Research Article

Conservation Laws and Self-Consistent Sources for an Integrable Lattice Hierarchy Associated with a Three-by-Three Discrete Matrix Spectral Problem

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A lattice hierarchy with self-consistent sources is deduced starting from a three-by-three discrete matrix spectral problem. The Hamiltonian structures are constructed for the resulting hierarchy. Liouville integrability of the resulting equations is demonstrated. Moreover, infinitely many conservation laws of the resulting hierarchy are obtained.

1. Introduction

Nonlinear integrable systems of the discrete version, treated as models of some physical phenomena, have attracted more and more attention in recent years. A well-known result is that a hierarchy of soliton equations can be generated through the isospectral compatibility condition of a pair of spectral problems [1]. By using the discrete trace identity the Hamiltonian forms of the soliton equations can be constructed [2]. Various methods have been developed to search for new integrable systems [3–5], integrable coupling systems [6], soliton solutions [7], and so on. However, the work of searching for new integrable systems associated with higher order matrix spectral problems is few. In [8, 9], the well-known method is used for the matrix spectral problems with 3×3 matrixes. In studying the integrability of discrete systems, the conservation laws play important roles. From the Lax pair of lattice soliton equations conservation laws can be deduced directly [10].

With the development of soliton theory, people began to focus on the soliton equations with self-consistent sources. Soliton equations with self-consistent sources are often used to express interactions between different solitary waves and are relevant to some problems of hydrodynamics, solid state

physics, plasma physics, and so on. Many integrable coupling systems with self-consistent sources in continuous cases are obtained [11–14]. In [15, 16], integrable discrete systems with self-consistent sources are given.

In the present paper, first, a new three-by-three discrete matrix spectral problem is proposed. By means of constructing a proper continuous time evolution equation and using the discrete zero curvature equation a hierarchy of lattice models is derived. Then the Hamiltonian forms of the resulting hierarchy are worked out by using the discrete trace identity. Further, the Liouville integrability of the discrete systems is demonstrated. Infinitely many conservation laws and self-consistent sources for the integrable systems are also obtained.

2. A New Three-by-Three Discrete Matrix Spectral Problem and Related Integrable Lattice Hierarchy

We first recall some presentations on a discrete integrable system. For a lattice function $f_n = f(n)$, the shift operator E, the inverse of E, and the operator E^k are defined by

$$Ef_n = f(n+1),$$

$$E^{-1} f_n = f(n-1),$$

$$E^k f_n = f(n+k), \quad k \in Z. \tag{1}$$

Let $u_n = u(n,t)$ be the potential vector. The variational derivative, the Gateaux derivative, the inner product, and the Poisson bracket are defined by

$$\widetilde{H}_n = \sum_{n \in z} H_n, \qquad \frac{\delta \widetilde{H}_n}{\delta u_n} = \sum_{m \in z} E^{-m} \left(\frac{\partial H_n}{\partial u_{n+m}} \right), \tag{2}$$

$$P'(u_n)[v_n] = \frac{\partial}{\partial \varepsilon} P(u_n + \varepsilon v_n) \Big|_{\varepsilon=0},$$

$$\langle f_n, g_n \rangle = \sum_{n \in z} (f_n, g_n)_{R^3}, \tag{3}$$

$$\{f_n, g_n\}_J = \left\langle \frac{\delta f_n}{\delta u_n}, J \frac{\delta g_n}{\delta u_n} \right\rangle = \sum_{n \in z} \left(\frac{\delta f_n}{\delta u_n}, J \frac{\delta g_n}{\delta u_n} \right)_{R^3}, \tag{4}$$

where P can be a vector function or an operator, f_n and g_n are vector functions, $(f_n, g_n)_{R^3}$ denotes the standard inner product of f_n and g_n in the Euclidean space R^3, and J is a Hamiltonian operator. A system of evolution equations $u_{n_{t_m}} = K_m(u_n)$ is called a Hamiltonian system, if there is a Hamiltonian operator J and a sequence of conserved functionals $\widetilde{H}_n^{(m)}$, $m = 1, 2, \ldots$, such that

$$u_{n_{t_m}} = K_m(u_n) = J \frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n}. \tag{5}$$

The functional $\widetilde{H}_n^{(m)}$ is called a Hamiltonian functional of the system, and we say that the system possesses Hamiltonian structures. As to a discrete Hamiltonian system, if there are infinitely many involutive conserved functionals, we say it is a Liouville integrable discrete Hamiltonian system.

