_{t_1}^{t_2} s\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)\left\|l\left(x_m\left(s\right)+\frac{e}{m}\right)\right\|ds \\
&\quad+t_1\int_{t_1}^{t_2}\left(1-s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)\left\|l\left(x_m\left(s\right)+\frac{e}{m}\right)\right\|ds\Bigg\} \\
&\leq \prod_{0<t_i<t}\left(1+\alpha_i\right) \\
&\quad\times\Bigg\{\left(t_2-t_1\right)\Bigg(\frac{1}{1-t_2}\int_0^{t_2}\left(1-s\right)s\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)l\left[\Omega\left(s\right)r,r+\frac{1}{m_0}\right]ds \\
&\quad+\frac{1}{t_2}\int_{t_2}^{1}s\left(1-s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)l\left[\Omega\left(s\right)r,r+\frac{1}{m_0}\right]ds\Bigg) \\
&\quad+\frac{1-t_1}{1-t_2}\int_{t_1}^{t_2}s\left(1-s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)l\left[\Omega\left(s\right)r,r+\frac{1}{m_0}\right]ds \\
&\quad+\int_{t_1}^{t_2}s\left(1-s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)l\left[\Omega\left(s\right)r,r+\frac{1}{m_0}\right]ds\Bigg\} \\
&\leq \prod_{0<t_i<t}\left(1+\alpha_i\right) \\
&\quad\times\Bigg\{\left(t_2-t_1\right)\Bigg(\frac{1}{1-t_2}\int_0^{t_2}G\left(s,s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)l\left[\Omega\left(s\right)r,r+\frac{1}{m_0}\right]ds \\
&\quad+\frac{1}{t_2}\int_{t_2}^{1}G\left(s,s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)l\left[\Omega\left(s\right)r,r+\frac{1}{m_0}\right]ds\Bigg) \\
&\quad+\frac{1-t_1}{1-t_2}\int_{t_1}^{t_2}G\left(s,s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)l\left[\Omega\left(s\right)r,r+\frac{1}{m_0}\right]ds \\
&\quad+\int_{t_1}^{t_2}G\left(s,s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)l\left[\Omega\left(s\right)r,r+\frac{1}{m_0}\right]ds\Bigg\} \\
&\leq \prod_{0<t_i<t}\left(1+\alpha_i\right) \\
&\quad\times\Bigg\{\left(t_2-t_1\right)\Bigg(\frac{1}{\zeta}\int_0^{t_2}G\left(s,s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)l\left[\Omega\left(s\right)r,r+\frac{1}{m_0}\right]ds \\
&\quad+\frac{1}{\zeta}\int_{t_2}^{1}G\left(s,s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)l\left[\Omega\left(s\right)r,r+\frac{1}{m_0}\right]\Bigg) \\
&\quad+\left(\frac{1-\zeta}{\zeta}+1\right)\int_{t_1}^{t_2}G\left(s,s\right)\frac{1}{\prod_{0<t_i<s}\left(1+\alpha_i\right)}\times g\left(s\right)l\left[\Omega\left(s\right)r,r+\frac{1}{m_0}\right]ds\Bigg\}
\end{aligned}$$

$$\leq \frac{\sigma^*}{\sigma_*}\left\{(t_2-t_1)\frac{2\eta}{\zeta} + \frac{1}{\zeta}\int_{t_1}^{t_2} G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right]ds\right\}. \quad (48)$$

This together with (39) implies that B is equicontinuous on J.

Case 2. $b = 0, d \neq 0, c \neq 0$, the boundary value condition is $x(0) = 0, cx(1) + dx'(1) = 0$. We need to show that $\{x_m(t)\}$ uniformly converge to θ with respect to $m \geq m_0$ as $t \to 0^+$. This can be obtained by the similar way in Case 1, so we omit it. Next, we show B is equicontinuous on $[\iota, 1]$ for any $\iota \in (0,1)$. In fact, for any $t_1, t_2 \in [\iota, 1], t_2 > t_1, m \geq m_0$, we have

