Research Article

Remarks on Remnants by Fermions' Tunnelling from Black Strings

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Hawking's calculation is unable to predict the final stage of the black hole evaporation. When effects of quantum gravity are taken into account, there is a minimal observable length. In this paper, we investigate fermions' tunnelling from the charged and rotating black strings. With the influence of the generalized uncertainty principle, the Hawking temperatures are not only determined by the rings, but also affected by the quantum numbers of the emitted fermions. Quantum gravity corrections slow down the increases of the temperatures, which naturally leads to remnants left in the evaporation.

1. Introduction

Hawking radiation is a quantum tunnelling phenomenon of particles across black holes' horizons. To describe this phenomenon, the semiclassical tunnelling method, which relies on calculating the imaginary part of emission particle's action, was put forward [1]. Adopting the WKB approximation, one can get the relationship between the tunnelling rate and the action of the classically forbidden trajectory of the particle. Here we adopt the canonically invariant expression [2–4]

$$\Gamma \propto \exp\left[-Im \oint p\, dr\right]. \tag{1}$$

This canonically invariant relation was first derived in [5–7].

The null geodesic method and the Hamilton-Jacobi method are usual methods employed to derive the imaginary part [8–12]. In the null geodesic method [8], we should first perform the Painleve coordinate transformation on a metric and then use canonical momenta and Hamilton canonical equations to get the imaginary part. When the variable background spacetime is taken into account, the corrected Hawking temperature is higher than the standard one. Therefore, the variable background spacetime implies the accelerated evaporation. The equation of motion of a massive particle is different from that of the massless one. The former obeys de Broglie wave function relation. Therefore, the phase velocity of the particle was adopted to research the tunnelling radiation of massive particles in the subsequent investigations [13, 14]. The embryonic form of the Hamilton-Jacobi method [11] was first found in [9, 10]. In this method, the action satisfies the Hamilton-Jacobi equation. Taking into account the property of the spacetime, one carries out separation of variables on the action. Then, inserting the separated variables into the Hamilton-Jacobi equation and solving it, one gets the imaginary part. Extending this work to the tunnelling radiation of fermions, the standard Hawking temperatures of the spherically symmetric and charged black holes were recovered [15]. Other works about fermions' tunnelling radiation are referred to in [16–24]. The standard temperatures were also recovered by anomaly cancellations [25–27].

Various theories of quantum gravity predict the existence of a minimal observable length [28–32]. This length can be implemented in the model of the generalized uncertainty principle (GUP)

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1 + \beta \Delta p^2\right], \tag{2}$$

where $\beta = \beta_0 (l_p^2/\hbar^2)$, β_0 is a dimensionless parameter, and l_p is the Planck length. The derivation of the GUP relies on the modified fundamental commutation relations. Kempf et al. first modified commutation relations [33] and got $[x_i, p_j] = i\hbar\delta_{ij}[1 + \beta p^2]$, where x_i and p_i are operators of position and momentum defined by

$$x_i = x_{0i}, \qquad p_i = p_{0i}\left(1 + \beta p^2\right), \qquad (3)$$

and x_{0i} and p_{0i} satisfy the canonical commutation relations $[x_{0i}, p_{0j}] = i\hbar\delta_{ij}$.

This modification plays an important role in quantum gravity. With considerations of modifications, the cosmological constant problem was discussed and the finiteness of the constant was derived in [34]. Using a new form of GUP, the Unruh effect has been analyzed in [35]. The quantum dynamics of the Friedmann-Robertson-Walker universe was gotten in [36]. The related predictions on postinflation preheating in the cosmology were derived in [37]. Using the modifications, the thermodynamics of the black holes were researched in [38–40] and the tunnelling radiation of scalar particles was investigated in [41]. In recent work [42], taking into account effects of quantum gravity, the authors modified the Dirac equation in curved spacetime and investigated fermions' tunnelling from the Schwarzschild black hole. They derived that the quantum correction slows down the increase of the Hawking temperature, which leads to the remnant.

In this paper, we extend this work to anti-de Sitter spacetime and investigate the tunnelling radiation of fermions from black strings, where effects of quantum gravity are taken into account. Black strings are cylindrically symmetric solutions of the Einstein-Maxwell equations with a negative cosmological constant. The solutions are asymptotically anti-de Sitter in transverse direction and along the axis. There are three Killing vectors, ∂_t, ∂_θ, and ∂_z, as the minimal symmetry. The AdS/CFT correspondence is an important topic in modern physics. Researches of anti-de Sitter spacetime are helpful to understand this correspondence. To incorporate effects of quantum gravity, we first modify the Dirac equation in curved spacetime by operators of position and momentum defined in [33] and then adopt the Hamilton-Jacobi method to get the imaginary parts of the action. The corrected Hawking temperatures are not only determined by the mass, charge, and angular momentum of the strings, but also affected by the quantum numbers (charge, angular momentum, mass, and energy) of the emitted fermions. Quantum gravity corrections slow down the increases of the Hawking temperatures. It is natural to lead to the remnants left in the evaporation.