In this paper, we consider the 3×3 discrete matrix spectral problem:

$$E\varphi_n = U_n(u_n, \lambda)\varphi_n, \quad U_n(u_n, \lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p_n & r_n & w_n + \lambda \end{pmatrix}, \tag{6}$$

in which $u_n = (p_n, r_n, w_n)^T$ is the potential, $p_n = p(n,t)$, $r_n = r(n,t)$, and $w_n = w(n,t)$ are real functions defined over $Z \times R$, λ is a spectral parameter, $\lambda_t = 0$, and $\varphi_n = (\varphi_1(n), \varphi_2(n), \varphi_3(n))^T$ is the eigenfunction.

To get a hierarchy of lattice models associated with (6), first we solve the stationary discrete zero curvature equation:

$$(E\Gamma_n)U_n - U_n\Gamma_n = 0, \tag{7}$$

where

$$\Gamma_n = \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & h_n & -a_n - e_n \end{pmatrix}. \tag{8}$$

Equation (8) gives

$$p_n c_{n+1} = d_n,$$

$$r_n c_{n+1} + a_{n+1} = e_n,$$

$$b_{n+1} - f_n + c_{n+1}(w_n + \lambda) = 0,$$

$$p_n f_{n+1} = g_n,$$

$$r_n f_{n+1} + d_{n+1} - h_n = 0, \tag{9}$$

$$f_{n+1}(w_n + \lambda) + e_{n+1} + a_n + e_n = 0,$$

$$-r_n d_n - (w_n + \lambda) g_n = p_n(a_n + a_{n+1} + e_{n+1}),$$

$$-r_n(a_{n+1} + e_{n+1} + e_n) + g_{n+1} = p_n b_n + h_n(w_n + \lambda),$$

$$-r_n f_n + h_{n+1} = p_n c_n + (a_{n+1} + e_{n+1} - a_n - e_n)(w_n + \lambda).$$

Substituting the expansions

$$a_n = \sum_{m=0}^{\infty} a_n^{(m)} \lambda^{-m}, \qquad b_n = \sum_{m=0}^{\infty} b_n^{(m)} \lambda^{-m},$$

$$c_n = \sum_{m=0}^{\infty} c_n^{(m)} \lambda^{-m}, \qquad d_n = \sum_{m=0}^{\infty} d_n^{(m)} \lambda^{-m},$$

$$e_n = \sum_{m=0}^{\infty} e_n^{(m)} \lambda^{-m}, \qquad f_n = \sum_{m=0}^{\infty} f_n^{(m)} \lambda^{-m}, \tag{10}$$

$$g_n = \sum_{m=0}^{\infty} g_n^{(m)} \lambda^{-m}, \qquad h_n = \sum_{m=0}^{\infty} h_n^{(m)} \lambda^{-m}$$

into (9), we get the recursion relation

$$p_n c_{n+1}^{(m)} = d_n^{(m)},$$

$$a_{n+1}^{(m)} - e_n^{(m)} + r_n c_{n+1}^{(m)} = 0,$$

$$b_{n+1}^{(m)} - f_n^{(m)} + w_n c_{n+1}^{(m)} = -c_{n+1}^{(m+1)},$$

$$g_n^{(m)} - p_n f_{n+1}^{(m)} = 0,$$

$$r_n f_{n+1}^{(m)} + d_{n+1}^{(m)} - h_n^{(m)} = 0,$$

$$w_n f_{n+1}^{(m)} + \left(e_{n+1}^{(m)} + e_n^{(m)} + a_n^{(m)} \right) = -f_{n+1}^{(m+1)},$$

