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Research Article

A Second Order Characteristic Method for Approximating Incompressible Miscible Displacement in Porous Media

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An approximation scheme is defined for incompressible miscible displacement in porous media. This scheme is constructed by two methods. Under the regularity assumption for the pressure, cubic Hermite finite element method is used for the pressure equation, which ensures the approximation of the velocity smooth enough. A second order characteristic finite element method is presented to handle the material derivative term of the concentration equation. It is of second order accuracy in time increment, symmetric, and unconditionally stable. The optimal L^2-norm error estimates are derived for the scalar concentration.

1. Introduction

In this paper, we will consider the following incompressible miscible displacement in porous media, which is governed by a coupled system of partial differential equations with initial and boundary values. The pressure is governed by an elliptic equation and the concentration is governed by a convection-diffusion equation [1–6] as follows:

$$-\nabla \cdot \left(\frac{k}{\mu}\nabla p\right) \equiv \nabla \cdot u = q, \quad x \in \Omega,\ t \in J, \tag{1.1a}$$

$$\phi\frac{\partial c}{\partial t} + u \cdot \nabla c - \nabla \cdot (D\nabla c) = (\tilde{c} - c)\tilde{q}, \quad x \in \Omega,\ t \in J, \tag{1.1b}$$

$$u \cdot \nu = (D(x)\nabla c) \cdot \nu = 0, \quad x \in \partial\Omega,\ t \in J, \tag{1.1c}$$

$$c(x,0) = c_0(x), \quad x \in \Omega, \tag{1.1d}$$

where Ω is a bounded domain in $\mathcal{R}^2$, $J = (0, T]$, and $\tilde{q} = \max\{q, 0\}$ is nonzero at injection wells only. The variables in (1.1a)–(1.1d) are the pressure $p(x,t)$ in the fluid mixture, the Darcy velocity $u = (u_1, u_2)'$, and the relative concentration $c(x,t)$ of the injected fluid. The ν is the unit outward normal vector on boundary $\partial\Omega$.

The coefficients and data in (1.1a)–(1.1d) are $k(x)$, the permeability of the porous media; $\mu(x)$, the viscosity of the fluid mixture; $q(x,t)$, representing flow rates at wells; $\phi(x)$, the porosity of the rock; $\tilde{c}(x,t)$, the injected concentration at injection wells ($q > 0$) and the resident concentration at production wells ($q < 0$); $\tilde{q} = \max(q, 0)$. Here, for the diffusion coefficient, we consider a dispersion-free case [5, 6] as follows:

$$D = \phi d_m I, \tag{1.2}$$

where d_m is the molecular diffusivity and I is a 2×2 identity matrix. Furthermore, a compatibility condition $\int_\Omega q(x,t)dx = 0$ must be imposed to determine the pressure.

The pressure equation is elliptic and easily handled, but the concentration equation is parabolic and normally convection dominated. It is well known that standard Galerkin scheme applied to the convection-dominated problems does not work well and produces excessive numerical diffusion or nonphysical oscillation. The characteristic method has been introduced to obtain better approximations for (1.1a)–(1.1d), such as characteristic finite element method [3–7], characteristic finite difference method [8], the modified of characteristic finite element method (MMOC-Galerkin) [9], and the Eulerian-Lagrangian localized adjoint method (ELLAM) [10].

We had considered a combined numerical approximation for (1.1) in [11]. Standard mixed finite element was used for Darcy velocity equation and a characteristics-mixed finite element method was presented for approximating the concentration equation. Characteristic approximation was applied to handle the convection term, and a lowest order mixed finite element spatial approximation was adopted to deal with the diffusion term. Thus, the scalar unknown concentration and the diffusive flux can be approximated simultaneously. This approximation conserves mass globally. The optimal L^2-norm error estimates were derived.

It should be pointed out that the works mentioned above all gave one order accuracy in time increment Δt. That is to say, the first order characteristic method in time was analyzed. As for higher order characteristic method in time, Rui and Tabata [12] used the second order Runge-Kutta method to approximate the material derivative term for convection-diffusion problems. The scheme presented was of second order accuracy in time increment Δt, symmetric, and unconditionally stable. Optimal error estimates were proved in the framework of L^2-theory. Numerical analysis of convection-diffusion-reaction problems with higher order characteristic/finite elements were analyzed in [13, 14], which extended the work [12]. The $l^\infty(L^2)$ error estimates of second order in time increment Δt were obtained.

The goal of this paper is to present a second order characteristic finite element method in time increment to handle the material derivative term of the concentration equation of (1.1a)–(1.1d). It is organized as follows. In Section 2, we formulate the second order characteristic finite element method for the concentration and cubic Hermite finite element method for the pressure, respectively. Then, we present a combined approximation scheme. In Section 3, we analyze the stability of the approximation scheme for the concentration equation. In Section 4, we derive the optimal-order L^2-norm error estimates for the scalar concentration. They are of second order accuracy in time increment, symmetric, and unconditionally stable. We conclude our results in Section 5.

2. Formulation of the Method

2.1. Statements and Assumptions

In this paper, we adopt notations and norms of usual Sobolev spaces. For periodic functions, we use the notation as in [4] as follows. Let:

$$W_p^k = \widetilde{W}_p^k(\Omega) = \left\{ \varphi : \ \frac{\partial^\alpha \varphi}{\partial x^\alpha} \in \ L^p(\Omega) \ \text{for} \ |\alpha| \le k, \ \text{periodic} \right\} \tag{2.1}$$

be the periodic Sobolev space on Ω with the usual norm. If $p = 2$, we write

$$H^k = \widetilde{H}^k(\Omega) = \widetilde{W}_2^k(\Omega) \tag{2.2}$$

with norm

$$\|\varphi\|_k = \|\varphi\|_{\widetilde{H}^k(\Omega)}, \qquad \|\varphi\| = \|\varphi\|_0 = \|\varphi\|_{L^2(\Omega)}. \tag{2.3}$$

Moreover, we adopt some notations for the functional spaces involved, which were used in [12–14]. For a Banach space X and a positive integer m, spaces $C^m([0,T],X)$ and $H^m((0,T),X)$ are abbreviated as $C^m(X)$ and $H^m(X)$, respectively, and endowed with norms

$$\|\varphi\|_{C^m(X)} := \max_{t\in[0,T]} \left\{ \max_{j=0,\ldots,m} \left\|\varphi^{(j)}(t)\right\|_X \right\}, \qquad \|\varphi\|_{H^m(X)} := \left(\int_0^T \sum_{j=0}^m \left\|\varphi^{(j)}(t)\right\|_X^2 dt \right)^{1/2}, \tag{2.4}$$

where $\varphi^{(j)}$ denotes the jth derivative of φ with respect to time. The Banach space Z^m is defined by

$$Z^m = \left\{ f \in C^j\left(H^{m-j}(\Omega)\right); \ j = 0,\ldots,m \right\}, \tag{2.5}$$

equipped with the norm

$$\|\varphi\|_{Z^m} := \max\left\{ \|\varphi\|_{C^j(H^{m-j})}; \ 0 \le j \le m \right\}. \tag{2.6}$$

We also require the following assumptions on the coefficients in (1.1) [3]. Let a_*, a^*, ϕ_*, ϕ^*, and K^* be positive constants such that

$$\begin{gathered} 0 < a_* \le \frac{k(x)}{\mu(x)} \le a^*, \quad 0 < \phi_* \le \phi(x) \le \phi^*, \\ \left|\widetilde{q}(x,t)\right| + \left|\frac{\partial \widetilde{q}}{\partial t}(x,t)\right| \le K^*. \end{gathered} \tag{C}$$

Other assumptions will be made in individual theorems as necessary.