The rest is organized as follows. In the next section, using operators of position and momentum defined in [33], we modify the Dirac equation in curved spacetime. In Section 3, we investigate the tunnelling radiation of charged fermions from the charged black string. The remnant is observed in the evaporation. In Section 4, the radiation of uncharged fermions in the rotating black string is discussed. Section 5 is devoted to our conclusion.

2. Generalized Dirac Equation

Here we adopt the modified fundamental commutation relation put forward in [33] to modify the Dirac equation in curved spacetime. Using (3), the square of momentum operators is gotten as

$$
\begin{aligned}
p^2 = p_i p^i &= -\hbar^2 \left[1 - \beta\hbar^2 \left(\partial_j \partial^j\right)\right] \partial_i \cdot \left[1 - \beta\hbar^2 \left(\partial^j \partial_j\right)\right] \partial^i \\
&\simeq -\hbar^2 \left[\partial_i \partial^i - 2\beta\hbar^2 \left(\partial^j \partial_j\right) \left(\partial^i \partial_i\right)\right].
\end{aligned}
\tag{4}
$$

The higher order terms of β are neglected in the last step. In the theory of quantum gravity, the generalized frequency is found as [43]

$$\tilde{\omega} = E\left(1 - \beta E^2\right), \qquad (5)$$

where E is the energy operator and is defined as $E = i\hbar\partial_t$. From the energy mass shell condition $p^2 + m^2 = E^2$, the generalized expression of the energy was derived [41, 43–45]. It is

$$\tilde{E} = E\left[1 - \beta\left(p^2 + m^2\right)\right]. \qquad (6)$$

The generalized Dirac equation without considerations of electromagnetic effects in the flat spacetime has been derived in [44] by the consequence of the GUP. In curved spacetime, the Dirac equation with an electromagnetic field takes on the form

$$i\gamma^\mu \left(\partial_\mu + \Omega_\mu + \frac{i}{\hbar}eA_\mu\right)\Psi + \frac{m}{\hbar}\Psi = 0, \qquad (7)$$

where $\Omega_\mu \equiv (i/2)\omega_\mu{}^{ab}\Sigma_{ab}$, $\omega_\mu{}^{ab}$ is the spin connection defined by the tetrad $e^\lambda{}_b$ and ordinary connection

$$\omega_\mu{}^a{}_b = e_\nu{}^a e^\lambda{}_b \Gamma^\nu_{\mu\lambda} - e^\lambda{}_b \partial_\mu e_\lambda{}^a. \qquad (8)$$

The Latin indices live in the flat metric η_{ab} while Greek indices are raised and lowered by the curved metric $g_{\mu\nu}$. The tetrad can be constructed from

$$
\begin{aligned}
g_{\mu\nu} &= e_\mu{}^a e_\nu{}^b \eta_{ab}, \quad \eta_{ab} = g_{\mu\nu} e^\mu{}_a e^\nu{}_b, \\
e^\mu{}_a e_\nu{}^a &= \delta^\mu_\nu, \quad e^\mu{}_a e_\mu{}^b = \delta^b_a.
\end{aligned}
\tag{9}
$$

In (7), Σ_{ab} is the Lorentz spinor generators defined by

$$\Sigma_{ab} = \frac{i}{4}\left[\gamma^a, \gamma^b\right], \quad \left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab}. \qquad (10)$$

Then one can construct γ^μ's in the curved spacetime as

$$\gamma^\mu = e^\mu{}_a \gamma^a, \quad \left\{\gamma^\mu, \gamma^\nu\right\} = 2g^{\mu\nu}. \qquad (11)$$

To get the generalized Dirac equation in the curved spacetime, we rewrite (7) as

$$-i\gamma^0 \partial_0 \Psi = \left(i\gamma^i \partial_i + i\gamma^\mu \Omega_\mu + i\gamma^\mu \frac{i}{\hbar}eA_\mu + \frac{m}{\hbar}\right)\Psi. \qquad (12)$$

