Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 627248, 4 pages
http://dx.doi.org/10.1155/2014/627248

Research Article

On Harmonious Labeling of Corona Graphs

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Received 9 January 2014; Accepted 12 March 2014; Published 3 April 2014

Academic Editor: Hui-Shen Shen

A graph G with q edges is said to be harmonious, if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. In this paper, we study the existence of harmonious labeling for the corona graphs of a cycle and a graph G and for the corona graph of K_2 and a tree.

1. Introduction

Harmonious graphs naturally arose in the study of modular version of error-correcting codes and channel assignment problems. Graham and Sloane [1] defined a (p, q)-graph G of order p and size q to be *harmonious*, if there is an injective function $f : V(G) \to \mathbb{Z}_q$, where $\mathbb{Z}_q$ is the group of integers modulo q, such that the induced function $f^* : E(G) \to \mathbb{Z}_q$, defined by $f^*(xy) = f(x) + f(y)$ for each edge $xy \in E(G)$, is a bijection.

The function f is called *harmonious labeling* and the image of f denoted by $\mathrm{Im}(f)$ is called the corresponding set of vertex labels.

When G is a tree or, in general for a graph G with $p = q+1$, exactly one label may be used on two vertices.

Graham and Sloane [1] proved that if a harmonious graph has an even number of edges q and the degree of every vertex is divisible by 2^k, then q is divisible by 2^{k+1}. This necessary condition is called the harmonious parity condition. They also proved that if f is harmonious labeling of a graph G of size q, then so is $af + b$ labeling, where a is an invertible element of $\mathbb{Z}_q$ and b is any element of $\mathbb{Z}_q$.

Chang et al. [2] define an injective labeling f of a graph G with q edges to be *strongly c-harmonious*, if the vertex labels are from the set $\{0, 1, \ldots, q-1\}$ and the edge labels are from the set $\{f^*(xy) = f(x) + f(y) : xy \in E(G)\} = \{c, c+1, \ldots, c+q-1\}$. Grace [3, 4] called such labeling *sequential*. In the case of a tree, Grace allows the vertex labels to range from 0 up to q. Strongly 1-harmonious graph is called strongly harmonious.

By taking the edge labels of a sequentially labeled graph with q edges modulo q, we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. More than 50 papers have been published on harmonious labeling and comprehensive information can be found in [5]. Similarly, labeling of special types of crown graphs is examined in [6].

In this paper, we study the existence of harmonious labeling for the graphs obtained by corona operation between a cycle and a graph G and also between K_2 and a tree or K_2 and a unicyclic graph.

2. Main Results

In this section, we present the results related to corona graphs. The corona operation between two graphs was introduced by Frucht and Harary [7]. Given two graphs G of order p and H, the *corona* of G with H, denoted by $G \odot H$, is the graph with $V(G \odot H) = V(G) \cup \bigcup_{i=1}^{p} V(H_i)$, and $E(G \odot H) = E(G) \cup \bigcup_{i=1}^{p} (E(H_i) \cup \{(v_i, u) : v_i \in V(G) \text{ and } u \in V(H_i)\})$. In other words, a corona graph is obtained from two graphs, G of order p and H, taking one copy of G and p copies of H and joining by an edge the ith vertex of G to every vertex in the ith copy of H.

Grace [4] showed that $C_{2n+1} \odot K_1$ is harmonious and conjectured that $C_{2n} \odot K_1$ is harmonious. This conjecture has been proved by Liu and Zhang [8] and Liu [9]. Singh in [10, 11] has proved that $C_n \odot K_2$ and $C_n \odot K_3$ are sequential for all odd $n > 1$. Santhosh [12] has shown that $C_n \odot P_4$ is sequential for all odd $n \geq 3$.

The *join* of two graphs G and H, denoted by $G + H$, is the graph where $V(G) \cap V(H) = \emptyset$ and each vertex of G is adjacent to all vertices of H. When $H = K_1$, this is the corona graph $K_1 \odot G$.

Graham and Sloane [1] showed harmonious labeling of the join of the path P_n and K_1, that is, the *fan* $F_n = P_n + K_1$, and harmonious labeling of the *double fan* $P_n + \overline{K_2}$. Later, Chang et al. [2] gave harmonious labeling of the join of the star S_n and K_1.

The next result shows that if join of a graph G and K_1 is strongly harmonious, then the corona of a cycle and the graph G admitted harmonious labeling.