2.2. A Cubic Hermite Element Method for the Pressure

The variational form for the pressure equation (1.1a) is equal to the following:

$$\left(\frac{k}{\mu}\nabla p, \nabla \chi\right) = (q(t), \chi), \quad \chi \in H^1(\Omega),\ t \in J. \tag{2.7}$$

For $h_p > 0$, we discretize (2.7) in space on a quasiuniform triangle mesh $\mathcal{T}_{h_p}$ of Ω with diameter of element $\leq h_p$. Let $W_{h_p} \subset L^2(\Omega)$ be a cubic Hermite finite element space for this mesh. The finite element method for the pressure, given at a time $t \in J$, consists of $P \in W_{h_p}$ such that

$$\left(\frac{k}{\mu}\nabla P, \nabla \chi\right) = (q(t), \chi), \quad \forall \chi \in W_{h_p}. \tag{2.8}$$

Since the left-hand side of (2.7) ((2.8), resp.) is a continuous and coercive bilinear form and the right-hand side of (2.7) ((2.8), resp.) is a continuous linear functional, existence and uniqueness of p (P, resp.) are ensured obviously [15].

By the theory of the cubic Hermit element, the finite element space W_{h_p} possess the following approximation property [15, 16]:

$$\inf_{\chi \in W_{h_p}} \|g - \chi\|_s \leq K h_p^{4-s} \|g\|_4, \quad s = 0, 1,\ \forall g \in H^4(\Omega), \tag{2.9}$$

where K is a positive constant independent of h_p.

Define Π: $C^2(\Omega) \rightarrow W_{h_p}$ is the interplant operator. Then, we have the following theorem.

Theorem 2.1. *Let p and P be the solutions of* (1.1a)–(1.1d) *and* (2.8)*, respectively, and assume $p \in H^4(\Omega)$. Then, there exists a positive constant K independent of h_p such that*

$$\|\nabla(p - P)\| \leq K h_p^3 \|p\|_4. \tag{2.10}$$

Proof. By Céa lemma, we have

$$\begin{aligned}
a\|\nabla(p - P)\|^2 &\leq \left(\frac{k}{\mu}\nabla(p - P), \nabla(p - P)\right) \\
&= \left(\frac{k}{\mu}\nabla(p - P), \nabla(p - \Pi p)\right) + \left(\frac{k}{\mu}\nabla(p - P), \nabla(\Pi p - P)\right) \\
&= \left(\frac{k}{\mu}\nabla(p - P), \nabla(p - \Pi p)\right) \\
&\leq a^*\|\nabla(p - P)\|\|\nabla(p - \Pi P)\|.
\end{aligned} \tag{2.11}$$

Then by (2.9), we can derive

$$\|\nabla(p-P)\| \le K h_p^3 \|p\|_4. \tag{2.12}$$

□

2.3. A Second Order Characteristic Method for the Concentration

Now, we define the characteristic lines associated with vector field u and recall some classical properties satisfied by them. Thus, for given $(x,t) \in \overline{\Omega} \times [0,T]$, the characteristic line through (x,t) is the vector function $X_e(x,t;\cdot)$ solving the following initial value problem:

$$\frac{\partial X_e}{\partial \tau}(x,t;\tau) = v(X_e(x,t;\tau),\tau), \quad X_e(x,t;\tau) = x, \tag{2.13}$$

where $v(X_e(x,t;\tau),\tau) := u(X_e(x,t;\tau),\tau)/\phi$.

Next, assuming they exist, we denote by F_e (by L, resp.) the gradient of X_e (of v, resp.) with respect to the space variable x, that is,

$$(F_e)_{rs}(x,t;\tau) := \frac{\partial (X_e)_r}{\partial x_s}(x,t;\tau), \qquad L_{rs}(x,t) := \frac{\partial v_r}{\partial x_s}(x,t). \tag{2.14}$$

We adopted some propositions and lemmas from [13].

Proposition 2.2. *If $v \in C^0(C^n(\overline{\Omega}))$ for $n \ge 1$ an integer, then $X_e \in C^0(\overline{\Omega} \times [0,T] \times [0,T])$ and it is C^n with respect to the x variable.*

In order to compute second order approximations of matrices F_e and F_e^{-1}, we need the following equations:

$$\begin{aligned} \frac{\partial F_e}{\partial \tau}(x,t;\tau) &= L(X_e(x,t;\tau),\tau)F_e(x,t;\tau), \\ \frac{\partial^2 F_e}{\partial \tau^2}(x,t;\tau) &= \nabla\left(\frac{\partial v}{\partial t} + Lv\right)(X_e(x,t;\tau),\tau)F_e(x,t;\tau). \end{aligned} \tag{2.15}$$

Proposition 2.3. *If $v \in C^0(C^1(\overline{\Omega}))$, then*

$$\|F_e(x,t;\tau)\| \le e^{\|v\|_{C^0(C^1(\overline{\Omega}))}|\tau - t|}, \quad \forall x \in \Omega,\ t,\tau \in [0,T]. \tag{2.16}$$

Proposition 2.4. *If $v \in C^0(C^2(\overline{\Omega})) \cap C^1(C^1(\overline{\Omega}))$, then F_e satisfies the Taylor expansions as*

$$\begin{aligned} F_e(x,t;s) &= I + (s-t)L(x,t) \\ &\quad + \int_s^t (\tau - s)\nabla\left(\frac{\partial v}{\partial t} + Lv\right)(X_e(x,t;\tau),\tau)F_e(x,t;\tau)d\tau, \end{aligned} \tag{2.17}$$

and its inverse, F_e^{-1}, satisfies the Liouville theorem as

$$F_e^{-1}(x,t;s) = I + (s-t)L(X_e(x,t;s),s) - \int_s^t (\tau - t)\nabla\left(\frac{\partial v}{\partial t} + Lv\right)(X_e(x,t;\tau),\tau)F_e(X_e(x,t;s),s;\tau)d\tau. \tag{2.18}$$

By using Liouville's theorem and the chain rule, we obtain

$$\begin{aligned}\frac{\partial}{\partial \tau}\det F_e^{-1}(x,t;\tau) &= -\det F_e^{-1}(x,t;\tau)\operatorname{div} v(X_e(x,t;\tau),\tau),\\ \frac{\partial^2}{\partial \tau^2}\det F_e^{-1}(x,t;\tau) &= -\det F_e^{-1}(x,t;\tau)\Big((\operatorname{div} v)^2(X_e(x,t;\tau),\tau) \\ &\quad \times \operatorname{div}\left(\frac{\partial v}{\partial t} + Lv\right)(X_e(x,t;\tau),\tau) \\ &\quad -\left(L\cdot L^T\right)(X_e(x,t;\tau),\tau)\Big).\end{aligned} \tag{2.19}$$

Proposition 2.5. *If $v \in C^0(C^1(\overline{\Omega}))$, then*

$$\det F_e(x,t;\tau) \le e^{\|v\|_{C^0(C^1(\overline{\Omega}))}|\tau - t|}, \quad \forall x \in \Omega,\ t,\tau \in [0,T]. \tag{2.20}$$

Proposition 2.6. *If $v \in C^0(C^2(\overline{\Omega})) \cap C^1(C^1(\overline{\Omega}))$, then* $\det F_e^{-1}$ *satisfies*

$$\det F_e^{-1}(x,t;s) = 1 - (s-t)\operatorname{div} v(x,t) + \int_s^t (\tau - s)\frac{\partial^2}{\partial \tau^2}\left(\det F_e^{-1}\right)(x,t;\tau)d\tau. \tag{2.21}$$

Variational Formulation

From the definition of the characteristic curves and by using the chain rule, it follows that

$$\phi\frac{dc}{d\tau}(X_e(x,t;\tau),\tau) = \phi\frac{\partial c}{\partial t}(X_e(x,t;\tau),\tau) + u(X_e(x,t;\tau),\tau)\cdot\nabla c(X_e(x,t;\tau),\tau). \tag{2.22}$$

By writing (1.1b) at point $X_e(x,t;\tau)$ and time τ and using (2.22), we have

$$\phi\frac{dc}{d\tau}(X_e(x,t;\tau),\tau) - \nabla\cdot(D\nabla c)(X_e(x,t;\tau),\tau) = \left((\tilde{c} - c)\tilde{q}\right)(X_e(x,t;\tau),\tau). \tag{2.23}$$

Before giving a week formulation of (2.23), we adopted a lemma from [13], which can be considered as Green's formula.

