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Research Article

New Laguerre Filter Approximators to the Grünwald-Letnikov Fractional Difference

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This paper presents a series of new results in modeling of the Grünwald-Letnikov discrete-time fractional difference by means of discrete-time Laguerre filers. The introduced Laguerre-based difference (LD) and combined fractional/Laguerre-based difference (CFLD) are shown to perfectly approximate its fractional difference original, for fractional order $\alpha \in (0, 2)$. This paper is culminated with the presentation of finite (combined) fractional/Laguerre-based difference (FFLD), whose excellent approximation performance is illustrated in simulation examples.

1. Introduction

Noninteger or fractional-order dynamic models have recently attracted a considerable research interest. Their specific properties can make them more adequate in modeling of selected industrial systems [1–4]. Our interest is in discrete-time representations of fractional-order systems, so we proceed with the Grünwald-Letnikov fractional-order difference (FD) [5–9]. An infinite-memory filter incorporated in FD may lead to a computational explosion. Therefore, a number of discrete-time FD-based systems have been modeled both via transfer function or difference equation models [10–13] and state space ones [7, 9, 14].

Various approximations to the fractional difference have been pursued. Since FD represents in fact (a sort of) an infinite impulse response (IIR) filter, one solution has been to least-squares (LS) fit an impulse/step response of a discrete-time integer-difference IIR filter to that of the associated FD [15–17]. These methods give digital rational approximations (IIR filters) to continuous fractional-order integrators and differentiators. The problem here is to propose a good structure of the integer-difference filter, possibly involving a low number of parameters. On the other hand, an LS fit of the FIR filter to FD has been analyzed in the frequency domain [18], with a high-order optimal filter providing a good approximation accuracy, at the cost of a remarkable computational effort, however.

Another approach relies on the approximation of the FD filter with its truncated, finite-memory version [14, 19, 20]. In analogy to finite impulse response (FIR) the term finite FD, or FFD, has been coined [21]. Additionally, a series of results in finite and infinite-memory modeling of a discrete-time FD by FFD-like models has been presented in [22].

An approach behind that research direction has been the employment of an approximating filter incorporating orthonormal basis functions (OBF) [21, 23]. Another attempt at the application of OBF in modeling of FD has been presented in [24]. This paper provides a nice theoretical background for those rather intuitive approaches to the OBF-based approximation of FD, in that the so-called Laguerre-based difference (LD) is shown to be equivalent, in some sense, to FD.

The proposed approximation method is solved for the model parameters in an analytical way. The paper is culminated with the introduction of a new model of FD, being an effective combination of FFD and finite LD (or FLD), whose excellent performance results from expert *a priori* knowledge used when constructing the model.

Having introduced the FD modeling problem, the Grünwald-Letnikov discrete-time fractional difference is recalled, together with its FFD approximation, in Section 2. Section 3 presents the OBF modeling problem, in particular via discrete-time Laguerre filters. Laguerre-based difference (LD) is covered in Section 4, followed by a Laguerre-based approximation to FD in terms of finite LD (FLD). Finally, Section 4 provides tools for selection of optimal Laguerre pole for FLD approximation and presents a series of simulations, which show the approximation efficiency of FLD modeling. Finally, combined fractional/Laguerre-based difference (CFLD) and its finite approximation called finite (combined) FLD (or FFLD) have been introduced in Section 5. That section also presents a method for selection of optimal Laguerre pole for FFLD and includes a series of simulation examples which present a high approximation accuracy of FFLD modeling. Conclusions of Section 6 summarize the achievements of the paper.

2. Fractional Discrete-Time Difference

In our considerations, we use a simple generalization of the familiar Grünwald-Letnikov difference [25], that is the fractional difference (FD) in discrete time t, described by the following equation [7–9]:

$$\Delta^{\alpha} x(t) = \sum_{j=0}^{t} P_j(\alpha) x(t) q^{-j} = x(t) + \sum_{j=1}^{t} P_j(\alpha) x(t) q^{-j} \quad t = 0, 1, \dots, \tag{2.1}$$

where the fractional order $\alpha \in (0, 2)$, q^{-1} is the backward shift operator and

$$P_j(\alpha) = (-1)^j C_j(\alpha) \tag{2.2}$$

with

$$C_j(\alpha) = \binom{\alpha}{j} = \begin{cases} 1 & j = 0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & j > 0. \end{cases} \tag{2.3}$$

Note that each element in (2.1) from time t back to 0 is nonzero so that each incoming sample of the signal $x(t)$ increases the complication of the model equation. In the limit, with $t \to +\infty$, we have an infinite number of FD components leading to computational explosion.