Using (4), (6), and (12) and neglecting the higher order terms of β, we get [41, 43–45]

$$-i\gamma^0 \partial_0 \Psi = \left(i\gamma^i \partial_i + i\gamma^\mu \Omega_\mu + i\gamma^\mu \frac{i}{\hbar} eA_\mu + \frac{m}{\hbar} \right)$$
$$\times \left(1 + \beta\hbar^2 \partial_j \partial^j - \beta m^2 \right) \Psi, \tag{13}$$

which is rewritten as

$$\left[i\gamma^0 \partial_0 + i\gamma^i \partial_i \left(1 - \beta m^2 \right) + i\gamma^i \beta\hbar^2 \left(\partial_j \partial^j \right) \partial_i \right.$$
$$+ \frac{m}{\hbar} \left(1 + \beta\hbar^2 \partial_j \partial^j - \beta m^2 \right)$$
$$+ i\gamma^\mu \frac{i}{\hbar} eA_\mu \left(1 + \beta\hbar^2 \partial_j \partial^j - \beta m^2 \right)$$
$$\left. + i\gamma^\mu \Omega_\mu \left(1 + \beta\hbar^2 \partial_j \partial^j - \beta m^2 \right) \right] \Psi = 0. \tag{14}$$

Thus the generalized Dirac equation is derived. When $A_\mu = 0$, it describes an equation without electromagnetic fields. In the following sections, we adopt (14) to describe fermions tunnelling from the charged and rotating black strings.

3. Fermions' Tunnelling from a Charged Black String

The 4-dimensional neutral black string solutions to Einstein-Maxwell equations with a negative cosmological constant were derived in [46]. Subsequently, the general solutions with electric charges were gotten [47]. In this section, we investigate charged fermions' tunnelling from a cylindrically symmetric black string. The black string solution is given by [47]

$$ds^2 = -f(r) dt^2 + \frac{1}{g(r)} dr^2 + r^2 d\theta^2 + \alpha^2 r^2 dz^2, \tag{15}$$

with the electromagnetic potential

$$A_\mu = (A_t, 0, 0, 0) = \left(\frac{2Q}{\alpha r}, 0, 0, 0 \right), \tag{16}$$

where $f(r) = g(r) = \alpha^2 r^2 - (4M/\alpha r) + (4Q^2/\alpha^2 r^2)$, $0 \le \theta \le 2\pi$, $\alpha^2 = -\Lambda/3$, and Λ is the negative cosmology constant. M and Q are the ADM mass and charge per unit length in the z direction, respectively. The above spacetime is asymptotically anti-de Sitter in the transverse directions and string directions. The singularity at $r = 0$ is enclosed by the horizon r_+ if the condition $Q^2 \le (3/4)M^{4/3}$ holds. The event horizon r_+ is located at

$$r_+ = \frac{1}{2} \left[\sqrt{2R} + \left(-2R + \frac{8M}{\alpha^3 \sqrt{2R}} \right) \right], \tag{17}$$

where

$$R = \left[\frac{M^2}{\alpha^6} + \left(\left(\frac{M^2}{\alpha^6} \right)^2 - \left(\frac{4Q^2}{3\alpha^4} \right)^3 \right)^{1/2} \right]^{1/3}$$
$$+ \left[\frac{M^2}{\alpha^6} - \left(\left(\frac{M^2}{\alpha^6} \right)^2 - \left(\frac{4Q^2}{3\alpha^4} \right)^3 \right)^{1/2} \right]^{1/3}. \tag{18}$$

The metric (15) describes a neutral black string solution when $Q = 0$.

For a spin-1/2 fermion, there are two states corresponding to spin-up and spin-down. Here we only investigate the state with spin-up. The investigation of the state with spin-down is parallel and the same result can be obtained. To describe the motion of a charge fermion, we suppose that the wave function takes on the form

$$\Psi = \begin{pmatrix} A \\ 0 \\ B \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar} I(t, r, \theta, z) \right), \tag{19}$$

where A and B are functions of t, r, θ, and z and I is the action of the fermion with spin-up state. To find gamma matrices, we should first construct a tetrad. It is straightforward to construct a tetrad from the metric (15). The tetrad is

$$e_\mu{}^a = \text{diag}\left(\sqrt{f}, 1/\sqrt{g}, r, \alpha r \right). \tag{20}$$

Then gamma matrices are gotten as

$$\gamma^t = \frac{1}{\sqrt{f(r)}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \gamma^\theta = \sqrt{g^{\theta\theta}} \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix},$$
$$\gamma^r = \sqrt{g(r)} \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \qquad \gamma^z = \sqrt{g^{zz}} \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}. \tag{21}$$