Lemma 2.7. *Let $X : \overline{\Omega} \to \overline{X(\Omega)}$, $X \in C^2(\overline{\Omega})$ be an invertible vector valued function. Let $F = \nabla X$ and assume that $F^{-1} \in C^1(\overline{\Omega})$. Then*

$$\int_\Omega \operatorname{div} \mathbf{w}(X(x))\psi(x)dx = \int_\Gamma F^{-T} n(x) \cdot \mathbf{w}(X(x))\psi(x)d\mathbf{A}_x - \int_\Omega F^{-1}\mathbf{w}(X(x)) \cdot \nabla\psi(x)dx - \int_\Omega \operatorname{div} F^{-T} \cdot \mathbf{w}(X(x))\psi(x)dx, \tag{2.24}$$

with $\mathbf{w} \in H^1(X(\Omega))$ a vector valued function and $\psi \in H^1(\Omega)$ a scalar function.

Now, we can multiply (2.23) by a test function $\psi \in H^1(\Omega)$, integrate in Ω, and apply the usual Green's formula and (2.24) with $X(x) = X_e(x,t;\tau)$, obtaining

$$\begin{aligned}
&\int_\Omega \phi \frac{dc}{d\tau}(X_e(x,t;\tau),\tau)\psi(x)dx + \int_\Omega F_e^{-1}(x,t;\tau)D\nabla c(X_e(x,t;\tau)) \cdot \nabla\psi(x)dx \\
&\quad + \int_\Omega \operatorname{div} F_e^{-T}(x,t;\tau) \cdot D\nabla c(X_e(x,t;\tau),\tau)\psi(x)dx \\
&= \int_\Omega ((\tilde{c}-c)\tilde{q})(X_e(x,t;\tau),\tau)\psi(x)dx.
\end{aligned} \tag{2.25}$$

2.4. The Combined Approximation Scheme

We now present our sequential time-stepping procedure that combines (2.8) and (2.25). Part J into $0 = t^0 < t^1 < \cdots < t^N = T$, with $\Delta t_c^n = t^n - t^{n-1}$. The analysis is valid for variable time steps, but we drop the superscript from Δt_c for convenience. For functions f on $\Omega \times J$, we write $f^n(x)$ for $f(x,t^n)$. As in [3], let us part J into pressure time steps $0 = t_0 < t_1 < \cdots < t_M = T$, with $\Delta t_p^m = t_m - t_{m-1}$. Each pressure step is also a concentration step, that is, for each m there exists n such that $t_m = t^n$, in general, $\Delta t_p > \Delta t_c$. We may vary Δt_p, but except for Δt_p^1 we drop the superscript. For functions f on $\Omega \times J$, we write $f_m(x)$ for $f(x,t_m)$; thus, subscripts refer to pressure steps and superscripts to concentration steps.

If concentration step t^n relates to pressure steps by $t_{m-1} < t^n \le t_m$, we require a velocity approximation for (2.25) based on U_{m-1} and earlier values. If $m \ge 2$, take the linear extrapolation of U_{m-1} and U_{m-2} defined by

$$EU^n = \left(1 + \frac{t^n - t_{m-1}}{t_{m-1} - t_{m-2}}\right)U_{m-1} - \frac{t^n - t_{m-1}}{t_{m-1} - t_{m-2}}U_{m-2}; \tag{2.26}$$

if $m = 1$, set

$$EU^n = U_0. \tag{2.27}$$

We retain the superscript on Δt_p^1 because EU^n is first-order correct in time during the first pressure step and second-order during later steps.

Let $X : (0,T) \to R^2$ be a solution of the ordinary differential equation

$$\frac{dX}{dt} = \widehat{v}(X,t), \quad \text{where } \widehat{v}(X,t) := \frac{EU(X,t)}{\phi}. \tag{2.28}$$

Then, we can write

$$\phi \frac{d}{dt} c(X,t) = \phi \frac{\partial c}{\partial t} + EU(X,t) \cdot \nabla c. \tag{2.29}$$

Subject to an initial condition $X(t^{n+1}) = x$, we get approximate values of X at t^n by the Euler method and the second order Runga-Kutta method, respectively,

$$\begin{aligned} X_E^n(x) &= x - \Delta t_c \widehat{v}^{n+1}(x), \\ X_{RK}^n(x) &= x - \Delta t_c \widehat{v}^{n+(1/2)}\left(x - \frac{\Delta t_c}{2}\widehat{v}^{n+1}(x)\right). \end{aligned} \tag{2.30}$$

Next, assuming they exist, we denote by F_E^n (resp., by L_E) the gradient of X_E^n (resp., of $\widehat{v}(x)$) with respect to the space variable x, that is, [13, 14]

$$F_E^n(x) := \nabla X_E^n(x) = I(x) - \Delta t_c L_E^{n+1}(x). \tag{2.31}$$

Hypothesis 2.8. For convenience, we assume that (1.1a)–(1.1d) is Ω-periodic ([3]), that is, all functions will be assumed to be spatially Ω-periodic throughout the rest of this paper.

This is physically reasonable, because no-flow conditions (1.1c) are generally treated by reflection, and, in general, interior flow patterns are much more important than boundary effects in reservoir simulation. Thus, the boundary conditions (1.1c) can be dropped.

Lemma 2.9. *Under Hypothesis 2.8, if* $\|\widehat{v}\|_{C^0(W^{1,\infty}(\Omega))}\Delta t_c < 1/2$*, we can see that*

$$X_E^n\left(\overline{\Omega}\right) = X_{RK}^n\left(\overline{\Omega}\right) = \overline{\Omega}. \tag{2.32}$$

Lemma 2.10. *Under Hypothesis 2.8, if* $\|\widehat{v}\|_{C^0(W^{1,\infty}(\Omega))}\Delta t_c < 1/2$*, we have*

$$\left(F_E^n\right)^{-1}(x) = I + \Delta t_c L_E^{n+1}(x) + (\Delta t_c)^2\left(L_E^{n+1}(x)\right)^2 + O\left((\Delta t_c)^3\right). \tag{2.33}$$

Corollary 2.11. *Under the assumptions of Lemma 2.10, for all* $x \in \Omega$*, we have*

$$\begin{aligned} \det\left(F_E^n\right)^{-1}(x) &= 1 + \Delta t_c \operatorname{div} \widehat{v}^{n+1}(x) + O\left((\Delta t_c)^2\right), \\ \left|\det\left(F_E^n\right)^{-1}(x)\right| &\le 1 + \Delta t_c \left\|\widehat{v}^{n+1}(x)\right\|_{C^0(W^{1,\infty}(\Omega))} + O\left((\Delta t_c)^2\right). \end{aligned} \tag{2.34}$$

The time difference for (2.25) will be combined with a standard finite element in the space variables. For $h_c > 0$, we discrete (2.25) in space on a quasi-uniform mesh $\mathcal{T}_{h_c}$ of Ω with diameter of element $\leq h_c$. Let $M_{h_c} \subset H^1(\Omega)$ be a finite element space.