Remark 2.1. Possible accounting for the sampling period T when transferring from a continuous-time derivative to the discrete-time difference results in dividing the right-hand side of (2.1) by T^α [19]. Operating without T^α as in the sequel corresponds to putting $T = 1$ or to the substitution of $P_j(\alpha)$ for $P_j\ (\alpha)/T^\alpha$, $j = 0, \ldots, t$.

2.1. Finite Fractional Difference

In [21], truncated or finite fractional difference (FFD) has (in analogy to FIR) been considered for practical, feasibility reasons, with the convergence to zero of the series $C_j(\alpha)$ enabling to assume $C_j(\alpha) \approx 0$ for some $j > \overline{J}$, where $\overline{J}$ is the number of backward signal samples used to calculate the fractional difference. We will further proceed with FFD, to be formally defined below.

Definition 2.2 (see [22]). Let the fractional difference (FD) be defined as in (2.1) to (2.3). Then the finite fractional difference (FFD) is defined as

$$\Delta^\alpha x(t, J) = x(t) + \sum_{j=1}^{J} P_j(\alpha) x(t) q^{-j}, \tag{2.4}$$

where $J = \min(t, \overline{J})$, and $\overline{J}$ is the upper bound for j when $t > \overline{J}$.

The FFD model has been analyzed in some papers under the heading of a practical implementation of FD [7, 26], a finite difference [14, 19], or a short-memory difference [20].

Remark 2.3. It is well known [22] that, equivalently to (2.1), FD can be rewritten as the limiting FFD (for $\overline{J} \to \infty$) in the form

$$\begin{aligned} \Delta^\alpha x(t) &= x(t) + \sum_{j=1}^{\infty} P_j(\alpha) x(t-j) \\ &= x(t) + X_{\mathrm{FD}}(t) \quad t = 0, 1, \ldots \end{aligned} \tag{2.5}$$

with $x(l) = 0$ for all $l < 0$.

FFD is known to suffer from the steady-state modeling error with respect to FD [22, 27], so special means have been designed to provide steady-state error-free modeling [22, 27].

3. Orthonormal Basis Functions

It is well known that an open-loop stable linear discrete-time IIR system governed by the transfer function:

$$G(z) = \sum_{j=1}^{\infty} g_j z^{-j} \tag{3.1}$$

with the impulse response $g_j = g(j)$, $j = 1, 2, \ldots$, can be described in the Laurent expansion form [28, 29]:

$$G(z) = \sum_{j=1}^{\infty} c_j L_j(z), \tag{3.2}$$

including a series of orthonormal basis functions (OBF) $L_j(z)$ and the weighting parameters c_j, $j = 1, 2, \ldots$, characterizing the model dynamics.

Various OBFs can be used in (3.2). Two commonly used sets of OBF are simple Laguerre and Kautz functions. These functions are characterized by the “dominant” dynamics of a system, which is given by a single real pole (p) or a pair of complex ones (p, p^*), respectively. In case of discrete-time Laguerre filters to be exploited hereinafter, the orthonormal functions

$$L_j(z) = L_j(z, p) = \frac{k}{z - p} \left[\frac{1 - pz}{z - p}\right]^{j-1} \quad j = 1, 2, \ldots \tag{3.3}$$

with $k = \sqrt{1 - p^2}$ and $p \in (-1, 1)$, consist of a first-order low-pass factor and $(j - 1)$th-order all-pass filters.

Remark 3.1. Depending on the domain context, we will use various arguments in $L_j(\cdot)$, for example, $L_j(z)$ in the z-domain and $L_j(q)$ or $L_j(q^{-1})$ in the time domain. The same concerns the arguments in $G(\cdot)$.

The coefficients c_j, $j = 1, 2, \ldots$, can be calculated form the scalar product of $G(z)$ and $L_j(z)$ [28] as follows:

$$c_j = \langle G(z), L_j(z) \rangle = \frac{1}{2\pi i} \oint_{\gamma} G^*(z) L_j(z) \frac{dz}{z}, \tag{3.4}$$

where $G^*(z)$ is the complex conjugate of $G(z)$ and γ is the unit circle. Note that $G(z)$ and $L_j(z)$, $j = 1, 2 \ldots$, must be analytic in γ. It is also possible to calculate the scalar product in the time domain

$$c_j = \langle g(t), l_j(t) \rangle = \sum_{t=1}^{\infty} g(t) l_j(t) \tag{3.5}$$

with $g(t) = G(q^{-1})\delta(t)$, $l_j(t) = L_j(q^{-1})\delta(t)$, $t = 1, 2, \ldots$, and $\delta(t)$ is the Kronecker delta.

4. Laguerre-Based Difference

In analogy to FD, let us firstly define a "sort of" a difference to be referred to as the Laguerre-based difference.