Theorem 1. *Let G be a graph of order p and size q. If $G + K_1$ is strongly harmonious with the 0 label on the vertex of K_1, then $C_n \odot G$ is harmonious for all odd $n \geq 3$.*

Proof. Let G be a (p, q)-graph and $G + K_1$ strongly harmonious with the 0 label on the vertex $x \in K_1$. Then, there exists labeling $f : V(G+K_1) \to \{0, \ldots, p+q-1\}$ such that $f(x) = 0$ and the edge labels are from the set $\{f^*(uv) = f(u) + f(v) : uv \in E(G + K_1)\} = \{1, 2, \ldots, p + q\}$.

Now, for n odd, $n \geq 3$, we consider the corona graph $C_n \odot G$ with $n(p+1)$ vertices and $\Gamma = n(p+q+1)$ edges. Denote the vertices and edges of the cycle C_n such that $V(C_n) = \{x_1, x_2, \ldots, x_n\}$ and $E(C_n) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\}$. By the symbol y^i, we denote a vertex in the ith copy of G, denoted by G_i, corresponding to the vertex y in G; that is, $y \in V(G)$ and $y^i \in V(G_i)$.

We define the vertex labeling $g : V(C_n \odot G) \to \{0, 1, \ldots, \Gamma - 1\}$ in the following:

$$g(x_i) = (p+q+1)(i-1), \quad \text{for } 1 \leq i \leq n,$$
$$g(y^i) = f(y) + (p+q+1)(i-1), \quad \text{for } 1 \leq i \leq n. \tag{1}$$

If we denote the join graph $G + K_1$ as $G + \{x\}$, then the set of all edge labels of the ith copy of $G + \{x\}$ consists of the consecutive integers $g^*(E(G_i + \{x_i\})) = \{2(p+q+1)(i-1) + 1, 2(p+q+1)(i-1)+2, \ldots, 2(p+q+1)(i-1)+p+q\}$, $1 \leq i \leq n$. For edge labels of the cycle C_n, we have $g^*(x_i x_{i+1}) = g(x_i) + g(x_{i+1}) = (p+q+1)(2i-1)$, for $1 \leq i \leq n-1$, and $g^*(x_n x_1) = g(x_n) + g(x_1) = (p+q+1)(n-1)$.

It is not difficult to see that, for $1 \leq i \leq (n-1)/2$, it is true that

(i) $1 + \max\{g^*(E(G_i + \{x_i\}))\} = (p+q+1)(2i-1) = g^*(x_i x_{i+1})$;

(ii) $1 + g^*(x_i x_{i+1}) = \min\{g^*(E(G_{((n+1)/2)+i} + \{x_{((n+1)/2)+i}\}))\} = (p+q+1)(2i-1) + 1 \pmod{\Gamma}$;

(iii) $1 + \max\{g^*(E(G_{((n+1)/2)+i} + \{x_{((n+1)/2)+i}\}))\} = 2(p+q+1)i \pmod{\Gamma}$ and it is equal to $g^*(x_{((n+1)/2)+i} x_{((n+1)/2)+i+1}) \pmod{\Gamma}$;

(iv) $1 + g^*(x_{((n+1)/2)+i} x_{((n+1)/2)+i+1}) \pmod{\Gamma}$ is equal to the $\min\{g^*(E(G_{i+1} + \{x_{i+1}\}))\} = 2(p+q+1)i + 1$.

Moreover, $1 + \max\{g^*(E(G_{(n+1)/2} + \{x_{(n+1)/2}\}))\} = 0 \pmod{\Gamma}$ and it is equal to $g^*(x_{(n+1)/2} x_{(n+3)/2}) = 0 \pmod{\Gamma}$.

Thus, under the induced mapping g^*, all the resulting edge labels are distinct and they get the consecutive integers from 0 up to $n(p+q+1) - 1 \pmod{\Gamma}$. This concludes the proof. ☐

Graham and Sloane [1] have proved that the fans $F_m = P_m + K_1$, $m \leq 7$, and the wheels $W_m = C_m + K_1$, $m \not\equiv 2 \pmod 3$, are strongly harmonious with the 0 label on the vertex of K_1. In light of these results and Theorem 1, we have the following corollaries.