In the above equations, $\sqrt{g^{\theta\theta}} = 1/r$ and $\sqrt{g^{zz}} = 1/\alpha r$. To apply the WKB approximation, we insert the wave function and the gamma matrices into the generalized Dirac equation and then divide by the exponential term and multiply by $\hbar$. The resulting equation of leading order in $\hbar$ is derived and decoupled into four equations

$$-iA\frac{1}{\sqrt{f}}\partial_t I - B\left(1 - \beta m^2 \right) \sqrt{g}\partial_r I$$
$$- Am\beta \left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{zz}(\partial_z I)^2 \right]$$
$$+ B\beta\sqrt{g}\partial_r I \left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{zz}(\partial_z I)^2 \right]$$
$$+ Am\left(1 - \beta m^2 \right) - iA\frac{eA_t}{\sqrt{f}}$$
$$\times \left[1 - \beta m^2 - \left(g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{zz}(\partial_z I)^2 \right) \right]$$
$$= 0,$$

$$iB\frac{1}{\sqrt{f}}\partial_t I - A\left(1 - \beta m^2\right)\sqrt{g}\partial_r I$$

$$- Bm\beta\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{zz}(\partial_z I)^2\right]$$

$$+ A\beta\sqrt{g}\partial_r I\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{zz}(\partial_z I)^2\right]$$

$$+ Bm\left(1 - \beta m^2\right) + iB\frac{eA_t}{\sqrt{f}}$$

$$\times\left[1 - \beta m^2 - \left(g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{zz}(\partial_z I)^2\right)\right]$$

$$= 0, \tag{22}$$

$$A\left\{-\left(1 - \beta m^2\right)\sqrt{g^{\theta\theta}}\partial_\theta I + \beta\sqrt{g^{\theta\theta}}\partial_\theta I\right.$$

$$\times\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{zz}(\partial_z I)^2\right]$$

$$- i\left(1 - \beta m^2\right)\sqrt{g^{zz}}\partial_z I + i\beta\sqrt{g^{zz}}\partial_z I$$

$$\left.\times\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{zz}(\partial_z I)^2\right]\right\} = 0, \tag{23}$$

$$B\left\{-\left(1 - \beta m^2\right)\sqrt{g^{\theta\theta}}\partial_\theta I + \beta\sqrt{g^{\theta\theta}}\partial_\theta I\right.$$

$$\times\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{zz}(\partial_z I)^2\right]$$

$$- i\left(1 - \beta m^2\right)\sqrt{g^{zz}}\partial_z I + i\beta\sqrt{g^{zz}}\partial_z I$$

$$\left.\times\left[g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{zz}(\partial_z I)^2\right]\right\} = 0.$$

Obviously, it is difficult to get the solution of the action I from the above equations. However, the action can be separated by the property of the black string. Considering the Killing vectors of the spacetime, the author separated the action as $I = -\omega t + W(r) + l\theta + Jz$ [17], where ω is the energy of the emitted fermion. From the above four equations, we carry out separation of variables as

$$I = -\omega t + W(r) + \Theta(\theta, z). \tag{24}$$

We first observe (23) and find that they are irrelevant to A and B and can be reduced to the same equation. Inserting (24) into (23) yields

$$\left(\sqrt{g^{\theta\theta}}\partial_\theta\Theta + i\sqrt{g^{zz}}\partial_z\Theta\right)$$

$$\times\left[1 - \beta m^2 - \beta g^{rr}(\partial_r W)^2 - \beta g^{\theta\theta}(\partial_\theta\Theta)^2 - \beta g^{zz}(\partial_z\Theta)^2\right]$$

$$= 0. \tag{25}$$

In the above equation, the summation of factors in the square brackets cannot be zero. Therefore, it should be

$$\sqrt{g^{\theta\theta}}\partial_\theta\Theta + i\sqrt{g^{zz}}\partial_z\Theta = 0, \tag{26}$$

which yields a complex function solution (other than the trivial constant solution) of Θ. However, this solution has no contribution to the tunnelling rate. Therefore, we will not consider its contribution in the calculation. Another important relation predicted by (26) is $g^{\theta\theta}(\partial_\theta\Theta)^2 + g^{zz}(\partial_z\Theta)^2 = 0$. Now we focus our attention on the first two equations. Inserting (24) into (22) and canceling A and B yield

$$A_6(\partial_r W)^6 + A_4(\partial_r W)^4 + A_2(\partial_r W)^2 + A_0 = 0, \tag{27}$$

where

$$A_6 = \beta^2 g^3 f,$$

$$A_4 = \beta g^2 f\left(m^2\beta - 2\right) - \beta^2 g^2 e^2 A_t^2,$$

$$A_2 = gf\left(1 - \beta m^2\right)\left(1 + \beta m^2\right)$$

$$\quad - 2\beta geA_t\left[\omega - eA_t\left(1 - \beta m^2\right)\right], \tag{28}$$

$$A_0 = -m^2 f\left(1 - \beta m^2\right)^2 - \left[\omega - eA_t\left(1 - \beta m^2\right)\right]^2.$$