Definition 4.1. Let c_j and $L_j(z)$, $j = 1, 2, \ldots$, be described as in (3.2) through (3.4). Then the Laguerre-based difference (LD) is defined as

$$\begin{aligned} \Delta_{\mathrm{LD}}^{\alpha} x(t) &= x(t) + \sum_{j=1}^{\infty} c_j L_j\left(q^{-1}\right) x(t) \\ &= x(t) + X_{\mathrm{LD}}(t) \quad t = 0, 1, \ldots \end{aligned} \tag{4.1}$$

with $x(l) = 0$ for all $l < 0$.

Since $x_{\mathrm{FD}}(t)$ in (2.5) represents a sort of IIR and so does $X_{\mathrm{LD}}(t)$ as in (4.1), the question arises whether there is relationship between $X_{\mathrm{FD}}(t)$ and $X_{\mathrm{LD}}(t)$ and, moreover, if yes then when it is possible to obtain $X_{\mathrm{LD}} = X_{\mathrm{FD}}$.

Now, a new fundamental result in this respect is announced as follows.

Theorem 4.2. *Let the FD be defined as in* (2.1) *through* (2.3) *or, equivalently, as in* (2.5)*, and let the LD be defined as in Definition 4.1. Then LD is identical with FD, that is,* $X_{LD}(t) \equiv X_{FD}(t)$*, if and only if*

$$c_j = \sum_{i=0}^{j-1} \binom{j-1}{i} \frac{k^{2i}(-p)^{j-1-i}}{i!} \left. \frac{d^i C_1(z)}{dz^i} \right|_{z=p} \quad j = 1, 2, \ldots \tag{4.2}$$

with $k = \sqrt{1-p^2}$, $p \in (-1, 1) \setminus \{0\}$ *being the dominant Laguerre pole and*

$$C_1(z) = k \frac{(1-z)^{\alpha} - 1}{z}. \tag{4.3}$$

Proof. See Appendix A. □

Remark 4.3. Note that, rather surprisingly, an actual value of $p \in (-1, 1) \setminus \{0\}$ is meaningless for the validity of Theorem 4.2. This intriguing fact has been confirmed in a plethora of our simulations, both in time and frequency domains. Well, on the other hand, the infinite expansion as in (3.2) can also perfectly model any rational transfer function irrespectively of an actual value of p.

Exemplary coefficients c_j, $j = 1, 2, 3$ as in (4.2) are given in Appendix B.

Remark 4.4. The coefficients c_j in (4.2) can as well be calculated in an experimental way on the basis of (3.5):

$$c_j = \left\langle \Delta^{\alpha} x(t)|_{x(t)=\delta(t)}, L_j\left(q^{-1}\right)\delta(t) \right\rangle = \sum_{t=1}^{\infty} P_t(\alpha) L_j\left(q^{-1}\right)\delta(t) \quad j = 1, 2, \ldots, \tag{4.4}$$

where $\delta(t)$ is the Kronecker delta.

Even though (2.5) and (4.1) are equivalent in the sense that $X_{\mathrm{FD}}(t) \equiv X_{\mathrm{LD}}(t)$ under the circumstances, the respective differences will still be referred to as FD and LD.

4.1. Finite Approximation of LD

Like for FD, we have an infinite number of LD components leading to computational explosion. In analogy to the presented finite fractional difference (FFD), the convergence to zero of the series c_j enables to assume $c_j \approx 0$ for some $j > M$, where M is the number of the Laguerre filters used to calculate the finite LD. We will further proceed with the finite Laguerre-based difference (FLD), to be formally defined below.

Definition 4.5. Let the Laguerre-based discrete-time difference (LD) be defined as in Definition 4.1. Then, the finite Laguerre-based difference (FLD) is defined as

$$\Delta_{\mathrm{FLD}}^{\alpha} x(t) = x(t) + \sum_{j=1}^{M} c_j L_j\left(q^{-1}\right) x(t), \quad t = 0, 1, \ldots, \tag{4.5}$$

where M is the number of the Laguerre filters used to calculate the difference FLD, and c_j, $j = 1, 2, \ldots, M$, are calculated as in (4.2).