$$\begin{aligned}
&\left\|x_m(t_2) - x_m(t_1)\right\| \\
&\leq \int_0^{t_1}\left|G^*(t_2,s) - G^*(t_1,s)\right|\left\|f\left(s, x_m(s)+\frac{e}{m}\right)\right\|ds \\
&\quad + \int_{t_1}^{t_2}\left|G^*(t_2,s) - G^*(t_1,s)\right|\left\|f\left(s, x_m(s)+\frac{e}{m}\right)\right\|ds \\
&\quad + \int_{t_2}^{1}\left|G^*(t_2,s) - G^*(t_1,s)\right|\left\|f\left(s, x_m(s)+\frac{e}{m}\right)\right\|ds \\
&\leq (t_2-t_1)\frac{c}{c(1-s)+d}\frac{\sigma^*}{\sigma_*} \\
&\quad \times \int_0^{t_1}\frac{s(c(1-s)+d)}{c+d}g(s)\left\|l\left(x_m(s)+\frac{e}{m}\right)\right\|ds \\
&\quad + 2\frac{\sigma^*}{\sigma_*}\int_{t_1}^{t_2}G(s,s)\,g(s)\left\|l\left(x_m(s)+\frac{e}{m}\right)\right\|ds \\
&\quad + (t_2-t_1)\frac{1}{s}\frac{\sigma^*}{\sigma_*} \\
&\quad \times \int_{t_2}^{1}\frac{s(c(1-s)+d)}{c+d}g(s)\left\|l\left(x_m(s)+\frac{e}{m}\right)\right\|ds \\
&\leq (t_2-t_1)\frac{c}{d}\frac{\sigma^*}{\sigma_*}\int_0^{t_1}G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right]ds \\
&\quad + 2\frac{\sigma^*}{\sigma_*}\int_{t_1}^{t_2}G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right]ds \\
&\quad + (t_2-t_1)\frac{1}{\iota}\frac{\sigma^*}{\sigma_*}\int_{t_2}^{1}G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right]ds \\
&\leq (t_2-t_1)\,\eta\frac{\sigma^*}{\sigma_*}\left(\frac{c}{d}+\frac{1}{\iota}\right) \\
&\quad + 2\frac{\sigma^*}{\sigma_*}\int_{t_1}^{t_2}G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right]ds.
\end{aligned} \quad (49)$$

This together with (39) implies that B is equicontinuous on J.

Case 3. $b = 0, d \neq 0, c = 0$, the boundary value condition is $x(0) = 0, x'(1) = 0$. The equicontinuity of B on J of this case can be obtained by the same way of Case 2, so we omit it.

Case 4. $d = 0, b \neq 0, a \neq 0$, the boundary value condition is $ax(0) - bx'(0) = 0, x(1) = 0$. Similarly, by Case 1, we can get that $\{x_m(t)\}$ uniformly converge to θ with respect to $m \geq m_0$ as $t \to 1^-$. For any $\varsigma \in (0,1)$, we need to show that B is equicontinuous on $[0, \varsigma]$. In fact, for any $t_1, t_2 \in [0, \varsigma], t_2 > t_1, m \geq m_0$, we have