Corollary 2. *Let $C_n \odot P_m$ be the corona graph of a cycle C_n and a path P_m. Then, $C_n \odot P_m$ is harmonious for all odd $n \geq 3$ and $1 \leq m \leq 7$.*

Corollary 3. *Let $C_n \odot C_m$ be the corona graph of two cycles. Then, $C_n \odot C_m$ is harmonious for all odd $n \geq 3$ and $m \not\equiv 2 \pmod 3$.*

Shee [13] has shown that the complete tripartite graph $K_{1,m,k} = K_{m,k} + K_1$, $m, k \geq 1$, is strongly harmonious, while Gnanajothi [14] proved that $K_{1,1,m,k} = K_{1,m,k} + K_1$, $m, k \geq 1$, is also strongly harmonious. In both cases, the vertex of K_1 is labeled by the 0 label. Thus, with respect to Theorem 1, we obtain the following.

Corollary 4. *For $m, k \geq 1$ and odd $n \geq 3$, the corona graph $C_n \odot K_{m,k}$ is harmonious.*

Corollary 5. *For $m, k \geq 1$ and odd $n \geq 3$, the corona graph $C_n \odot K_{1,m,k}$ is harmonious.*

Let one consider the graphs obtained by corona operation between the single edge K_2 and a tree.

Theorem 6. *If T is a strongly c-harmonious tree of odd size q and $c = (q+1)/2$, then the corona graph $K_2 \odot T$ is also strongly c-harmonious.*

Proof. Let T be a tree of size q with strongly c-harmonious labeling $f : V(T) \to \{0, 1, \ldots, q\}$, where the edge labels are from the set of consecutive integers $\{f^*(e) : e \in E(T)\} = \{c, c+1, \ldots, c+q-1\}$.

Consider the corona graph $K_2 \odot T$ with vertices $x_1, x_2 \in V(K_2)$ and vertices $y^i \in V(T_i)$, $i = 1, 2$, corresponding to the vertices $y \in T$, where the vertex x_i is incident to every vertex in T_i for $i = 1, 2$.

Define now new vertex labeling $g : V(K_2 \odot T) \to \{0, 1, \ldots, 4q+2\}$ such that

$$g(x_i) = \begin{cases} c + q, & \text{for } i = 1, \\ q + 1, & \text{for } i = 2, \end{cases}$$
$$g(y^i) = \begin{cases} f(y), & \text{for } i = 1 \text{ and every } y \in T, \\ f(y) + c + q + 1, & \text{for } i = 2 \text{ and every } y \in T. \end{cases} \tag{2}$$

Thus, $\mathrm{Im}(g) = \{0, 1, 2, \ldots, q, q+1\} \cup \{c+q, c+q+1, c+q+2, \ldots, c+2q, c+2q+1\}$ and, for the edge labels, we have

$$\begin{aligned}
&\{g^*(e) : e \in E(T_1)\} = \{c, c+1, c+2, \ldots, c+q-1\},\\
&\{g^*(x_1y^1) = g(x_1) + g(y^1) : y^1 \in V(T_1)\}\\
&\quad = \{c+q, c+q+1, \ldots, c+2q\},\\
&g^*(x_1x_2) = g(x_1) + g(x_2) = c+2q+1,\\
&\{g^*(x_2y^2) = g(x_2) + g(y^2) : y^2 \in V(T_2)\}\\
&\quad = \{c+2q+2, c+2q+3, \ldots, c+3q+2\},\\
&\{g^*(e) : e \in E(T_2)\}\\
&\quad = \{3c+2q+2, 3c+2q+3, \ldots, 3c+3q+1\}.
\end{aligned} \tag{3}$$

We can see that edge labels form the set of consecutive integers from c up to $3c+3q+1$ if and only if $\max\{g^*(x_2y^2) = g(x_2) + g(y^2) : y^2 \in V(T_2)\} + 1 = \min\{g^*(e) : e \in E(T_2)\}$; that is, $c = (q+1)/2$. □

We know that every caterpillar Cat_p admits strongly c-harmonious labeling. As an illustration, Figure 1 provides an example of the strongly 5-harmonious labeling of Cat_{10}.

As an immediate consequence of Theorem 6, we can state the following corollary.

Corollary 7. *Let Cat_{q+1} be a caterpillar of odd size q. If Cat_{q+1} admits strongly $(q+1)/2$-harmonious labeling, then the corona graph $K_2 \odot \mathrm{Cat}_{q+1}$ also admits strongly $(q+1)/2$-harmonious labeling.*

Theorem 8. *Let G be a unicyclic graph of odd size q. If G is a strongly c-harmonious and $c = (q-1)/2$, then the corona graph $K_2 \odot G$ is also strongly c-harmonious.*

Proof. Let G be a connected (p, q)-graph containing exactly one cycle. Clearly, $p = q$. Let $f : V(G) \to \{0, 1, \ldots, q-1\}$ be strongly c-harmonious labeling with the edge labels from the set of consecutive integers $\{f^*(e) : e \in E(G)\} = \{c, c+1, \ldots, c+q-1\}$.