A characteristic discretization of the weak form (2.25) is given by $\{C^n\}_{n=1}^N \in M_{h_c}$ such that

$$\left(\phi\frac{C^{n+1} - C^n \circ X_{RK}^n}{\Delta t_c}, \varphi\right) + \left(D\frac{\nabla C^{n+1} + (\nabla C^n) \circ X_E^n}{2}, \nabla\varphi\right)$$
$$+ \frac{\Delta t_c}{2}\left(D(L^n \nabla C^n) \circ X_E^n, \nabla\varphi\right) + \frac{\Delta t_c}{2}\left((\nabla \operatorname{div} \widehat{v}^n \cdot D\nabla C^n) \circ X_E^n, \varphi\right)$$
$$+ \frac{1}{2}\left(\widetilde{q}^{n+1} C^{n+1} + (\widetilde{q}^n C^n) \circ X_E^n, \varphi\right) = \frac{1}{2}\left(\widetilde{q}^{n+1} \widetilde{c}^{n+1} + (\widetilde{q}^n \widetilde{c}^n) \circ X_E^n, \varphi\right), \quad \forall \varphi \in M_{h_c}, \tag{2.35a}$$

$$C^0 = \widetilde{C}^0, \quad \forall x \in \Omega, \tag{2.35b}$$

where $g \circ X_E^n$ and $g \circ X_{RK}^n$ are compositions

$$(g \circ X_E^n)(x) = g(X_E^n(x)), \qquad (g \circ X_{RK}^n)(x) = g(X_{RK}^n(x)), \tag{2.36}$$

$\widetilde{C}^0$ is an initial approximation of exact solution $c_0(x)$ into M_{h_c}, which will be defined in Section 5.

At each pressure time step t_m, we define

$$U_m = \frac{k}{\mu} \nabla P_m, \tag{2.37}$$

$$C^* = \min(\max(C, 0), 1) \tag{2.38}$$

is the truncation of C to $[0, 1]$. Then at t_m, (2.8) is the following:

$$\left(\frac{k}{\mu} \nabla P_m, \nabla \chi\right) = (q_m, \chi), \quad \forall \chi \in W_{h_p}, \tag{2.39a}$$

$$(P_m, 1) = 0. \tag{2.39b}$$

The steps of calculation are as follows.

Step 1. C^0 known $\to$ solve (U_0, P_0) by (2.37), (2.39a) and (2.39b);

Step 2. by (2.35a) and (2.35b) to solve C^1 $\to$ then by (2.35a) and (2.35b) to solve C^2;

Step 3. analogously, $\{C^{j-1}\}_{j=1}^{n_1}$ known $\to$ $\{C^j\}_{j=1}^{n_1}$ such that $t^{n_1} = t_1$;

Step 4. then by (2.37), (2.39a) and (2.39b) for (U_1, P_1);

Step 5. calculate the approximations in turn analogously to get the pressure, velocity, and concentration at other time step, respectively.

Throughout the analysis, K will denote a generic positive constant, independent of h_c, h_p, Δt_c, and Δt_p, but possibly depending on constants in (C). Similarly, ε will denote a generic small positive constant.

3. Stability for the Concentration Equation

In this section, we derive the stability for the concentration equation. For a given series of functions $\{\varphi\}_{n=0}^{N}$, we define the following norms and seminorm:

$$\begin{gathered}
\|\varphi\|_{l^\infty(L^2)} = \max\{\|\varphi^n\|;\ 0 \le n \le N\}, \\
\|\varphi^n\|_{l^2(L^2)} = \left\{\Delta t_c \sum_{n=0}^{N} \|\varphi^n\|^2\right\}^{1/2}, \\
\|\varphi\|'_{l^2(H^1)} = \left\{\Delta t_c \sum_{n=0}^{N-1} \left\|\nabla\varphi^{n+1} + (\nabla\varphi^n)\circ X_E^n\right\|_D^2\right\}^{1/2}, \\
\|\varphi\|_\phi^2 = (\phi\varphi,\varphi), \qquad \|\varphi\|_D^2 = (D\varphi,\varphi), \qquad \|\varphi\|_D^2 = (q(x,t)\varphi,\varphi).
\end{gathered} \tag{3.1}$$

In the following sections, we use positive constants as

$$c_0 = c_0\left(\|EU\|_{C^0(L^\infty)}\right), \qquad c_1 = c_1\left(\|EU\|_{l^\infty(W^{(1,\infty)})}\right), \qquad c_2 = c_2\left(\|EU\|_{C^0(W^{(2,\infty)})\cap C^1(L^\infty)}\right). \tag{3.2}$$

In our analysis, we need some lemmas.

Lemma 3.1. *Under the definitions* (2.26) *and* (2.30), *for $n = 0,\ldots,N$ and $\varphi \in L^2(\Omega)$, it holds that*

$$\|\varphi\circ X_i^n\|^2 \le (1 + c_1\Delta t_c)\|\varphi\|^2, \quad i = 1,2. \tag{3.3}$$

Proof. We only need to show a proof in the case $i = 1$. Let J_1 be the Jacobian matrix of the transformation $y = X_E^n(x) = x - v^n(x)\Delta t_c$ as

$$J_1 = \begin{pmatrix} 1 - \dfrac{\partial v_1^n(x)}{\partial x_1}\Delta t_c & -\dfrac{\partial v_1^n(x)}{\partial x_2}\Delta t_c \\[2ex] -\dfrac{\partial v_2^n(x)}{\partial x_1}\Delta t_c & 1 - \dfrac{\partial v_2^n(x)}{\partial x_2}\Delta t_c \end{pmatrix}. \tag{3.4}$$

According to the proof of (3.23) in [4], we have

$$\left|\frac{\partial v_j^n(x)}{\partial x_i}\Delta t_c\right| \le K h^{-1}\Delta t_c. \tag{3.5}$$

Since $\Delta t_c = o(h_c)$, for h_c sufficiently small, we see that

$$|\det J_1 - 1| \le c_1 \Delta t_c. \tag{3.6}$$

Following (3.6), we have

$$\left\|\varphi \circ X_E^n\right\|^2 = \int_\Omega \varphi\left(X_E^n(x)\right)^2 dx = \int_\Omega \varphi(y)^2 (\det J_1)^{-1} dy. \tag{3.7}$$

We complete the proof. □

Suppose that $\{C^n\}_{n=0}^N \subset H^1(\Omega)$ and $\tilde{c} \in C^0(L^2)$ be given. We define linear forms $\mathcal{A}_h^{n+1/2}$ and $\mathcal{F}_h^{n+1/2}$ on M_{h_c} for $n = 0, \ldots, N-1$ by

$$\begin{aligned}
\left\langle \mathcal{A}_h^{n+1/2} C, \varphi \right\rangle \equiv{}& \left(\phi \frac{C^{n+1} - C^n \circ X_{RK}^n}{\Delta t_c}, \varphi\right) + \left(D \frac{\nabla C^{n+1} + (\nabla C^n) \circ X_E^n}{2}, \nabla\varphi\right) \\
&+ \frac{\Delta t_c}{2}\left(D(L^n \nabla C^n) \circ X_E^n, \nabla\varphi\right) + \frac{\Delta t_c}{2}\left((\nabla \operatorname{div} \hat{v}^n \cdot D\nabla C^n) \circ X_E^n, \varphi\right) \\
&+ \frac{1}{2}\left(\tilde{q}^{n+1} C^{n+1} + (\tilde{q}^n C^n) \circ X_E^n, \varphi\right), \\
\left\langle \mathcal{F}_h^{n+1/2}, \varphi \right\rangle \equiv{}& \frac{1}{2}\left(\tilde{q}^{n+1} \tilde{c}^{n+1} + (\tilde{q}^n \tilde{c}^n) \circ X_E^n, \varphi\right),
\end{aligned} \tag{3.8}$$

where $\varphi \in M_{h_c}$.