$$\begin{aligned}
&\left\|x_m(t_2) - x_m(t_1)\right\| \\
&\leq \int_0^{t_1}\left|G^*(t_2,s) - G^*(t_1,s)\right|\left\|f\left(s, x_m(s)+\frac{e}{m}\right)\right\|ds \\
&\quad + \int_{t_1}^{t_2}\left|G^*(t_2,s) - G^*(t_1,s)\right|\left\|f\left(s, x_m(s)+\frac{e}{m}\right)\right\|ds \\
&\quad + \int_{t_2}^{1}\left|G^*(t_2,s) - G^*(t_1,s)\right|\left\|f\left(s, x_m(s)+\frac{e}{m}\right)\right\|ds \\
&\leq (t_2-t_1)\frac{1}{1-s}\frac{\sigma^*}{\sigma_*}\int_0^{t_1}G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right]ds \\
&\quad + 2\frac{\sigma^*}{\sigma_*}\int_{t_1}^{t_2}G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right]ds \\
&\quad + (t_2-t_1)\frac{a}{as+b}\frac{\sigma^*}{\sigma_*}\int_{t_2}^{1}G(s,s)\,g(s) \\
&\qquad \times l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right]ds \\
&\leq (t_2-t_1)\frac{1}{1-\varsigma}\frac{\sigma^*}{\sigma_*}\int_0^{t_1}G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right]ds \\
&\quad + 2\frac{\sigma^*}{\sigma_*}\int_{t_1}^{t_2}G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right]ds \\
&\quad + (t_2-t_1)\frac{b}{a}\frac{\sigma^*}{\sigma_*}\int_{t_2}^{1}G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right]ds \\
&\leq (t_2-t_1)\,\eta\frac{\sigma^*}{\sigma_*}\left(\frac{1}{1-\varsigma}+\frac{b}{a}\right) \\
&\quad + 2\frac{\sigma^*}{\sigma_*}\int_{t_1}^{t_2}G(s,s)\,g(s)\,l\left[\Omega(s)\,r, r+\frac{1}{m_0}\right]ds.
\end{aligned} \quad (50)$$

This together with (39) implies that B is equicontinuous on J.

Case 5. $d = 0, b \neq 0, a = 0$, the boundary value condition is $x'(0) = 0, x(1) = 0$.

Case 6. $a = 0, bd \neq 0, c \neq 0$, the boundary value condition is $x'(0) = 0, cx(1) + dx'(1) = 0$.

Case 7. $a \neq 0, bd \neq 0, c = 0$, the boundary value condition is $ax(0) - bx'(0) = 0, x'(1) = 0$.

Case 8. $bd \neq 0, ac \neq 0$, the boundary value condition is $ax(0) - bx'(0) = 0, cx(1) + dx'(1) = 0$.

By the very similar methods in Cases 2 and 4, we can prove that B is equicontinuous on J in Cases 5–8. Hence, the proofs are omitted. Next we show that B is relatively compact. It is easy to see that

$$\begin{aligned}
&\alpha(B(t)) \\
&\leq \alpha\left(\left\{\prod_{0<t_i<t}(1+\alpha_i)\int_0^1 \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)}G(t,s)\right.\right. \\
&\qquad\qquad \left.\left.\times f\left(s, x_m(s)+\frac{e}{m}\right)ds : m \geq m_0\right\}\right) \\
&\leq 2\prod_{0<t_i<t}(1+\alpha_i)\int_0^1 \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)}G(t,s) \\
&\qquad\qquad \times \alpha\left(\left\{f\left(s, x_m(s)+\frac{e}{m}\right) : m \geq m_0\right\}\right)ds \\
&\leq 2\frac{\sigma^*}{\sigma_*}L\int_0^1 G(t,s)\,\alpha(B(s))\,ds.
\end{aligned} \tag{51}$$

Hence, we have

$$\begin{aligned}
\alpha_{pc}(B) &\leq 2\frac{\sigma^*}{\sigma_*}L\int_0^1 G(t,s)\,ds\,\alpha_{pc}(B) \\
&\leq 2\frac{\sigma^*}{\sigma_*}L\int_0^1 G(s,s)\,ds\,\alpha_{pc}(B) \\
&\leq 2LM_0\frac{\sigma^*}{\sigma_*}\alpha_{pc}(B).
\end{aligned} \tag{52}$$

Observing $L < \sigma_*/2M_0\sigma^*$, so $\alpha_{pc}(B) = 0$, which means B is relatively compact.

It follows from Lemma 5 that there is a convergent subsequence of $\{x_m\}$, and without loss of generality, we may assume that $\{x_m\}$ itself converges to some $x_r \in \partial\overline{Q}_r$, and $\lim_{m\to\infty}\lambda_m = \lambda_r \in [0,1]$. Hence (40) and the dominated convergence theorem imply that

$$x_r(t) = \lambda_r\int_0^1 G^*(t,s)\,f(s, x_r(s))\,ds = \lambda_r A x_r(t), \quad t \in J. \tag{53}$$

Obviously, $\lambda_r > 0$. Consequently, $Ax_r = \lambda_r^{-1}x_r = \mu_r x_r$, where $\mu_r = \lambda_r^{-1} \geq 1$.

