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A PAIR OF BIORTHOGONAL POLYNOMIALS FOR THE SZEGÖ-HERMITE WEIGHT FUNCTION

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ABSTRACT. A pair of polynomial sequences $\{S_n^{\mu}(x;k)\}$ and $\{T_m^{\mu}(x;k)\}$ where $S_n^{\mu}(x;k)$ is of degree n in x^k and $T_m^{\mu}(x;k)$ is of degree m in x, is constructed. It is shown that this pair is biorthogonal with respect to the Szegö-Hermite weight function $|x|^{2\mu}\exp(-x^2)$, $(\mu > -1/2)$ over the interval $(-\infty,\infty)$ in the sense that

$$\int_{-\infty}^{\infty} |x|^{2\mu} \exp(-x^2)\, S_n^{\mu}(x;k)\, T_m^{\mu}(x;k)dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$$

where m,n = 0,1,2, ... and k is an odd positive integer.

Generating functions, mixed recurrence relations for both these sets are obtained. For $k = 1$, both the above sets get reduced to the orthogonal polynomials introduced by professor Szegö.

KEYS WORDS AND PHRASES. Szego-Hermite weight function, Biorthogonal pair, Generating functions, Recurrence relations, etc.
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1. INTRODUCTION.

The biorthogonality conditions are useful in the computations involving the penetration of gamma rays through matter as well as in determining the moments of a hypergeometric distribution function. The notion of biorthogonality dates back to Didon [1] and Deruyts [2]. The questions of constructing biorthogonal pairs of polynomials corresponding to the weight functions of classical orthogonal polynomials were taken up by Konhauser [3] for the Laguerre weight function $x^{\alpha}\, e^{-x}$, by Toscano [4], Chai [5], Carlitz [6] and Madhekar and Thakare [7] for the Jacobi weight function $(1-x)^{\alpha}\, (1+x)^{\beta}$ and by Thakare and Madhekar [4] for the Hermite weight function $\exp(-x^2)$. The Szegö-Hermite polynomials $H_n^{\mu}(x)$ are orthogonal w.r.t. the Szegö-Hermite weight function $|x|^{2\mu}\exp(-x^2)$, $(\mu > -1/2)$ over the interval $(-\infty,\infty)$ and these are found useful in connection with Gauss-Jacobi mechanical quadrature, see Szegö [8]. For $\mu = 0$, Szegö-Hermite polynomials are just the classical Hermite polynomials.

2. A BIORTHOGONAL SYSTEM.

We shall construct a pair of biorthogonal polynomials w.r.t. the Szego-Hermite weight function $|x|^{2\mu}\exp(-x^2)$, $\mu > -1/2$. Consider the following pair of polynomial sequences.

$$S_n^{\mu}(x;k) = 2^n\Gamma((kn + k - k\varepsilon)/2 + \mu + \varepsilon)$$
$$\cdot \sum_{j=0}^{[1/2]} (-1)^j \binom{[n/2]}{j} x^{nk-2kj} / \Gamma((kn+1+\varepsilon)/2 - kj + \mu). \tag{2.1}$$

$$T_n^{\mu}(x;k) = (-1)^{[n/2]}2^n \sum_{r=0}^{[n/2]} x^{n-2r}/([n/2]-r)! \sum_{s=0}^{[n/2]-r} (-1)^s \binom{[n/2]-r}{s}$$
$$\cdot ((2s+(k+1)\varepsilon + 2\mu+1)/2k)_{[n/2]}, \tag{2.2}$$

where the value of ε is 0 or 1 according to even or odd nature of n. Throughout this paper ε always has this meaning; and [p] is the greatest integer less than or equal to p.

It is fairly easy to verify after reverting the order of summation for even and odd integers that

$$S_{2n}^{\mu}(x;k) = (-1)^n 2^{2n}\, \Gamma(kn+\mu+k/2) \sum_{j=0}^{n} (-1)^j \binom{n}{j} x^{2kj}/\Gamma(kj+\mu+1/2),$$
$$= (-1)^n 2^{2n}\, n!\, [\Gamma(kn+\mu+k/2)/\Gamma(kn+\mu+1/2)]\, Z_n^{\mu-1/2}(x^2;k); \tag{2.3}$$

$$S_{2n+1}^{\mu}(x;k) = (-1)^n\, 2^{2n+1}\, \Gamma(kn+\mu+1+k/2) \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{2kj+k}}{\Gamma(kj+\mu+1+k/2)}$$
$$= (-1)^n 2^{2n+1} n!\, x^k\, Z_n^{\mu+k/2}(x^2;k); \tag{2.4}$$

$$T_{2n}^{\mu}(x;k) = (-1)^n 2^{2n} \sum_{r=0}^{n} \frac{2^{2r}}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} ((s+\mu+1/2)/k)_n,$$
$$= (-1)^n 2^{2n}\, n!\, Y_n^{\mu-1/2}(x^2;k), \tag{2.5}$$

$$T_{2n+1}^{\mu}(x;k) = (-1)^n 2^{2n+1} \sum_{r=0}^{n} (x^{2r+1}/r!) \sum_{s=0}^{r} (-1)^s \binom{r}{s} ((s+\mu+1+k/2)/k)_n,$$
$$= (-1)^n 2^{2n+1}\, n!\, x\, Y_n^{\mu+k/2}(x^2;k). \tag{2.6}$$