Lemma 3.2. *Let $\{C^n\}_{n=0}^N$ be a solution of* (2.35a) *and* (2.35b). *Then it holds that*

$$\begin{aligned}
\left\langle \mathcal{A}_h^{n+1/2} C, C^{n+1} \right\rangle \ge{}& D_{\Delta t_c}\left(\frac{1}{2}\|C^n\|_\phi^2 + \frac{\Delta t_c}{4}\|\nabla C^n\|_D^2 + \frac{\Delta t_c q_*}{4}\|C^n\|^2\right) \\
&+ \frac{1}{2\Delta t_c}\left\|C^{n+1} - C^n \circ X_{RK}^n\right\|_\phi^2 \\
&+ \frac{1}{4}\left\|\nabla C^{n+1} + \nabla C^n \circ X_E^n\right\|_D^2 + \frac{1}{4q_*}\left\|\tilde{q}^{n+1} C^{n+1} + (\tilde{q}^n C^n) \circ X_E^n\right\|^2 \\
&- \left\{\frac{c_1}{2}\|C^n\|_\phi^2 + \frac{c_1 \Delta t_c}{4}\|\nabla C^n\|_D^2 - q_*\|C^n\|^2 + \frac{1 + c_1 \Delta t_c}{q_*}\|\tilde{q}^n C^n\|^2 \right. \\
&\left. + \frac{c_1 \Delta t_c}{2}\left(\left\|\nabla C^{n+1}\right\|_D^2 + \|\nabla C^n\|_D^2 + \left\|C^{n+1}\right\|_D^2\right)\right\},
\end{aligned} \tag{3.9}$$

where $D_{\Delta t_c}$ is the forward difference operator defined by

$$D_{\Delta t_c} \varphi^n = \frac{(\varphi^{n+1} - \varphi^n)}{\Delta t_c}. \tag{3.10}$$

Proof. Substituting $\varphi = C^{n+1}$ into (3.8), we have

$$\begin{aligned}\left\langle \mathcal{A}_h^{n+1/2} C, C^{n+1} \right\rangle \equiv{}& \left(\phi \frac{C^{n+1} - C^n \circ X_{RK}^n}{\Delta t_c}, C^{n+1}\right) + \left(D \frac{\nabla C^{n+1} + (\nabla C^n) \circ X_E^n}{2}, \nabla C^{n+1}\right) \\ &+ \frac{\Delta t_c}{2}\left(D(L^n \nabla C^n) \circ X_E^n, \nabla C^{n+1}\right) + \frac{\Delta t_c}{2}\left((\nabla \operatorname{div} \widehat{v}^n \cdot D \nabla C^n) \circ X_E^n, C^{n+1}\right) \\ &+ \frac{1}{2}\left(\widetilde{q}^{n+1} C^{n+1} + (\widetilde{q}^n C^n) \circ X_E^n, C^{n+1}\right) \\ ={}& I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{3.11}$$

Lemma 3.1 implies that

$$\begin{aligned} I_1 &= \left(\phi \frac{C^{n+1} - C^n \circ X_{RK}^n}{\Delta t_c}, \frac{C^{n+1} + C^n \circ X_{RK}^n}{2} + \frac{C^{n+1} - C^n \circ X_{RK}^n}{2}\right) \\ &\geq D_{\Delta t_c}\left(\frac{1}{2}\left\|C^{n+1}\right\|_\phi^2\right) - \frac{c_1}{2}\|C^n\|_\phi^2 + \frac{1}{2\Delta t_c}\left\|C^{n+1} - C^n \circ X_{RK}^n\right\|_\phi^2, \end{aligned} \tag{3.12}$$

$$\begin{aligned} I_2 &= \left(D \frac{\nabla C^{n+1} + (\nabla C^n) \circ X_E^n}{2}, \frac{\nabla C^{n+1} + (\nabla C^n) \circ X_E^n}{2} + \frac{\nabla C^{n+1} - (\nabla C^n) \circ X_E^n}{2}\right) \\ &\geq D_{\Delta t_c}\left(\frac{\Delta t_c}{4}\|\nabla C^n\|_D^2\right) - \frac{c_1 \Delta t_c}{4}\|\nabla C^n\|_D^2 + \frac{1}{4}\left\|\nabla C^{n+1} + (\nabla C^n) \circ X_E^n\right\|_D^2. \end{aligned} \tag{3.13}$$

Next, by using $c_1 \Delta t_c < 1$, we obtain

$$\begin{aligned} I_3 &= \frac{\Delta t_c}{2}\left(D(L^n \nabla C^n) \circ X_E^n, \nabla C^{n+1}\right) \\ &\leq \frac{\Delta t_c}{2}\sqrt{1 + c_1 \Delta t_c}\|L^n \nabla C^n\|_D \left\|\nabla C^{n+1}\right\|_D \\ &\leq \frac{c_1 \Delta t_c}{2}\left\{\|\nabla C^n\|_D^2 + \left\|\nabla C^{n+1}\right\|_D^2\right\}. \end{aligned} \tag{3.14}$$

Then when $I_3 \geq 0$ and $I_3 < 0$, we have

$$I_3 \geq -\frac{c_1 \Delta t_c}{2}\left\{\|\nabla C^n\|_D^2 + \left\|\nabla C^{n+1}\right\|_D^2\right\}. \tag{3.15}$$

Similarly, for I_4, we obtain the estimate

$$I_4 \geq -\frac{c_1 \Delta t_c}{2}\left\{\|\nabla C^n\|_D^2 + \left\|C^{n+1}\right\|_D^2\right\}. \tag{3.16}$$

Analogous computations to term I_2 give

$$\begin{aligned}
I_5 &= \left(\frac{\tilde{q}^{n+1}C^{n+1} + (\tilde{q}^n C^n)\circ X_1^n}{2}, \frac{\tilde{q}^{n+1}C^{n+1} + (\tilde{q}^n C^n)\circ X_1^n}{2\tilde{q}^{n+1}} + \frac{\tilde{q}^{n+1}C^{n+1} - (\tilde{q}^n C^n)\circ X_1^n}{2\tilde{q}^{n+1}} \right) \\
&\geq \frac{1}{4}\left(q_* \left\| C^{n+1} \right\|^2 - \frac{1}{q_*}\left\| (\tilde{q}^n C^n)\circ X_1^n \right\|^2 \right) + \frac{1}{4q_*}\left\| \tilde{q}^{n+1}C^{n+1} + (\tilde{q}^n C^n)\circ X_1^n \right\|^2 \\
&\geq D_{\Delta t_c}\left(\frac{\Delta t_c q_*}{4}\|C^n\|^2 \right) + \frac{q_*}{4}\|C^n\|^2 - \frac{1 + c_1\Delta t_c}{q_*}\left\|\tilde{q}^n C^n\right\|^2 + \frac{1}{4q_*}\left\| \tilde{q}^{n+1}C^{n+1} + (\tilde{q}^n C^n)\circ X_E^n \right\|^2 \\
&\geq D_{\Delta t_c}\left(\frac{\Delta t_c q_*}{4}\|C^n\|^2 \right) + \frac{q_*}{4}\|C^n\|^2 - \frac{2}{q_*}\left\|\tilde{q}^n C^n\right\|^2 + \frac{1}{4q_*}\left\| \tilde{q}^{n+1}C^{n+1} + (\tilde{q}^n C^n)\circ X_E^n \right\|^2,
\end{aligned} \tag{3.17}$$

which completes the proof. □

From Lemma 3.1, we have

$$\begin{aligned}
\left\langle \mathcal{F}_h^{n+1/2}, C^{n+1} \right\rangle &= \frac{1}{2}\left(\tilde{q}^{n+1}\tilde{c}^{n+1} + (\tilde{q}^n \tilde{c}^n)\circ X_E^n, C^{n+1} \right) \\
&\leq \frac{1}{2\phi}\left\| C^{n+1} \right\|_\phi^2 + \frac{1}{2}\left\{ \left\| \tilde{q}^{n+1}\tilde{c}^{n+1} \right\|^2 + (1 + c_1\Delta t_c)\left\| \tilde{q}^n \tilde{c}^n \right\|^2 \right\}.
\end{aligned} \tag{3.18}$$