The case $\liminf_{t\to 0+}(h(t)/t) = +\infty$ can be proved similarly, so it is omitted. Then the theorem is proved. □

Theorem 8. *Let conditions (H_1), (H_2), and (H_4) be satisfied; then there exists a positive number M such that for any given $r > M/\min_{t\in J}\Omega(t)$, the impulsive SBVP (1) has a nontrivial eigenfunction $x_r \in PC[J,P]$ with $\|x_r\|_{PC} = r$ corresponding to eigenvalue $\mu_r \geq 1$.*

Proof. By (H_4), there exist $\varepsilon_0 > 0$ and $M > 0$ such that

$$\int_{a_1}^{b_1} G(t,s)\,(\xi(s) - \varepsilon_0)\,ds > \frac{\sigma^*}{\sigma_*}, \tag{54}$$

and $\phi(f(t,x)) \geq (\xi(t) - \varepsilon_0)\phi(x)$ for $t \in J_1 = [a_1, b_1]$, $x \in P$ with $\|x\| > M$.

Choose $r > M/\min_{t\in J}\Omega(t)$. If there exists $x_0 \in \partial\overline{Q}$ such that $x_0 \geq A_m x_0$, then

$$\|x_0(s)\| \geq \Omega(s)\|x_0\|_{pc} = \Omega(s)\,r > M, \quad s \in J. \tag{55}$$

Hence, for $t \in J_1$, we have

$$\begin{aligned}
&\phi(x_0(t)) \\
&\geq \prod_{0<t_i<t}(1+\alpha_i)\int_0^1 \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} \\
&\qquad\qquad \times G(t,s)\,\phi\left(f\left(s, x_0(s)+\frac{e}{m}\right)\right)ds \\
&\geq \prod_{0<t_i<t}(1+\alpha_i)\int_{J_1} \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} \\
&\qquad\qquad \times G(t,s)\,(\xi(s)-\varepsilon_0)\,\phi\left(x_0(s)+\frac{e}{m}\right)ds \\
&\geq \prod_{0<t_i<t}(1+\alpha_i)\int_{J_1} \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} \\
&\qquad\qquad \times G(t,s)\,(\xi(s)-\varepsilon_0)\,\phi(x_0(s))\,ds \\
&\geq \frac{\sigma_*}{\sigma^*}\int_{J_1} G(t,s)\,(\xi(s)-\varepsilon_0)\,\phi(x_0(s))\,ds.
\end{aligned} \tag{56}$$

Obviously,

$$\inf_{t\in J_1}\|x_0(t)\| \geq \Omega(t)\|x_0\|_{PC} = \Omega(t)\,r > 0. \tag{57}$$

The continuity of $x_0(t)$ implies that $\phi(x_0(t))$ is continuous on J_1. Hence, this together with (57) and (H_4) implies that $\min_{t\in J_1}\phi(x_0(t)) > 0$. Observing (54) and (56), we can obtain a contradiction. Applying Lemma 4, we know that the fixed point index $i(A_m, Q_r, Q) = 0$. Since $i(\theta, Q_r, Q) = 1$, it follows from the homotopy invariance of fixed point index for strict set contraction that there exist $x_m \in \partial\overline{Q}_r$ and $0 < \lambda_m < 1$ such that

$$x_m = \lambda_m A_m x_m, \quad \text{i.e. } A_m x_m = \lambda_m^{-1}x_m = \mu_m x_m, \tag{58}$$

where $\mu_m = \lambda_m^{-1} > 1$. The rest of the proof is completely similar to that of Theorem 7. So it is omitted. □

Theorem 9. *Let conditions $(H_1), (H_2)$, and (H_5) be satisfied; then there exists a positive number δ such that for any given $r \in (0, \delta)$, the impulsive SBVP (1) has a nontrivial eigenfunction $x_r \in PC[J, P]$ with $\|x_r\|_{PC} = r$ corresponding to eigenvalue $\mu_r \geq 1$.*