Neglect the higher order terms of β and solve (27) at the event horizon. Thus the imaginary part of the radial action is

$$ImW_\pm = \pm\int\frac{dr}{\sqrt{gf}}\sqrt{\left[\omega - eA_t\left(1 - \beta m^2\right)\right]^2 + m^2 f}$$

$$\times\left(1 + \beta m^2 + \beta\frac{\widetilde{\omega}_0^2 - eA_t\widetilde{\omega}_0}{f}\right) \tag{29}$$

$$= \pm\pi\frac{\omega - eA_{t+}}{f'}\left(1 + \beta\xi\right),$$

where $+(-)$ denote the outgoing (ingoing) solutions, $f' = 2\alpha^2 r_+ + (4M/\alpha r_+^2) - (8Q^2/\alpha^2 r_+^3)$, $\xi = (3/2)m^2 + ((2m^2 + 1)/2\omega_0) + (2eA_{t+}/f' r_+) - (2/3)(\omega_0/f' r_+)$, $\widetilde{\omega}_0 = \omega - eA_t$, $\omega_0 = \omega - eA_{t+}$, and $A_{t+} = 2Q/\alpha r_+$ is the electromagnetic potential at the event horizon. Using the relation of the roots of $f = 0$, it is easily proved that $\xi > 0$. The tunnelling rate of the charged fermion at the event horizon is

$$\Gamma \propto \exp\left[-Im\oint p_r dr\right]$$

$$= \exp\left[-Im\left(\int p_r^{out}dr - \int p_r^{in}dr\right)\right] \tag{30}$$

$$= \exp\left[\mp 2Im\int p_r^{out,in}dr\right].$$

Here, $p_r = \partial_r W$, and out(in) correspond to $+(-)$. Thus the tunnelling rate is gotten as

$$\Gamma \propto \exp\left[-2\pi\frac{\omega - eA_{t+}}{f'}\left(1 + \beta\xi\right)\right]. \tag{31}$$

However, the temporal contribution to the tunneling amplitude was missed in the above calculation [2, 5–7]. We use Kruskal coordinates (T, R) to find this temporal contribution. The region exterior is described by

$$T = e^{\kappa r_*}\sinh(\kappa t), \qquad R = e^{\kappa r_*}\cosh(\kappa t), \tag{32}$$

where $r_* = r + (1/2\kappa)\ln((r - r_+)/r_+)$ is the tortoise coordinate and κ is the surface gravity. The interior region is given by

$$T = e^{\kappa r_*}\cosh(\kappa t), \qquad R = e^{\kappa r_*}\sinh(\kappa t). \qquad (33)$$

To find the temporal contribution, we connect these two patches across the horizon. Rotate the time t as $t \rightarrow t - (\pi/2)i\kappa$. As pointed out in [2–4], this rotation would lead to an additional imaginary contribution coming from the temporal part, namely, $Im[(\omega - eA_{t_+})\Delta t^{\text{out,in}}] = (1/2)\pi(\omega - eA_{t_+})\kappa$. So the total temporal contribution is $Im[(\omega - eA_{t_+})\Delta t] = \pi(\omega - eA_{t_+})\kappa$. Therefore, the tunnelling rate with the consideration of the temporal contribution is

$$\Gamma \propto \exp\left[-\frac{1}{\hbar}\left(Im\left((\omega - eA_{t_+})\Delta t\right) + Im\oint p_r dr\right)\right]$$
$$= \exp\left[-4\pi\frac{\omega - eA_{t_+}}{f'}\left(1 + \frac{1}{2}\beta\xi\right)\right]. \qquad (34)$$

This is the Boltzmann factor with the Hawking temperature at the event horizon taking

$$T = \frac{f'}{4\pi(1 + \beta\xi)} = T_0\left(1 - \frac{1}{2}\beta\xi\right), \qquad (35)$$

where $T_0 = (1/2\pi)(\alpha^2 r_+ + (2M/\alpha r_+^2) - (4Q^2/\alpha^2 r_+^3))$ is the standard Hawking temperature of the black string.

It is shown that the corrected temperature appears and is lower than the standard one. The correction is not only determined by the mass and charge of the black string, but also affected by the quantum number (mass, charge, and energy) of the emitted fermion. Quantum gravity correction slows down the increase of the Hawking temperature caused by the evaporation. Finally, the black string is in a balance state. At this state, the evaporation stops and the remnant is produced.