Combining (3.18) with (3.8), we get

$$\begin{aligned}
D_{\Delta t_c}&\left(\frac{1}{2}\|C^n\|_\phi^2 + \frac{\Delta t_c}{4}\|\nabla C^n\|_D^2 + \frac{\Delta t_c q_*}{4}\|C^n\|^2 \right) + \frac{1}{2\Delta t_c}\left\| C^{n+1} - C^n\circ X_{RK}^n \right\|_\phi^2 \\
&+ \frac{1}{4}\left\| \nabla C^{n+1} + \nabla C^n \circ X_E^n \right\|_D^2 + \frac{1}{4q_*}\left\| \tilde{q}^{n+1}C^{n+1} + (\tilde{q}^n C^n)\circ X_E^n \right\|^2 + q_*\|C^n\|^2 \\
&\leq \frac{c_1}{2}\|C^n\|_\phi^2 + \frac{c_1\Delta t_c}{4}\|\nabla C^n\|_D^2 + \frac{2}{q_*}\left\|\tilde{q}^n C^n\right\|^2 \\
&+ \frac{c_1\Delta t_c}{2}\left(\left\|\nabla C^{n+1}\right\|_D^2 + \|\nabla C^n\|_D^2 + \left\| C^{n+1} \right\|_D^2 \right) + \frac{1}{2\phi}\left\| C^{n+1} \right\|_\phi^2 \\
&+ \frac{1}{2}\left\{ \left\| \tilde{q}^{n+1}\tilde{c}^{n+1} \right\|^2 + (1 + c_1\Delta t_c)\left\| \tilde{q}^n \tilde{c}^n \right\|^2 \right\},
\end{aligned} \tag{3.19}$$

which completes the proof of the stability by virtue of Gronwall's inequality.

Theorem 3.3 (stability). *Let $\{C^n\}_{n=0}^N$ be the solution of* (2.35a) *and* (2.35b) *subject to the initial value C^0. Then there exists a positive constant $c_1 = c_1(\|EU\|_{l^\infty(W^{1,\infty})})$ independent of h_c and Δt_c such that*

$$\begin{aligned}&\|C\|_{l^\infty(L^2)} + \sqrt{\Delta t_c}\|\nabla C\|_{l^\infty(L^2)} + \sqrt{\Delta t_c}\|C\|_{l^\infty(L^2)} + \|C\|'_{l^2(H^1)}\\ &\quad\le c_1\left\{\left\|C^0\right\| + \sqrt{\Delta t_c}\left\|\nabla C^0\right\|^2 + \|\tilde{q}\tilde{c}\|_{l^2(L^2)}\right\}.\end{aligned} \tag{3.20}$$

4. Error Estimate Theorem

Now, we turn to derive an optimal priori error estimate in L^2-norm for the concentration of approximation (2.35a) and (2.35b). In order to state error estimates, we need the following Lagrange interpolation operator [15] $\Pi_h : C^0(\overline{\Omega}) \to M_{h_c}$.

Lemma 4.1. *There exists a positive integer k such that*

$$\|\Pi_h c - c\|_s \le Kh^{k+1-s}\|c\|_{k+1}, \quad s = 0, 1,\ \forall c \in H^{k+1}(\Omega) \cap C^0\left(\overline{\Omega}\right). \tag{4.1}$$

Let $e^n = C^n - \Pi_h c^n$ and $\eta^n = c^n - \Pi_h c^n$. By Lemma 4.1, it holds that

$$\|\eta^n\|_1 \le Kh^k\|c^n\|_{k+1}, \qquad \|D_{\Delta t_c}\eta^n\| \le \frac{Kh^k}{\sqrt{\Delta t_c}}\left\|\frac{\partial c}{\partial t}\right\|_{L^2((t^n,t^{n+1});H^k)}. \tag{4.2}$$

Let $c \in C^1(L^2) \cap C^0(H^2)$, $u \in C^0(L^\infty)$, and $\tilde{q} \in C^0(L^2)$ be given. Corresponding to $\mathcal{A}_h^{n+1/2}$ and $\mathcal{F}_h^{n+1/2}$, we introduce linear forms $\mathcal{A}^{n+1/2}$ and $\mathcal{F}^{n+1/2}$ on $(H^1(\Omega))'$ for $n = 0, \ldots, N-1$ by (2.25) as follows:

$$\begin{aligned}\left\langle \mathcal{A}^{n+1/2}c, \varphi\right\rangle \equiv{}& \left(\phi\left(\frac{dc}{dt}\right)^{n+(1/2)} \circ X_e^{n+(1/2)}, \varphi\right) + \left(\left(F_e^{n+(1/2)}\right)^{-1}\left(D\nabla c^{n+(1/2)}\right) \circ X_e^{n+(1/2)}, \nabla\varphi\right)\\ &+ \left(\operatorname{div}\left(F_e^{n+(1/2)}\right)^{-T}\left(D\nabla c^{n+(1/2)}\right) \circ X_e^{n+(1/2)}, \varphi\right)\\ &+ \left(\left(\tilde{q}^{n+(1/2)}c^{n+(1/2)}\right) \circ X_e^{n+(1/2)}, \varphi\right),\\ \left\langle \mathcal{F}^{n+1/2}, \varphi\right\rangle \equiv{}& \left(\left(\tilde{q}^{n+(1/2)}\tilde{c}^{n+(1/2)}\right) \circ X_e^{n+(1/2)}, \varphi\right), \quad \forall\varphi \in H^1(\Omega).\end{aligned} \tag{4.3}$$

If c is the solution of (1.1a)–(1.1d), we have for $n = 0, \ldots, N-1$

$$\left(\mathcal{A}^{n+(1/2)}c, \varphi\right) = \left(\mathcal{F}^{n+(1/2)}, \varphi\right), \quad \forall\varphi \in H^1(\Omega). \tag{4.4}$$

We decompose e as

$$\mathcal{A}_h^{n+(1/2)}e = \left(\mathcal{F}_h^{n+(1/2)} - \mathcal{F}^{n+(1/2)}\right) + \left(\mathcal{A}^{n+(1/2)} - \mathcal{A}_h^{n+(1/2)}\right)c + \mathcal{A}_h^{n+(1/2)}\eta. \tag{4.5}$$

In order to estimate the terms of the right-hand side of (4.5), we need the following lemmas.

Lemma 4.2 (see [13]). *Assume the above Hypotheses hold, and that the coefficients of the problem* (1.1a)–(1.1d) *satisfy* $v \in C^0(W^{3,\infty}(\Omega)) \cap C^1(W^{2,\infty}(\Omega))$, $q \in W^{2,\infty}(\Omega)$, *and* $\|\widehat{v}\|_{C^0(W^{1,\infty}(\Omega))}\Delta t_c < 1/2$. *Let the solution of* (4.4) *satisfy* $c \in Z^3$, $\nabla c \in Z^3$. *Then, for each* $n = 0, 1, \ldots, N-1$, *there exists function* $\xi_A^{n+(1/2)} : \Omega \to R$, *such that*

$$\left\langle \left(\mathcal{A}^{n+(1/2)} - \mathcal{A}_h^{n+(1/2)}\right)c, \varphi \right\rangle = \left(\xi_A^{n+(1/2)}, \varphi\right), \quad \varphi \in H^1(\Omega). \tag{4.6}$$

Moreover, $\xi_A^{n+(1/2)} \in L^2(\Omega)$ *and the following estimate holds:*

$$\left\|\xi_A^{n+(1/2)}\right\|_0 \le \widetilde{c}_1(\Delta t_c)^2\left(\|c\|_{Z^3} + \|\nabla c\|_{Z^3}\right), \tag{4.7}$$

where $\widetilde{c}$ *denotes a constant independent of* Δt_c.