Here $Z_n^{\alpha}(x;k)$ and $Y_n^{\alpha}(x;k)$ is a pair of Konhauser [3] biorthogonal polynomials w.r.t. the Laguerre weight function $x^{\alpha}\exp(-x)$ over $(0,\infty)$ and are given by

$$Z_n^{\alpha}(x;k) = \frac{\Gamma(kn + \alpha + 1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + \alpha + 1)} \tag{2.7}$$

$$Y_n^\alpha(x;k) = \frac{1}{n!} \sum_{r=0}^{n} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} ((s+\alpha+1)/k)_n; \text{ see Carlitz [9]} \tag{2.8}$$

where $\alpha > -1$, and k is a postive integer, and

$$\int_0^\infty x^\alpha e^{-x} Z_n(x;k)\, Y_m^\alpha(x;k)\, dx = \frac{\Gamma(kn+\alpha+1)}{n!} \delta(n,m) \quad \text{with} \quad \delta(n,m) \tag{2.9}$$

the Kronecker's delta. Using [10] one readily obtains the following biorthogonality condition for the sets $\{S_n^\mu(x;k)\}$ and $\{T_m^\mu(x;k)\}$:

$$\int_{-\infty}^{\infty} |x|^{2\mu} \exp(-x^2)\, S_n^\mu(x;k)\, T_m^\mu(x;k)\, dx$$

$$= 2^{2n} [n/2]!\, \Gamma(\mu+\varepsilon+(kn+k-k\varepsilon)/2) \quad \delta(n,m) . \tag{2.10}$$

An independent proof of (2.10) is also possible by using the identity of Carlitz [9, p. 249]:

$$(-j)_m = \sum_{r=0}^{m} \binom{kj+c+m-r}{m-r} \sum_{s=0}^{m-r} (-1)^s \binom{m-r}{s} ((s+c+1)/k)_m .$$

One has to note, however, that k is involved in $S_n^\mu(x;k)$ and $T_m^\mu(x;k)$ must be an odd positive integer in view of the existence theorem for biorthogonality due to Konhauser [10, p.255].

One readily obtains

$$\Gamma(kn+k+\mu+1/2)\, S_{2n+1}^\mu(x;k) = 2x^k\, \Gamma(kn+\mu+1+k/2)\, S_{2n}^{\mu+(k+1)/2}(x;k), \text{ and} \tag{2.11}$$

$$T_{2n+1}^\mu(x;k) = 2x\, T_{2n}^{\mu+(k+1)/2}(x;k), \tag{2.12}$$

$$D\, S_{2n}^\mu(x;k) = 4\, nk\, x^{k-1} \frac{\Gamma(kn+\mu+k/2)}{\Gamma(kn+\mu+1/2)} S_{2n-1}^{\mu+(k-1)/2}(x;k) . \tag{2.13}$$

3. SOME PROPERTIES.

Using the relationship (2.3) to (2.6) it is fairly easy to obtain many results for the Szegö-Hermite biorthogonal pair of polynomials from the known results for the Konhauser biorthogonal sets. The results stated below could also be proved directly. Recall the Calvez and Ge'nin [11] generating function in the form (see also Srivastava [12]):

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^\alpha(x;k)\, t^n = R^{(1+\alpha+mk)} \exp\{x(1-R)\}\, Y_m^\alpha(xR;k), \tag{3.1}$$

where m is any integer ≥ 0 and $R = (1-t)^{-1/k}$. By handling even and odd cases separately, from (2.5) and (2.6) respectively, one obtains

$$\sum_{n=0} T_{2m+n}^\mu (x;k) t^n/[n/2]! \tag{3.2}$$

$$= VU^{(\mu+mk+(1+k)/2)} [U^{-k}\, T_{2m}^\mu(xU;k) + t\, T_{2m+1}^\mu(xU;k)] \text{ where } U=(1+4t^2)^{-1/2k} \text{ and}$$

$V = \exp\{x^2[1-(1+4t^2)^{-1/k}]\}$. The special case with m=0 is worth noting. Using (3.2) for even case and then applying (2.12) one obtains in a combined form the recurrence relation for the second set

$$T_n^{\mu}(x;k) = \sum_{m=0}^{[n/2]} (-1)^m\, 2^{2m} \binom{n/2}{m} \left(\frac{\mu-\lambda}{k}\right)_m T_{n-2m}^{\lambda}(x;k),\ \lambda \neq \mu \text{ and } \lambda,\mu > -1/2. \quad (3.3)$$