$$-p_n \left(a_n^{(m)} + a_{n+1}^{(m)} \right) - w_n g_n^{(m)} - p_n e_{n+1}^{(m)} - r_n d_n^{(m)} = g_n^{(m+1)},$$

$$-r_n \left(a_{n+1}^{(m)} + e_n^{(m)} + e_{n+1}^{(m)} \right) - w_n h_n^{(m)} + g_{n+1}^{(m)} - p_n b_n^{(m)} = h_n^{(m+1)},$$

$$h_{n+1}^{(m)} + w_n \left(a_n^{(m)} - a_{n+1}^{(m)} + e_n^{(m)} - e_{n+1}^{(m)} \right) - p_n c_n^{(m)} - r_n f_n^{(m)}$$

$$= a_{n+1}^{(m+1)} - a_n^{(m+1)} + e_{n+1}^{(m+1)} - e_n^{(m+1)} \tag{11}$$

and the initial requirement

$$a_{n+1}^{(0)} + e_{n+1}^{(0)} = a_n^{(0)} + e_n^{(0)},$$

$$g_n^{(0)} = h_n^{(0)} = d_n^{(0)} = f_{n+1}^{(0)} = c_{n+1}^{(0)} = 0. \tag{12}$$

The initial values are taken as $b_n^{(0)} = c_n^{(0)} = f_n^{(0)} = 0$, so we can get $e_n^{(0)} = a_{n+1}^{(0)}$, $a_n^{(0)} = e_{n+1}^{(0)}$, and then we take the initial value $a_n^{(0)} = e_n^{(0)} = -1/3$. Note that the definition of the inverse operator of $D = (E - 1)$ does not yield any arbitrary constant in computing $a_n^{(m)}$ and $e_n^{(m)}$, $m \geq 1$. Thus, the recursion relation (11) uniquely determines $a_n^{(m)}$, $b_n^{(m)}$, $c_n^{(m)}$, $d_n^{(m)}$, $e_n^{(m)}$, $f_n^{(m)}$, $g_n^{(m)}$, and $h_n^{(m)}$, $m \geq 1$, and the first few quantities are given by

$$a_n^{(1)} = b_n^{(1)} = c_n^{(1)} = d_n^{(1)} = e_n^{(1)} = 0,$$

$$f_n^{(1)} = 1, \qquad g_n^{(1)} = p_n, \qquad h_n^{(1)} = r_n,$$

$$a_n^{(2)} = 0, \qquad b_n^{(2)} = 1, \qquad c_n^{(2)} = 1, \tag{13}$$

$$d_n^{(2)} = p_n, \qquad e_n^{(2)} = r_n, \qquad f_n^{(2)} = -w_n,$$

$$g_n^{(2)} = -p_n w_n, \qquad h_n^{(2)} = p_{n+1} - w_n r_n, \dots .$$

Let

$$V_n^{(m)}$$

$$= \begin{pmatrix} \sum_{i=0}^{m} a_n^{(i)} \lambda^{m-i} & \sum_{i=0}^{m} b_n^{(i)} \lambda^{m-i} & \sum_{i=0}^{m} c_n^{(i)} \lambda^{m-i} \\ \sum_{i=0}^{m} d_n^{(i)} \lambda^{m-i} & \sum_{i=0}^{m} e_n^{(i)} \lambda^{m-i} & \sum_{i=0}^{m} f_n^{(i)} \lambda^{m-i} \\ \sum_{i=0}^{m} g_n^{(i)} \lambda^{m-i} & \sum_{i=0}^{m} h_n^{(i)} \lambda^{m-i} & -\sum_{i=0}^{m} a_n^{(i)} \lambda^{m-i} - \sum_{i=0}^{m} e_n^{(i)} \lambda^{m-i} \end{pmatrix},$$

$$m \geq 0, \tag{14}$$

and then take a modification

$$\eta_n^{(m)} = \begin{pmatrix} f_{n-1}^{(m+1)} & c_n^{(m+1)} & 0 \\ 0 & f_n^{(m+1)} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m \geq 0. \tag{15}$$