Lemma 4.3 (see [13]). *Assume the above Hypotheses hold, and that the coefficients of the problem* (1.1a)–(1.1d) *satisfy* $v \in C^0(W^{2,\infty}(\Omega)) \cap C^1(W^{1,\infty}(\Omega))$, $q \in W^{2,\infty}(\Omega)$, *and* $\|\widehat{v}\|_{C^0(W^{1,\infty}(\Omega))}\Delta t_c < 1/2$. *Let the solution of* (4.4) *satisfy* $c \in Z^3$, $\nabla c \in Z^3$. *Then, for each* $n = 0, 1, \ldots, N-1$, *there exists function* $\xi_f^{n+(1/2)} : \Omega \to R$, *such that*

$$\left\langle \left(F^{n+(1/2)} - \mathcal{F}_h^{n+(1/2)}, \varphi \right\rangle = \left(\xi_f^{n+(1/2)}, \varphi\right), \quad \varphi \in H^1(\Omega). \tag{4.8}$$

Moreover, $\xi_f^{n+(1/2)} \in L^2(\Omega)$ *and the following estimate holds:*

$$\left\|\xi_f^{n+(1/2)}\right\|_0 \le \widetilde{c}_1(\Delta t_c)^2\left\|\widetilde{q}\widetilde{c}\right\|_{Z^2}, \tag{4.9}$$

where $\widetilde{c}_1$ *denotes a constant independent of* Δt_c.

Lemma 4.4. *Let* $\eta = c - \Pi_h c$. *There exists*

$$\begin{aligned}\left\langle \mathcal{A}_h^{n+1/2}\eta, e^{n+1}\right\rangle \le{}& \frac{1}{8}\left\|\nabla e^{n+1} + \nabla e^n \circ X_E^n\right\|^2 \\ &+ \frac{1}{2}\left\{\left(D\nabla\eta^{n+1}, \nabla e^{n+1}\right) - \left(D\nabla\eta^n, \nabla e^n\right)\right\} \\ &+ c_1\left\{h^{2k}\left(\frac{1}{\Delta t_c}\left\|\frac{\partial c}{\partial t}\right\|^2_{L^2\left(t^n, t^{n+1}; H^k\right)} + \|c\|^2_{C^0\left(H^{k+1}\right)}\right)\right. \\ &\left. + h^{2k+2}\|c\|^2_{C^0\left(H^{k+1}\right)} + \Delta t_c\left(\|\nabla e^n\|^2 + \left\|\nabla e^{n+1}\right\|^2\right)\right\}.\end{aligned} \tag{4.10}$$

Proof. From the definition of $\mathcal{A}_h^{n+1/2}$, we have

$$\begin{aligned}\left\langle \mathcal{A}_h^{n+1/2}\eta, e^{n+1}\right\rangle \equiv{}& \left(\phi\frac{\eta^{n+1} - \eta^n \circ X_{RK}^n}{\Delta t_c}, e^{n+1}\right) + \left(D\frac{\nabla\eta^{n+1} + (\nabla\eta^n) \circ X_E^n}{2}, \nabla e^{n+1}\right) \\ &+ \frac{\Delta t_c}{2}\left(D\left(L^n\nabla\eta^n\right) \circ X_E^n, \nabla e^{n+1}\right) + \frac{\Delta t_c}{2}\left(\left(\nabla \operatorname{div} v^n \cdot D\nabla\eta^n\right) \circ X_E^n, e^{n+1}\right) \\ &+ \frac{1}{2}\left(\tilde{q}^{n+1}\eta^{n+1} + \left(\tilde{q}^n\eta^n\right) \circ X_E^n, e^{n+1}\right) \\ ={}& I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{4.11}$$

Since it holds that

$$\left\|\phi\frac{\eta^{n+1} - \eta^n \circ X_{RK}^n}{\Delta t_c}\right\| \le \frac{c_1}{\sqrt{\Delta t_c}}\left\{\left\|\frac{\partial\eta}{\partial t}\right\|_{L^2(t^n, t^{n+1}; L^2)} + \|\eta\|_{L^2(t^n, t^{n+1}; H^1)}\right\}, \tag{4.12}$$

we have

$$I_1 \le \frac{(1 + c_1\Delta t_c)(1 + 2c_1)^2 K^2 h^{2k}}{2\Delta t_c}\left(\left\|\frac{\partial c}{\partial t}\right\|^2_{L^2\left(t^n, t^{n+1}; H^k\right)} + \|c\|^2_{L^2\left(t^n, t^{n+1}; H^{k+1}\right)}\right) + \frac{1}{2}\left\|e^{n+1}\right\|^2. \tag{4.13}$$

To estimate I_2, we divide it into three parts as

$$I_2 = \frac{1}{2}\left[\left(D\nabla\eta^{n+1}, \nabla e^{n+1}\right) - \left(D\nabla\eta^n, \nabla e^n\right)\right] + I_{21} + I_{22}, \tag{4.14}$$

where

$$I_{21} = \frac{1}{2}\left[\left((D\nabla\eta^n)\circ X_E^n, \nabla e^{n+1} + \nabla e^n \circ X_E^n\right)\right],$$
$$I_{22} = \frac{1}{2}\left[(D\nabla\eta^n, \nabla e^n) - ((D\nabla\eta^n)\circ X_E^n, \nabla e^n \circ X_E^n)\right]. \tag{4.15}$$

The term I_{21} is estimated as

$$I_{21} \le \frac{1}{8}\left\|\nabla e^{n+1} + \nabla e^n \circ X_E^n\right\|^2 + 2\left\|(D\nabla\eta^n)\circ X_E^n\right\|^2$$
$$\le \frac{1}{8}\left\|\nabla e^{n+1} + \nabla e^n \circ X_E^n\right\|^2 + (1 + c_1\Delta t_c)K(d^*)^2 h^{2k}\|c^n\|_{H^{k+1}}^2. \tag{4.16}$$

Using the transformation $y = X_E^n(x)$, we have

$$I_{22} = \frac{1}{2}\int_\Omega D\nabla\eta^n \cdot \nabla e^n \left\{1 - \det\left(\frac{\partial y}{\partial x}\right)^{-1} dy\right\} dx$$
$$\le \frac{c_1(d^*)^2\Delta t_c}{4}\left(K^2h^{2k}\|c^n\|_{H^{k+1}}^2 + \|\nabla e^n\|^2\right). \tag{4.17}$$

It is clear that

$$I_3 \le \frac{(1 + c_1\Delta t_c)c_1^2(d^*)^2\Delta t_c K^2 h^{2k}}{4}\|c^n\|_{H^{k+1}}^2 + \frac{1}{4}\left\|\nabla e^{n+1}\right\|^2, \tag{4.18}$$

$$I_4 \le \frac{(1 + c_1\Delta t_c)c_1^2(d^*)^2\Delta t_c K^2 h^{2k}}{4}\|c^n\|_{H^{k+1}}^2 + \frac{1}{4}\left\|\nabla e^{n+1}\right\|^2. \tag{4.19}$$

Term I_5 can be divided it into three parts like term I_2 as

$$I_5 = I_{51} + I_{52} + I_{53}, \tag{4.20}$$

where

$$I_{51} = \frac{1}{2}\left[\left(\sqrt{\tilde{q}^{n+1}}\eta^{n+1}, \sqrt{\tilde{q}^{n+1}}e^{n+1}\right) - \left(\sqrt{\tilde{q}^n}\eta^n, \sqrt{\tilde{q}^n}e^n\right)\right],$$
$$I_{52} = \frac{1}{2}\left[\left(\left(\sqrt{\tilde{q}^n}\eta^n\right)\circ X_E^n, \sqrt{\tilde{q}^n}\circ X_E^n\left(e^{n+1} + e^n \circ X_E^n\right)\right)\right], \tag{4.21}$$
$$I_{53} = \frac{1}{2}\left[\left(\sqrt{\tilde{q}^n}\eta^n, \sqrt{\tilde{q}^n}e^n\right) - \left(\left(\sqrt{\tilde{q}^n}\eta^n\right)\circ X_E^n, \left(\sqrt{\tilde{q}^n}e^n\right)\circ X_E^n\right)\right].$$