Taking $\mu = 0$, and n even in (3.3) and using the biorthogonality condition (2.10) we have the integral

$$\int_{-\infty}^{\infty} |x|^{2\lambda} \exp(-x^2)\, S_{2m}^{\lambda}(x;k)\, T_{2n}(x;k)\, dx \quad (3.4)$$

$= (-1)^n\, 4^{m+n}\, (-n)_m (-\lambda/k)_{n-m} \Gamma(km+\lambda+k/2)$ where with $\mu = 0$, $T_{2n}(x;k)$ is the second biorthogonal set suggested by the Hermite polynomials; see Thakare and Madhekar [4]. The integral (3.4) says that $T_{2n}(x;k)$ are othogonal to $|x|^{2\lambda} S_{2m}^{\lambda}(x;k)$ w.r.t. the weight function $\exp(-x^2)$ when $n > m+\lambda/k$.

Consider the generating function first given by Genin and Calvez [13]; (see also Karande and Thakare [14], Prabhakar [15]):

$$\sum_{n=0}^{\infty} (c)_n Z_n^{\alpha}(x;k)\, t^n/(1+\alpha)_{kn} = (1-t)^{-c}\ {}_1F_k \left[\begin{matrix} c\ ; \\ \Delta(k,1+\alpha); \end{matrix}\ tx^k/(1-t)k^k\right] \quad (3.5)$$

where $|t| < 1$ and $\Delta(m,\delta)$ stands for the sequence of parameters δ/m, $(\delta+1)/m$, ..., $(\delta+m-1)/m$, $(m>1)$. Using (2.3) one obtains from (3.5), an expression involving even $S_{2n}^{\mu}(x;k)$ which after putting to use relation (2.11) gives a corresponding relation for odd $S_{2n+1}^{\mu}(x;k)$. This resulting expression further with the help of the relation

$$\sum_{n=0}^{\infty} (c)_n S_{2n+1}^{\mu}(x;k) t^{2n+1}/n!\, (\mu+k/2)_{nk} \quad (3.6)$$

$$= t(k+2\mu+k\theta)/(k+2\mu) \sum_{n=0}^{\infty} (c)_n S_{2n+1}^{\mu}(x;k)\, t^{2t}/n!(\mu+1+k/2)_{nk}, \text{ where } \theta = t,\ d/dt$$

yields

$$\sum_{n=0}^{\infty} \frac{(c)_n}{n!(\mu+k/2)_{nk}} S_{2n+1}^{\mu}(x;k) t^{2n+1} = 2tx^k\, U^{-2k(1+c)} \left(U^{-2k} - \frac{8ckt^2}{k+2}\right) \cdot \quad (3.7)$$

$$\cdot\ {}_1F_k \left[\begin{matrix} c; \\ \Delta(k,\mu+1+k/2); \end{matrix}\ W\right] + \frac{16\, ckt^3 x^{3k} U^{2k(c+2)}}{(k+2\mu)\ (1+\mu+k/2)_k}\ {}_1F_k \left[\begin{matrix} c+1; \\ \Delta(k,1+\mu+3k/2; \end{matrix}\ W\right]$$

where $W = 4x^{2k}t^2/(1+4t^2)k^k$.

In fact, one obtains after combining even case with (3.7) the following generating function for the first biorthogonal set $\{S_n^{\mu}(x;k)\}$:

$$\sum_{n=0}^{\infty} \frac{(c)_{[n/2]}}{[n/2]!\ (\mu+k/2)_{k[n/2]}} S_n^{\mu}(x;k) t^n = \frac{(\mu+k/2)}{(\mu+1/2)} U^{2kc}\ {}_1F_k \left[\begin{matrix} c; \\ \Delta(k,\mu+1/2); \end{matrix}\ W\right] \quad (3.8)$$

$$+ 2tx^k\, U^{2k(1+c)} \left(U^{-2k} - \frac{8ckt^2}{k+2\mu}\right) {}_1F_k \left[\begin{matrix} c; \\ \Delta(k,1+\mu+k/2); \end{matrix}\ W\right]$$

$$+ \frac{16\ ckt^3\ x^{3k}\ U^{2k(c+2)}}{(k+3\mu)\ (1+\mu+k/2)_k}\ {}_1F_k \left[\begin{matrix} c+1; \\ \Delta(k,1+\mu+3k/2; \end{matrix}\ W\right].$$

We finally state the differential equation satisfied by the first set $\{S_n^\mu(x;k)\}$ in the form

$$[x^2(xD+2\mu+1+\epsilon)]^k \{x^{1-2k} (D-\epsilon k/x)\ S_n^\mu(x;k)\} \tag{3.9}$$

$= (2x^2)^k \{x\ D\ S_n^\mu(x;k) - nk\ S_n^\mu(x;k)\}$, and a differential recurrence relation for the second set

$$k\ T_{n+2}^\mu(x;k) = -2xD\ T_n^\mu(x;k) - 2(1+m1+2\mu-2x^2)\ T_n^\mu(x;k)\ . \tag{3.10}$$

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