It is of interest to discuss the corrected area entropy. The entropy can be derived by the first law of thermodynamics with the corrected temperature (35). However, the expression is complicated, so we do not write it here. The corrected temperatures were also gotten in [48–53]. When $\beta = 0$, the standard Hawking temperature is recovered [17, 47].

4. Fermions' Tunnelling from a Rotating Black String

In this section, we investigate uncharged fermions' tunnelling from the event horizon of a rotating black string. Therefore, effects of the electromagnetic field in the generalized Dirac equation are not taken into account here. The rotating black string solution in a spacetime asymptotically anti-de Sitter in the radial direction was derived by Lemos and Zanchin [54]. The solution is

$$ds^2 = -\left(\alpha^2 r^2 - \frac{4M\left(1 - a^2\alpha^2/2\right)}{\alpha r}\right)dt^2$$
$$+ \left(\alpha^2 r^2 - \frac{4M\left(1 - (3/2)a^2\alpha^2\right)}{\alpha r}\right)^{-1}dr^2$$
$$- \frac{8Ma\sqrt{1 - a^2\alpha^2/2}}{\alpha r}dt\,d\varphi$$
$$+ \left(r^2 - \frac{4Ma^2}{\alpha r}\right)d\varphi^2 + \alpha^2 r^2 dz^2, \qquad (36)$$

where $\alpha^2 = -\Lambda/3$, Λ is the negative cosmological constant, and a is the angular momentum per unit mass. It is defined that $a^2\alpha^2 = 1 - \epsilon/M$ and $\epsilon = \sqrt{M^2 - 8J^2\alpha^2/9}$. M and J are the mass and angular momentum line densities of the spacetime, respectively. The relation between J and a is given by $J = (3/2)Ma\sqrt{1 - a^2\alpha^2/2}$. For convenience of the investigation, the metric (36) is rewritten as

$$ds^2 = -\Delta\left(\gamma dt - \frac{\delta}{\alpha^2}d\varphi\right)^2 + r^2\left(\gamma d\varphi - \delta dt\right)^2$$
$$+ \frac{dr^2}{\Delta} + \alpha^2 r^2 dz^2, \qquad (37)$$

where

$$\Delta = \alpha^2 r^2 - \frac{b}{\alpha r}, \quad b = 4M\left(1 - \frac{3a^2\alpha^2}{2}\right),$$
$$\gamma = \sqrt{\frac{2 - a^2\alpha^2}{2 - 3a^2\alpha^2}}, \quad \delta = \frac{a\alpha^2}{\sqrt{1 - (3/2)a^2\alpha^2}}. \qquad (38)$$

The event horizon is located at $r_+ = \alpha^{-1}b^{1/3}$ which is given for $\Delta = 0$. To describe the fermion's tunnelling from the event horizon, one can directly construct a tetrad and gamma matrices from the metric (37). For simplicity, to construct the tetrad and gamma matrices, we perform the dragging coordinate transformation

$$\varphi = \phi + \Omega t, \quad \Omega = \frac{-\Delta\gamma\delta\alpha^2 + r^2\gamma\delta\alpha^4}{-\Delta\delta^2 + r^2\gamma^2\alpha^4}, \qquad (39)$$

on the metric (37) and get

$$ds^2 = -F(r)dt^2 + \frac{1}{G(r)}dr^2 + g_{\phi\phi}d\phi^2 + g_{zz}dz^2$$
$$= -\frac{\Delta r^2\left(\alpha^2\gamma^2 - \delta^2\right)^2}{-\Delta\delta^2 + \alpha^4 r^2\gamma^2}dt^2 + \frac{1}{\Delta}dr^2$$
$$+ \left(-\frac{\Delta\delta^4}{\alpha^4} + r^2\gamma^2\right)d\phi^2 + \alpha^2 r^2 dz^2. \qquad (40)$$

Here we still only investigate the state with spin-up. Assume that the wave function of the fermion with spin-up state shares the same expression as (19), namely,

$$\Psi = \begin{pmatrix} A \\ 0 \\ B \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar} I\left(t, r, \phi, z\right)\right). \tag{41}$$

The tetrad is easily constructed as

$$e_\mu{}^a = \mathrm{diag}\left(\sqrt{F}, 1/\sqrt{G}, \sqrt{g_{\phi\phi}}, \sqrt{g_{zz}}\right). \tag{42}$$