Now we set

$$V_n^{[m]} = V_n^{(m)} + \eta_n^{(m)}, \quad m \geq 0. \tag{16}$$

Then we introduce the auxiliary spectral problems associated with the spectral problem (6):

$$\varphi_{n_{t_m}} = V_n^{[m]} \varphi_n, \quad m \geq 0. \tag{17}$$

The compatibility conditions of (6) and (17) are

$$U_{n_{t_m}} = \left(E V_n^{[m]} \right) U_n - U_n V_n^{[m]}, \quad m \geq 0, \tag{18}$$

which give rise to the following hierarchy of integrable lattice equations:

$$p_{n_{t_m}} = g_n^{(m+1)} - p_n f_{n-1}^{(m+1)},$$

$$r_{n_{t_m}} = h_n^{(m+1)} - r_n f_n^{(m+1)} - p_n c_n^{(m+1)}, \tag{19}$$

$$w_{n_{t_m}} = \left(a_{n+1}^{(m+1)} - a_n^{(m+1)} \right) + \left(e_{n+1}^{(m+1)} - e_n^{(m+1)} \right),$$

$$(m \geq 0).$$

So the discrete spectral problem (6) and (17) constitute the Lax pairs of (19), and (19) are a hierarchy of Lax integrable lattice equations. It is easy to verify that the first lattice equation in (19), when $m = 1$, under $t_1 \to t$, is

$$p_{n_t} = p_n \left(w_{n-2} - w_n \right),$$

$$r_{n_t} = r_n \left(w_{n-1} - w_n \right) + \left(p_{n+1} - p_n \right), \tag{20}$$

$$w_{n_t} = r_{n+1} - r_n.$$

The Lax pair of (20) is (6) and the time evolution law for φ_n is as follows:

$$\varphi_{n_t} = V_n^{[1]} \varphi_n$$

$$= \begin{pmatrix} -\dfrac{\lambda}{3} - w_{n-2} + \dfrac{2}{3} & 1 & 0 \\ 0 & -\dfrac{\lambda}{3} + \dfrac{2}{3} - w_{n-1} & 1 \\ p_n & r_n & \dfrac{2}{3}\lambda + \dfrac{2}{3} \end{pmatrix} \varphi_n. \tag{21}$$

Now we would like to derive the Hamiltonian structures for (19).

Set $R_n = \Gamma_n U_n^{-1}$; through a direct calculation, we get

$$\mathrm{Tr}\left(R_n \frac{\partial U_n}{\partial \lambda} \right) = \frac{g_n}{p_n}, \qquad \mathrm{Tr}\left(R_n \frac{\partial U_n}{\partial p_n} \right) = \frac{a_n}{p_n},$$

$$\mathrm{Tr}\left(R_n \frac{\partial U_n}{\partial r_n} \right) = \frac{d_n}{p_n}, \qquad \mathrm{Tr}\left(R_n \frac{\partial U_n}{\partial w_n} \right) = \frac{g_n}{p_n}. \tag{22}$$

By the discrete trace identity [2]

$$\frac{\delta}{\delta u_n^i} \sum_{n \in Z} \mathrm{Tr}\left(R_n \frac{\partial U_n}{\partial \lambda} \right) = \lambda^{-\varepsilon} \left(\frac{\partial}{\partial \lambda} \right) \lambda^\varepsilon \mathrm{Tr}\left(R_n \frac{\partial U_n}{\partial u_n^i} \right), \tag{23}$$

$$i = 1, 2, 3,$$

we have

$$\left(\frac{\delta}{\delta p_n}, \frac{\delta}{\delta r_n}, \frac{\delta}{\delta w_n} \right)^T \sum_{n \in Z} \left(\frac{g_n}{p_n} \right) = \lambda^{-\varepsilon} \left(\frac{\partial}{\partial \lambda} \right) \lambda^\varepsilon \begin{pmatrix} \dfrac{a_n}{p_n} \\ \dfrac{d_n}{p_n} \\ \dfrac{g_n}{p_n} \end{pmatrix}, \tag{24}$$

where ε is a constant to be found.