Proof. By (H_5), there exist $\varepsilon_1 > 0$ and $\delta > 0$ such that

$$\int_{J_1} G(t, s)(\xi(s) - \varepsilon_1)\, ds > 1, \tag{59}$$

and $\phi(f(t, x)) \geq (\xi(t) - \varepsilon_1)\phi(x)$ for $t \in J_1 = [a_1, b_1], x \in P$ with $\|x\| < \delta$. Choose $r \in (0, \delta)$ and m such that $(1/m) < \delta - r$. If there exists $x_0 \in \partial\overline{Q}_r$ such that $x_0 \geq A_m x_0$, then for $t \in J_1$, we have

$$\begin{aligned}
&\phi(x_0(t)) \\
&\geq \prod_{0<t_i<t}(1+\alpha_i)\int_0^1 \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} \times G(t, s)\,\phi\left(f\left(s, x_0(s) + \frac{e}{m}\right)\right) ds \\
&\geq \prod_{0<t_i<t}(1+\alpha_i)\int_{J_1} \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} \times G(t, s)(\xi(s) - \varepsilon_1)\,\phi\left(x_0(s) + \frac{e}{m}\right) ds \\
&\geq \prod_{0<t_i<t}(1+\alpha_i)\int_{J_1} \frac{1}{\prod_{0<t_i<s}(1+\alpha_i)} \times G(t, s)(\xi(s) - \varepsilon_1)\,\phi(x_0(s))\, ds \\
&\geq \frac{\sigma_*}{\sigma^*}\int_{J_1} G(t, s)(\xi(s) - \varepsilon_1)\,\phi(x_0)\, ds.
\end{aligned} \tag{60}$$

Obviously,

$$\inf_{t\in J_1}\|x_0(t)\| \geq \Omega(t)\|x_0\|_{PC} = \Omega(t)\, r > 0. \tag{61}$$

The continuity of $x_0(t)$ implies that $\phi(x_0(t))$ is continuous on J_1. Hence, this together with (61) and (H_5) implies that $\min_{t\in J_1}\phi(x_0(t)) > 0$. Observing (59) and (60), then we can obtain a contradiction. Applying Lemma 4, we know that the fixed point index $i(A_m, Q_r, Q) = 0$. Since $i(\theta, Q_r, Q) = 1$, it follows from the homotopy invariance of fixed point index for strict set contraction that there exist $x_m \in \partial\overline{Q}_r$ and $0 < \lambda_m < 1$ such that

$$x_m = \lambda_m A_m x_m, \quad \text{i.e. } A_m x_m = \lambda_m^{-1} x_m = \mu_m x_m, \tag{62}$$

where $\mu_m = \lambda_m^{-1} > 1$. The rest of the proof is completely similar to that of Theorem 7. So it is omitted. □

Example 10. Consider the following impulsive SBVP:

$$\begin{aligned}
&\mu x_n'' + \frac{\pi}{\sqrt{t(1-t)}}\left(\frac{1}{3n}\left(\sqrt{t}x_n + e^{2+x_{n+1}}\right) + \frac{\arctan t}{\sqrt{n}\,\|x\|}\right) = 0, \\
&\qquad t \in (0, 1), \quad t \neq \frac{1}{5}; \\
&\Delta x_n|_{t=1/5} = \frac{2}{5}x_n, \\
&2x_n(0) - x_n'(0) = \theta; \\
&x_n(1) = \theta, \quad (n = 1, 2, 3, \ldots).
\end{aligned} \tag{63}$$

Conclusion. There exists a positive number δ such that for any given $r \in (0, \delta)$, impulsive SBVP (63) has a nontrivial, nonnegative eigenfunction x_{rn} $(n = 1, 2, 3, \ldots)$ satisfying $\sup_n x_{rn}(t) < \infty$ for $0 \leq t \leq 1$ and $\max_{t\in[0,1]}\sup_n x_{rn}(t) = r$ corresponding to $\mu_r \geq 1$.