Now gamma matrices take on the form

$$\gamma^t = \frac{1}{\sqrt{F(r)}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \gamma^\phi = \sqrt{g^{\phi\phi}}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix},$$

$$\gamma^r = \sqrt{G(r)}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \qquad \gamma^z = \sqrt{g^{zz}}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}. \tag{43}$$

In the above equations, $g^{\phi\phi} = \alpha^4/(-\Delta\delta^4 + \alpha^4 r^2\gamma^2)$, $g^{zz} = 1/\alpha^2 r^2$. Inserting the wave function and the gamma matrices into the generalized Dirac equation and adopting the same process as the above section, we get

$$-iA\frac{1}{\sqrt{F}}\partial_t I - B\left(1 - \beta m^2\right)\sqrt{G}\partial_r I$$

$$- Am\beta\left[g^{rr}(\partial_r I)^2 + g^{\phi\phi}(\partial_\phi I)^2 + g^{zz}(\partial_z I)^2\right]$$

$$+ B\beta\sqrt{G}\partial_r I\left[g^{rr}(\partial_r I)^2 + g^{\phi\phi}(\partial_\phi I)^2 + g^{zz}(\partial_z I)^2\right]$$

$$+ Am\left(1 - \beta m^2\right) = 0,$$

$$iB\frac{1}{\sqrt{F}}\partial_t I - A\left(1 - \beta m^2\right)\sqrt{G}\partial_r I$$

$$- Bm\beta\left[g^{rr}(\partial_r I)^2 + g^{\phi\phi}(\partial_\phi I)^2 + g^{zz}(\partial_z I)^2\right]$$

$$+ A\beta\sqrt{G}\partial_r I\left[g^{rr}(\partial_r I)^2 + g^{\phi\phi}(\partial_\phi I)^2 + g^{zz}(\partial_z I)^2\right]$$

$$+ Bm\left(1 - \beta m^2\right) = 0,$$

$$A\left\{-\left(1 - \beta m^2\right)\sqrt{g^{\phi\phi}}\partial_\phi I + \beta\sqrt{g^{\phi\phi}}\partial_\phi I\right.$$

$$\times\left[g^{rr}(\partial_r I)^2 + g^{\phi\phi}(\partial_\phi I)^2 + g^{zz}(\partial_z I)^2\right]$$

$$- i\left(1 - \beta m^2\right)\sqrt{g^{zz}}\partial_z I + i\beta\sqrt{g^{zz}}\partial_z I$$

$$\left.\times\left[g^{rr}(\partial_r I)^2 + g^{\phi\phi}(\partial_\phi I)^2 + g^{zz}(\partial_z I)^2\right]\right\} = 0, \tag{44}$$

$$B\left\{-\left(1 - \beta m^2\right)\sqrt{g^{\phi\phi}}\partial_\phi I + \beta\sqrt{g^{\phi\phi}}\partial_\phi I\right.$$

$$\times\left[g^{rr}(\partial_r I)^2 + g^{\phi\phi}(\partial_\phi I)^2 + g^{zz}(\partial_z I)^2\right]$$

$$- i\left(1 - \beta m^2\right)\sqrt{g^{zz}}\partial_z I + i\beta\sqrt{g^{zz}}\partial_z I$$

$$\left.\times\left[g^{rr}(\partial_r I)^2 + g^{\phi\phi}(\partial_\phi I)^2\right] + g^{zz}(\partial_z I)^2\right\} = 0. \tag{45}$$

It is also difficult to solve the action I from the above equations. We first observe the last two equations. They can be reduced into the same equation and yield $\sqrt{g^{\phi\phi}}\partial_\phi I + i\sqrt{g^{zz}}\partial_z I = 0$. This implies

$$g^{\phi\phi}(\partial_\phi I)^2 + g^{zz}(\partial_z I)^2 = 0. \tag{46}$$

Now our interest is the first two equations which determine the Hawking temperature of the black string. Considering the properties of the metrics (36) and (40), we carry out separation of variables as

$$I = -\left(\omega - j\Omega\right)t + W\left(r, z\right) + j\phi, \tag{47}$$

where ω and j are the energy and angular momentum of the emitted fermion, respectively. Inserting (47) into (44) and canceling A and B yield

$$B_6(\partial_r W)^6 + B_4(\partial_r W)^4 + B_2(\partial_r W)^2 + B_0 = 0, \tag{48}$$

where

$$B_6 = \beta^2 G^3 F,$$

$$B_4 = \beta G^2 F\left(m^2\beta - 2\right),$$

$$B_2 = GF\left[\left(1 - \beta m^2\right)^2 + 2\beta m^2\left(1 - m^2\beta\right)\right],$$

$$B_0 = -m^2\left(1 - \beta m^2\right)^2 F - \left(\omega - j\Omega\right)^2. \tag{49}$$