By substituting

$$a_n = \sum_{m=0}^{\infty} a_n^{(m)} \lambda^{-m},$$

$$d_n = \sum_{m=0}^{\infty} d_n^{(m)} \lambda^{-m}, \tag{25}$$

$$g_n = \sum_{m=0}^{\infty} g_n^{(m)} \lambda^{-m}$$

into (24) and equating the coefficients of λ^{-m-1}, we have

$$\left(\frac{\delta}{\delta p_n}, \frac{\delta}{\delta r_n}, \frac{\delta}{\delta w_n} \right)^T \sum_{n \in Z} \left(\frac{g_n^{(m+1)}}{-m p_n} \right) = \begin{pmatrix} \dfrac{a_n^{(m)}}{p_n} \\[2mm] \dfrac{d_n^{(m)}}{p_n} \\[2mm] \dfrac{g_n^{(m)}}{p_n} \end{pmatrix}, \quad m > 0. \tag{26}$$

Now we can rewrite (19) as follows:

$$\begin{pmatrix} p_n \\ r_n \\ w_n \end{pmatrix}_{t_m} = J \begin{pmatrix} \dfrac{a_n^{(m)}}{p_n} \\[2mm] \dfrac{d_n^{(m)}}{p_n} \\[2mm] \dfrac{g_n^{(m)}}{p_n} \end{pmatrix}, \tag{27}$$

where

$$J = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix},$$

$$J_{11} = p_n \left(E^{-1} - E + E^{-2} - E^2 \right) p_n,$$

$$J_{12} = p_n \left(E^{-2} + E^{-1} - E - 1 \right) r_n,$$

$$J_{13} = p_n \left(E^{-2} - 1 \right) w_n,$$

$$J_{21} = r_n \left(E^{-1} + 1 - E - E^2 \right) p_n, \tag{28}$$

$$J_{22} = r_n \left(E^{-1} - E \right) r_n + p_n E^{-1} w_n - w_n E p_n,$$

$$J_{23} = E p_n - p_n E^{-2} - r_n \left(1 - E^{-1} \right) w_n,$$

$$J_{31} = w_n \left(1 - E^2 \right) p_n,$$

$$J_{32} = E^2 p_n - p_n E^{-1} + w_n (1 - E) r_n,$$

$$J_{33} = E r_n - r_n E^{-1}.$$

It is easy to verify that the operator J is a Hamiltonian operator. So the lattice systems (19) can be rewritten as the hierarchy of discrete Hamiltonian equation (27). Set

$$\widetilde{H}_n^{(m)} = \sum_{n \in Z} \frac{g_n^{(m+1)}}{-m p_n}, \quad m \geq 1, \tag{29}$$

$$\frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n} = \phi_n \frac{\delta \widetilde{H}_n^{(m-1)}}{\delta u_n}. \tag{30}$$

From the recursion relation (11) we can get the recursion operator ϕ_n in (30):

$$\phi_n = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix},$$

$$\phi_{11} = \frac{1}{p_n} \left(E^2 - 1 \right)^{-1} w_n \left(1 - E^2 \right) p_n$$
$$\quad + \frac{1}{p_n} (E+1)^{-1} r_n \left(E p_n + p_n E^{-1} \right)^{-1} r_n (E-1) p_n,$$

$$\phi_{12} = \frac{1}{p_n} \left(E^2 - 1 \right)^{-1} \left[E^2 p_n - p_n E^{-1} + w_n (1 - E) r_n \right]$$
$$\quad - \frac{1}{p_n} (E+1)^{-1} r_n \left(E p_n + p_n E^{-1} \right)^{-1}$$
$$\quad \times \left(p_n E^{-1} w_n - w_n E p_n \right),$$

$$\phi_{13} = \frac{1}{p_n} \left(E^2 - 1 \right)^{-1} \left(E r_n - r_n E^{-1} \right)$$
$$\quad - \frac{1}{p_n} (E+1)^{-1} r_n \left(E p_n + p_n E^{-1} \right)^{-1} \left(E p_n - p_n E^{-2} \right),$$