Proof. Let $E = l^\infty = \{x = (x_1, \ldots, x_n, \ldots) : \sup_n|x_n| < \infty\}$ with norm $\|x\| = \sup_n|x_n|$ and $P = \{x = (x_1, \ldots, x_n, \ldots) \in l^\infty : x_n \geq 0, n = 1, 2, 3, \ldots\}$. Then, P is a normal solid cone of E with normal constant 1, and system (63) can be regarded as an impulsive SBVP in $E = l^\infty$ of the form (1), where $x(t) = (x_1(t), x_2(t), x_3(t), \ldots)$, $f(t, x) = (f_1, f_2, f_3, \ldots)$, and

$$f_n(t, x) = \frac{\pi}{\sqrt{t(1-t)}}\left(\frac{1}{3n}\left(\sqrt{t}x_n + e^{2+x_{n+1}}\right) + \frac{\arctan t}{\sqrt{n}\,\|x\|}\right). \tag{64}$$

It is easy to see that $f(t, x)$ is singular at $t = 0, 1$ and $x = \theta$.

Let $g(t) = \pi/\sqrt{t(1-t)}$ and $l(x) = (l_1(x), l_2(x), l_3(x), \ldots)$, where

$$l_n(x) = \frac{1}{3n}\left(x_n + e^{2+x_{n+1}}\right) + \frac{\pi}{4\sqrt{n}\,\|x\|}. \tag{65}$$

So we have

$$\begin{aligned}
&\|f(t, x)\| \leq g(t)\|l(x)\|, \quad t \in (0, 1), \ x \in P \setminus \{\theta\}, \\
&l[r, R] = \sup_{x\in\overline{P}_R\setminus P_r}\|l(x)\| < R + e^{2+R} + \frac{\pi}{4r} < +\infty, \\
&\qquad \forall R > r > 0.
\end{aligned} \tag{66}$$

By virtue of $\int_0^1(\pi/\sqrt{t(1-t)})dt = \pi^2$, $G(s, s) = (1/3)(1 + 2s)(1 - s)$, and

$$\begin{aligned}
&M_0 = \max_{s\in J} G(s, s) = \frac{3}{8}, \\
&\Omega(s) = \min\left\{\frac{1+2s}{3}, 1 - s\right\} \geq \frac{1}{3}(1+2s)(1-s),
\end{aligned} \tag{67}$$

we can obtain

$$\begin{aligned}
&\int_0^1 G(s, s)\, g(s)\, l[\Omega(s) r, R]\, ds \\
&\leq \frac{3\pi^2}{8}\left(R + e^{2+R}\right) + \frac{\pi^3}{4r} < +\infty.
\end{aligned} \tag{68}$$

So condition (H_1) is satisfied. On the other hand, by the diagonal method of choosing subsequence, we can see that $L = 0$ in condition (H_2); that is, condition (H_2) holds.

We now show that condition (H_3) is satisfied for

$$J_0 = \left[\frac{1}{2}, \frac{9}{10}\right], \quad h(t) = \frac{1}{t},$$
$$u_0 = \left(2\pi \arctan\frac{1}{2}, \frac{2}{\sqrt{2}}\pi \arctan\frac{1}{2}, \frac{2}{\sqrt{3}}\pi \arctan\frac{1}{2}, \ldots\right). \tag{69}$$

In fact,

$$f_n(t,x) \geq \frac{\pi}{\sqrt{t(1-t)}} \frac{\arctan t}{\sqrt{n}\|x\|} \geq \frac{2}{\sqrt{n}}\pi \arctan\frac{1}{2} \cdot \frac{1}{\|x\|}, \quad \forall t \in \left[\frac{1}{2}, \frac{9}{10}\right], \tag{70}$$

which means that condition (H_3) is satisfied for

$$u_0 = \left(2\pi \arctan\frac{1}{2}, \frac{2}{\sqrt{2}}\pi \arctan\frac{1}{2}, \frac{2}{\sqrt{3}}\pi \arctan\frac{1}{2}, \ldots\right),$$
$$J_0 = \left[\frac{1}{2}, \frac{9}{10}\right], \quad h(t) = \frac{1}{t}; \tag{71}$$

that is, condition (H_3) is satisfied.

Hence, our conclusion follows from Theorem 7. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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