Neglecting the higher order terms of β and solving (48) at the event horizon, we get the solution of W. Thus the imaginary part of W is

$$ImW_\pm = \pm\int dr\sqrt{\frac{m^2 F + \left(\omega - j\Omega\right)^2}{GF}}$$

$$\times\left(1 + \beta m^2 + \beta\frac{\left(\omega - j\Omega\right)^2}{F}\right) \tag{50}$$

$$= \pm\pi\frac{\left(\omega - j\Omega_+\right)\alpha^4 r_+^2\gamma}{\left(2\alpha^4 r_+^3 + b\right)\left(\alpha^2\gamma^2 - \delta^2\right)}\left(1 + \beta\chi\right),$$

where $+(-)$ are the outgoing (ingoing) solutions, $\chi = (3/2)m^2 + 3j(\omega - j\Omega_+)\delta/(\alpha^2\gamma^2 - \delta^2)\gamma r_+^2 - (3/2)((\omega - j\Omega_+)^2\delta^2/(\alpha^2\gamma^2 - \delta^2)r_+^2)$, and $\Omega_+ = \delta/\gamma$ is the angular velocity at the event horizon. It is not difficult to prove that $\chi > 0$.

To find the temporal contribution, we use the Kruskal coordinates (T, R). The region exterior to the string $(r > r_+)$ is described by

$$T = e^{\kappa_+ r_*} \sinh(\kappa_+ t), \qquad R = e^{\kappa_+ r_*} \cosh(\kappa_+ t), \qquad (51)$$

where $r_* = r + (1/2\kappa_+)\ln((r - r_+)/r_+)$ and κ_+ denote the surface gravity. The interior region is

$$T = e^{\kappa_+ r_*} \cosh(\kappa_+ t), \qquad R = e^{\kappa_+ r_*} \sinh(\kappa_+ t). \qquad (52)$$

Adopting the same process as the above section, we get that the total temporal contribution is $Im[(\omega - j\Omega_+)\Delta t] = \pi(\omega - j\Omega_+)\kappa_+$. Therefore, the tunnelling rate is

$$
\begin{aligned}
\Gamma &\propto \exp\left[-\frac{1}{\hbar}\left(Im\left((\omega - j\Omega_+)\Delta t\right) + Im \oint p_r dr\right)\right] \\
&= \exp\left[-\frac{4\pi(\omega - j\Omega_+)\alpha^4 r_+^2 \gamma}{(2\alpha^4 r_+^3 + b)(\alpha^2\gamma^2 - \delta^2)}\left(1 + \frac{1}{2}\beta\chi\right)\right].
\end{aligned}
\qquad (53)
$$

Equation (53) is the Boltzmann factor of the Hawking temperature at the event horizon taking

$$T = \frac{(2\alpha^4 r_+^3 + b)(\alpha^2\gamma^2 - \delta^2)}{4\pi\alpha^4 r_+^2 \gamma (1 + (1/2)\beta\chi)} = T_0\left(1 - \frac{1}{2}\beta\chi\right), \qquad (54)$$

where $T_0 = (2\alpha^4 r_+^3 + b)(\alpha^2\gamma^2 - \delta^2)/4\pi\alpha^4 r_+^2\gamma$ is the standard Hawking temperature. Obviously, the corrected Hawking temperature is lower than the standard one. The correction is related not only to the mass and angular momentum of the black string but also to the quantum number (mass, angular momentum, and energy) of the emitted fermion. Due to $\chi > 0$, there is a balance point. At this point, the evaporation stops and the remnant is left.

5. Conclusion

In this paper, taking into account the influence of quantum gravity, we modified the Dirac equation in curved spacetime by the modified fundamental commutation relations put forward in [33]. Then the tunnelling radiation of fermions from the event horizons of the charged and rotating black strings was investigated. The corrected Hawking temperatures were gotten. In the charged spacetime, the correction is related not only to the mass and charge of the black string but also to the quantum number (mass, charge, and energy) of the emitted fermion. In the rotating spacetime, the quantum number (mass, angular momentum, and energy) of the emitted fermion and the mass and angular momentum of the black string affect the Hawking temperature at the same time. Due to the quantum gravity corrections, the evaporation of the black strings slows down. Finally, the evaporation stops and the remnants are left. The remnants in the final state were also discussed in [55–58]. It is of interest to discuss remnants. A review of this topic can be found in [59].

When $\beta = 0$, the standard Hawking temperatures are recovered.

Conflict of Interests

Acknowledgment

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