$$\phi_{21} = - \left(E p_n + p_n E^{-1} \right)^{-1} r_n (E-1) p_n,$$

$$\phi_{22} = \left(E p_n + p_n E^{-1} \right)^{-1} \left(p_n E^{-1} w_n - r_n E p_n \right),$$

$$\phi_{23} = \left(E p_n + p_n E^{-1} \right)^{-1} \left(E p_n - p_n E^{-2} \right),$$

$$\phi_{31} = -p_n - E p_n - E^2 p_n,$$

$$\phi_{32} = -E r_n - r_n,$$

$$\phi_{33} = -w_n. \tag{31}$$

Therefore, we have

$$M = J \phi_n = \begin{pmatrix} 0 & 0 & p_n \left(1 - E^{-2} \right) \\ 0 & E p_n - p_n E^{-1} & r_n \left(1 - E^{-1} \right) \\ \left(E^2 - 1 \right) p_n & (E-1) r_n & 0 \end{pmatrix}. \tag{32}$$

It is easy to verify that $M^* = -M$; moreover, we can prove that

$$\left\{ \widetilde{H}_n^{(m)}, \widetilde{H}_n^{(l)} \right\}_J = 0, \quad m, l \geq 1,$$

$$\left(\widetilde{H}_n^{(m)} \right)_{t_l} = \left\langle \frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n}, u_{nt_l} \right\rangle = \left\langle \frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n}, J \frac{\delta \widetilde{H}_n^{(l)}}{\delta u_n} \right\rangle \quad (33)$$

$$= \left\{ \widetilde{H}_n^{(m)}, \widetilde{H}_n^{(l)} \right\}_J = 0, \quad m, l \geq 1.$$

So we get the following.

Proposition 1. $\{ \widetilde{H}_n^{(m)} \}_{m \geq 1}$ *defined by* (29) *forms an infinite set of conserved functionals of the hierarchy* (19), *and* $\widetilde{H}_n^{(m)}, m \geq 1$, *are involution in pairs with respect to the Poisson bracket* (4).

Theorem 2. *The lattice equations in* (19) *are all discrete Liouville integrable Hamiltonian systems.*

3. Infinitely Many Conservation Laws

We can get the following alternative form from (6) and (21):

$$\varphi_3 (n+1) = p_n \varphi_3 (n-2) + r_n \varphi_3 (n-1) + (w_n + \lambda) \varphi_3 (n),$$
$$(34)$$

$$\varphi_{3_t} (n) = p_n \varphi_3 (n-2) + r_n \varphi_3 (n-1) + \frac{2}{3} (1 + \lambda) \varphi_3 (n).$$
$$(35)$$

Set

$$\theta_n = \theta (n) = \frac{\varphi_3 (n)}{\varphi_3 (n+1)}, \quad (36)$$

and we can obtain

$$-(\ln \theta_n)_t = (E - 1) \left[p_n \theta_{n-2} \theta_{n-1} + r_n \theta_{n-1} + \frac{2}{3} (1 + \lambda) \right]. \quad (37)$$

Equation (34) can be written as follows:

$$\lambda \theta_n = 1 - p_n \theta_{n-2} \theta_{n-1} \theta_n - r_n \theta_{n-1} \theta_n - w_n \theta_n. \quad (38)$$

Then, expanding θ_n in the power series of $1/\lambda$:

$$\theta_n = \sum_{j=1}^{\infty} \theta_n^{(j)} \lambda^{-j}, \quad (39)$$

and substituting it into (38), we can obtain all the coefficients $\theta_n^{(j)}$. Substitute them into (37) and due to $\theta_n^{(1)} = 1$ and $\lambda_t = 0$, we can get the following fact:

$$\left[\sum_{k=1}^{\infty} (-1)^k \frac{1}{k} \left(\sum_{j=1}^{\infty} \frac{\theta_n^{(j+1)}}{\lambda^j} \right)^k \right]_t$$