Thus, the same kind of computations used for I_2 leads to the following estimate:

$$
\begin{aligned}
I_5 \leq I_{51} &+ \left(\frac{c_1 c_3 \Delta t_c}{4} + (1 + c_1 \Delta t_c) c_3\right) K^2 h^{2k} \|c^n\|^2_{H^{k+1}} \\
&+ \frac{(2c_1 + c_1^2 \Delta t_c) c_3 \Delta t_c}{8} \left(\left\| \sqrt{\tilde{q}^n} \eta^n \right\|^2 + \left\| \sqrt{\tilde{q}^{n+1}} \eta^{n+1} \right\|^2 \right) \\
&+ \frac{1}{8} \left\| \sqrt{\tilde{q}^{n+1}} \eta^{n+1} + \left(\sqrt{\tilde{q}^n} \eta^n \right) \circ X_E^n \right\|^2 .
\end{aligned} \tag{4.22}
$$

Gathering (4.11)–(4.19) together, we complete the proof. □

We now turn to estimate e^n. From the definition of $\mathcal{A}_h^{n+(1/2)}$ and $\mathcal{A}^{n+(1/2)}$, we see

$$
\begin{aligned}
\left\langle \mathcal{A}_h^{n+(1/2)} e, e^{n+1} \right\rangle = \left\langle \mathcal{A}_h^{n+(1/2)} \eta, e^{n+1} \right\rangle + \left\langle \left(\mathcal{A}^{n+(1/2)} - \mathcal{A}_h^{n+(1/2)} \right) c, e^{n+1} \right\rangle \\
+ \left\langle \left(\mathcal{F}_h^{n+(1/2)} - \mathcal{F}^{n+(1/2)} \right), e^{n+1} \right\rangle .
\end{aligned} \tag{4.23}
$$

From Lemmas 4.2, 4.3 and 4.4, we obtain

$$
\begin{aligned}
D_{\Delta t_c} &\left(\frac{1}{2} \left\| e^{n+1} \right\|_\phi^2 + \frac{\Delta t_c}{4} \|\nabla e^n\|_D^2 + \frac{\Delta t_c}{4} \|e^n\|_q^2 \right) + \frac{1}{2\Delta t_c} \left\| e^{n+1} - e^n \circ X_{RK}^n \right\|^2 \\
&+ \frac{1}{8} \left\| \nabla e^{n+1} + \nabla e^n \circ X_E^n \right\|_D^2 + \frac{1}{4} \left\| \tilde{q}^{n+1} e^{n+1} + \tilde{q}^n e^n \circ X_E^n \right\|^2 \\
&\leq c_1 \left\{ \left\| e^{n+1} \right\|^2 + \Delta t_c \left\| \nabla e^{n+1} \right\|_D^2 + \|e^n\|^2 + \Delta t_c \|\nabla e^n\|_D^2 + \Delta t_c \|e^n\|_q^2 \right\} \\
&+ \frac{1}{2} \left\{ \left(D \nabla \eta^{n+1}, \nabla e^{n+1} \right) - \left(D \nabla \eta^n, \nabla e^n \right) \right\} \\
&+ c_2 \left\{ h^{2k} \left(\frac{1}{\Delta t_c} \left\| \frac{\partial c}{\partial t} \right\|^2_{L^2\left(t^n, t^{n+1}; H^k\right)} + \|c\|^2_{C^0\left(H^{k+1}\right)} \right) + h^{2k+2} \|c\|^2_{C^0\left(H^{k+1}\right)} \right. \\
&\qquad \left. + (\Delta t_c)^4 \left(|||\nabla c|||_3^2 + |||c|||_3^2 \right) \right\} .
\end{aligned} \tag{4.24}
$$

Summing up the above equation about time from $t = 0$ to t^n, we get

$$a_{n+1} + \Delta t_c \sum_{j=0}^{n} b_j \leq c_1 \Delta t_c \sum_{j=0}^{n} a_j + \frac{\Delta t_c}{2} \left\{ \left(D\nabla \eta^{n+1}, \nabla e^{n+1} \right) - \left(D\nabla \eta^0, \nabla e^0 \right) \right\}$$
$$+ c_2 \left\{ h^{2k} \left(\frac{1}{\Delta t_c} \left\| \frac{\partial c}{\partial t} \right\|^2_{L^2(t^n, t^{n+1}; H^k)} + \|c\|^2_{C^0(H^{k+1})} \right) + h^{2k+2} \|c\|^2_{C^0(H^{k+1})} \right. \quad (4.25)$$
$$\left. + (\Delta t_c)^4 \left(|||\nabla c|||_3^2 + |||c|||_3^2 \right) \right\},$$

where

$$a_j = \frac{1}{2} \left\| e^j \right\|^2_{\phi} + \frac{\Delta t_c}{4} \left\| \nabla e^j \right\|^2_D + \frac{\Delta t_c}{4} \left\| e^j \right\|^2_q,$$
$$b_j = \frac{1}{2\Delta t_c} \left\| e^{j+1} - e^j \circ X^j_{RK} \right\|^2 + \frac{1}{8} \left\| \nabla e^{j+1} + \nabla e^j \circ X^j_E \right\|^2_D + \frac{1}{4} \left\| \tilde{q}^{j+1} e^{j+1} + \tilde{q}^j e^j \circ X^j_E \right\|^2. \quad (4.26)$$

Using the estimates

$$\frac{\Delta t_c}{2} \left(D\nabla \eta^j, \nabla e^j \right) \leq \frac{\Delta t_c}{8} \left\| \nabla e^j \right\|^2 + c_1 \Delta t_c h^{2k} \left\| c^j \right\|^2_{k+1} \quad (4.27)$$

for $j = 0$ and $n + 1$, and Gronwall's inequality, we derive the following error estimate.

Theorem 4.5 (error estimate). *Let $\{C^n\}_{n=0}^N$ be the solution of* (2.35a) *and* (2.35b) *subject to the initial value C^0. Then there exists a positive constant $c_1 = c_1(\|EU\|_{C^0(W^{1,\infty})})$ independent of h_c and Δt_c such that*

$$\|c - C\|_{l^\infty(L^2)} + \sqrt{\Delta t_c} \|\nabla (c - C)\|_{l^\infty(L^2)} + \sqrt{\Delta t_c} \|c - C\|_{l^\infty(L^2)} + \|c - C\|'_{l^2(H^1)}$$
$$\leq c_2 \left\{ h^k \left(\left\| \frac{\partial c}{\partial t} \right\|_{L^2(h^k)} + \|c\|_{C^0(H^{k+1})} \right) + (\Delta t_c)^2 \left(|||\Delta c|||_3^2 + |||c|||_3 \right) \right\}. \quad (4.28)$$

5. Conclusions

We have presented an approximation scheme for incompressible miscible displacement in porous media. This scheme was constructed by two methods. This paper is our sequential research work. Cubic Hermite finite element method for the pressure equation was used to ensure the higher regularity of the approximation velocity U. A second order characteristic finite element method was presented to handle the material derivative term of the concentration equation. We analyzed the stability of the approximation scheme and derived the optimal-order L^2-norm error estimates for the scalar concentration. They are of second order accuracy in time increment, symmetric, and unconditionally stable.

In the paper, the matrix F_e of the gradient of the characteristic line X_e and the inverse matrix F_e^{-1} were used to approximate the diffusion term. The properties of these matrices were very important and derived by the complicated theoretical analysis. This paper is the first step of our sequential research work. At current stage, we consider a simple case only for the theoretical aim, which was not related with actual petroleum applications. So, we consider the diffusion coefficient independent of the velocity u only in this paper. We will consider the more actual model in petroleum applications later.

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