If x_1 and x_2 are the vertices of K_2 and if by the symbol y^i we mean a vertex in the ith copy of G corresponding to the vertex $y \in V(G)$, then sets of vertices and edges of the corona graph $K_2 \odot G$ are as follows: $V(K_2 \odot G) = V(K_2) \cup V(G_1) \cup V(G_2)$, $E(K_2 \odot G) = \{x_1x_2\} \cup E(G_1) \cup \{x_1y^1 : y^1 \in V(G_1)\} \cup E(G_2) \cup \{x_2y^2 : y^2 \in V(G_2)\}$.

Define new vertex labeling $g : V(K_2 \odot G) \to \{0, 1, \ldots, 4q\}$ in the following:

$$\begin{aligned}
g(x_i) &= \begin{cases} c+q, & \text{for } i = 1,\\ q, & \text{for } i = 2,\end{cases}\\
g(y^i) &= \begin{cases} f(y), & \text{for } i = 1 \text{ and every } y \in G,\\ f(y) + c + q + 1, & \text{for } i = 2 \text{ and every } y \in G.\end{cases}
\end{aligned} \tag{4}$$

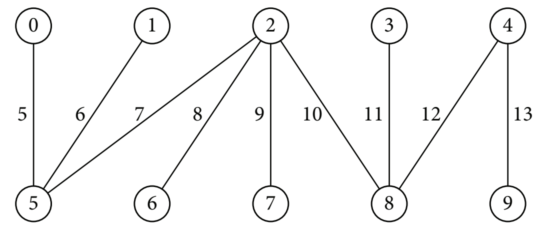

Figure 1: Strongly 5-harmonious labeling of the caterpillar Cat_{10}.

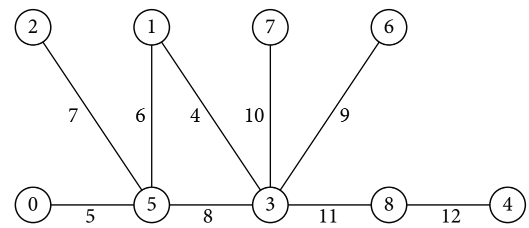

Figure 2: Strongly 4-harmonious labeling of a unicyclic graph.

The image of the vertex labeling g is a union of two sets of consecutive integers $\mathrm{Im}(g) = \{0, 1, 2, \ldots, q\} \cup \{c+q, c+q+1, c+q+2, \ldots, c+2q\}$. Observe that the edge labels are

$$\begin{aligned}
&\{g^*(e) : e \in E(G_1)\} = \{c, c+1, c+2, \ldots, c+q-1\},\\
&\{g^*(x_1y^1) = g(x_1) + g(y^1) : y^1 \in V(G_1)\}\\
&\quad = \{c+q, c+q+1, \ldots, c+2q-1\},\\
&g^*(x_1x_2) = g(x_1) + g(x_2) = c+2q,\\
&\{g^*(x_2y^2) = g(x_2) + g(y^2) : y^2 \in V(T_2)\}\\
&\quad = \{c+2q+1, c+2q+2, \ldots, c+3q\},\\
&\{g^*(e) : e \in E(T_2)\}\\
&\quad = \{3c+2q+2, 3c+2q+3, \ldots, 3c+3q+1\}.
\end{aligned} \tag{5}$$

The edge labels form the set of consecutive integers from c up to $3c+3q+1$ if and only if $c+3q+1 = 3c+2q+2$. It is true if $c = (q-1)/2$. Thus, the labeling g is strongly $(q-1)/2$-harmonious labeling of the corona graph $K_2 \odot G$. □

An example of the strongly 4-harmonious unicyclic graph is presented in Figure 2.

We know that every odd cycle C_{2n+1} admits strongly n-harmonious labeling. As consequence of Theorem 8, we have the following.

Corollary 9. *The corona graph $K_2 \odot C_{2n+1}$, $n \geq 1$, is strongly n-harmonious.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions leading to improvement of this paper. The research for this paper was supported by Slovak VEGA Grant 1/0130/12.

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