$$= (E - 1)$$

$$\times \left[p_n \sum_{j=1}^{\infty} \frac{\theta_{n-2}^{(j)}}{\lambda^j} \sum_{j=1}^{\infty} \frac{\theta_{n-1}^{(j)}}{\lambda^j} + r_n \sum_{j=1}^{\infty} \frac{\theta_{n-1}^{(j)}}{\lambda^j} + \frac{2}{3} (1 + \lambda) \right],$$
$$(40)$$

from which an infinite number of conservation laws can be determined by equating the powers of $1/\lambda$. The following are the first three of them:

$$w_{n_t} = (E - 1) r_n,$$

$$\left(-r_n - \frac{1}{2} w_n^2 \right)_t = (E - 1) \left(-w_{n-1} r_n + p_n \right),$$

$$\left(p_n - r_n w_{n-1} - w_n r_n + \frac{1}{3} w_n^3 \right)_t$$

$$= (E - 1) \left[r_n \left(r_{n-1} + w_{n-1}^2 \right) - p_n \left(w_{n-1} + w_{n-2} \right) \right].$$
$$(41)$$

We can get other conservation laws in the hierarchy (19) similarly.

4. Self-Consistent Sources for the Lattice Hierarchy (19)

In this section, we will construct the lattice hierarchy (19) with self-consistent sources. Consider the auxiliary linear problems

$$E \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix} = U_n \left(u_n, \lambda_j \right) \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix},$$

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_{t_m} = V_n^{[m]} \left(u_n, \lambda_j \right) \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix},$$
$$(42)$$

and, based on the results in [17], we show the following equation:

$$\frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n} + \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u_n} = 0, \quad (43)$$

where

$$\frac{\delta \lambda_j}{\delta u_n} = \frac{1}{2} \mathrm{Tr} \left(\psi_j \frac{\partial U \left(u_n, \lambda_j \right)}{\partial u_n} \right),$$

$$\psi_j = \begin{pmatrix} \varphi_{1j} \varphi_{1j} & \varphi_{1j} \varphi_{2j} & \varphi_{1j} \varphi_{3j} \\ \varphi_{2j} \varphi_{1j} & \varphi_{2j} \varphi_{2j} & \varphi_{2j} \varphi_{3j} \\ \varphi_{3j} \varphi_{1j} & \varphi_{3j} \varphi_{2j} & \varphi_{3j} \varphi_{3j} \end{pmatrix}, \quad j = 1, 2, \ldots, N.$$
$$(44)$$

Through a direct computation, we obtain the lattice hierarchy with self-consistent sources as follows:

$$
\begin{pmatrix} p_n \\ r_n \\ w_n \end{pmatrix}_{t_m} = J \begin{pmatrix} \dfrac{\delta \widetilde{H}_n^{(m)}}{\delta p_n} + \displaystyle\sum_{j=1}^{N} \dfrac{\delta \lambda_j}{\delta p_n} \\[2ex] \dfrac{\delta \widetilde{H}_n^{(m)}}{\delta r_n} + \displaystyle\sum_{j=1}^{N} \dfrac{\delta \lambda_j}{\delta r_n} \\[2ex] \dfrac{\delta \widetilde{H}_n^{(m)}}{\delta w_n} + \displaystyle\sum_{j=1}^{N} \dfrac{\delta \lambda_j}{\delta w_n} \end{pmatrix}
$$

$$
= J \begin{pmatrix} \dfrac{\delta \widetilde{H}_n^{(m)}}{\delta p_n} \\[2ex] \dfrac{\delta \widetilde{H}_n^{(m)}}{\delta r_n} \\[2ex] \dfrac{\delta \widetilde{H}_n^{(m)}}{\delta w_n} \end{pmatrix} + J \begin{pmatrix} \dfrac{1}{2}\displaystyle\sum_{j=1}^{N} \varphi_{1j}\varphi_{3j} \\[2ex] \dfrac{1}{2}\displaystyle\sum_{j=1}^{N} \varphi_{2j}\varphi_{3j} \\[2ex] \dfrac{1}{2}\displaystyle\sum_{j=1}^{N} \varphi_{3j}\varphi_{3j} \end{pmatrix}, \quad m \geq 1.
\tag{45}
$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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