Of course, an introduction of the bound M in FLD will lead to generation of an approximation error as compared to the original FD/LD. Define this error in the time domain as

$$\varepsilon_{\mathrm{FLD}}(t, M) = \Delta_{\mathrm{FLD}}^{\alpha} x(t) - \Delta^{\alpha} x(t). \tag{4.6}$$

The energy of the sequence, $\varepsilon_{\mathrm{FLD}}(t, M)$, $t = 1, 2, \ldots$, is given by

$$\|\varepsilon_{\mathrm{FLD}}(t, M)\|^2 = \langle \varepsilon_{\mathrm{FLD}}(t, M), \varepsilon_{\mathrm{FLD}}(t, M) \rangle = \sum_{t=0}^{\infty} \varepsilon_{\mathrm{FLD}}^2(t, M), \tag{4.7}$$

where $\langle \cdot \rangle$ is the scalar product. For the considered FLD, the value of $\|\varepsilon_{\mathrm{FLD}}(t; M)\|^2$ can be easily computed as (compare [28]) follows:

$$\|\varepsilon_{\mathrm{FLD}}(t, M)\|^2 = \|g(t)\|^2 - \sum_{j=1}^{M} c_j^2 = \sum_{j=1}^{\infty} P_j^2(\alpha) - \sum_{j=1}^{M} c_j^2 = \sum_{j=M+1}^{\infty} c_j^2, \tag{4.8}$$

with $P_j(\alpha)$ as in (2.2) and (2.3), and c_j as in (4.2). The value of $\|\varepsilon_{\mathrm{FLD}}(t; M)\|^2$ depends on three parameters: the limit M, the fractional order α, and the dominant Laguerre pole p. Accounting for the fact that increasing the limit M enhances the complexity of the FLD model, "costless" optimization of the FLD model with respect to $\|\varepsilon_{\mathrm{FLD}}(t; M)\|^2$ can only be realized by selection of a Laguerre pole p. So, in contrast to LD, selection of an optimal Laguerre pole p in the FLD model is important from the accuracy point of view.

4.2. Selection of Dominant Laguerre Pole

A choice of an optimal Laguerre pole has been given a due research attention [28, 30, 31]. Here, selection of a dominant Laguerre pole p can be obtained by optimization:

$$p = \arg\min_{p} \|\varepsilon_{\mathrm{FLD}}(t, M)\|^2. \tag{4.9}$$

The optimal Laguerre poles for various values of M and $\alpha \in (0, 1)$ are presented in Figure 1. Since FLD is not quite effective for $\alpha \in (1, 2)$, which will be commented in the sequel, we refrain from showing analogous results for that range of α.

On the basis of a plethora of simulations, in Appendix G we propose an approximation of an optimal Laguerre pole for FLD as a (heuristic) function of $\alpha \in (0, 1)$ and M.

Example 4.6. Consider the fractional difference FD and its FLD model, with $p = p_{\mathrm{opt}} = p_{\mathrm{opt}}(M, p)$ selected as in (4.9). Figure 2 presents Bode plots for the FLD models versus FD = LD, with $\alpha = 0.9$ and various values of M. Table 1 presents the approximation errors $\|\varepsilon_{\mathrm{FLD}}(t, M)\|^2$ of the FLD model for various values of M and α. It can be seen from Table 1 that (1) unsurprisingly, increasing the value of M increases an approximation accuracy of the FLD model, (2) generally, for the same values of M the approximation accuracy of FLD modeling is higher for greater α (excluding the area where α is close to (1)).

Qualitatively, the above results are quite similar to those for the FFD model [22, 27]. However, higher values of α lead to reduction of the approximation error for the FFD model much faster as compared to the FLD one. So, FLD is effective (and, in fact, more effective than FFD) for $\alpha \in (0, 1)$. This is illustrated in Table 2 which shows the values of $\overline{J}$ in the FFD model, providing an equivalent approximation accuracy to the FLD model with specified M, for various values of M and α. For instance, for $\alpha = 0.1$, the FLD model with $M = 27$ is equivalent, in terms of the approximation accuracy, to the FFD model with $\overline{J} = 445$, but for $\alpha = 1.5$ is equivalent to the FFD one with $\overline{J} = 103$.

Taking into account that the FLD model (1) needs *a priori* knowledge about the optimal Laguerre pole and (2) is more complex than FFD from the computational point of view, the FLD model can be recommended for $\alpha < 1$ only.

Let us finally show some interesting feature related with the FLD model.

Example 4.7. Consider the fractional difference FD and its FLD model as in Example 4.6. The approximation errors for the FLD model with various values of α and consecutive values of M are presented in Table 3.

It can be seen from that table that the adjacent values of M provide the same approximation accuracy for the FLD model. It is interesting to note that for $\alpha \in (0, 1)$ we obtain the same approximation errors for the pairs $M = \{1, 2\}; M = \{3, 4\}; M = \{5, 6\} \ldots$, but for $\alpha \in (1, 2)$ the same errors are obtained for the pairs $M = \{2, 3\}; M = \{4, 5\}; M = \{6, 7\} \ldots$. Accounting for the computational aspect, we, thus, recommend to use odd values of M for $\alpha \in (0, 1)$ and even values of M for $\alpha \in (1, 2)$.

It is worth mentioning that when in the above examples p_{opt} is substituted by its approximation computed as in Appendix C, the approximation errors are hardly distinguishable from those of Tables 1 and 3.

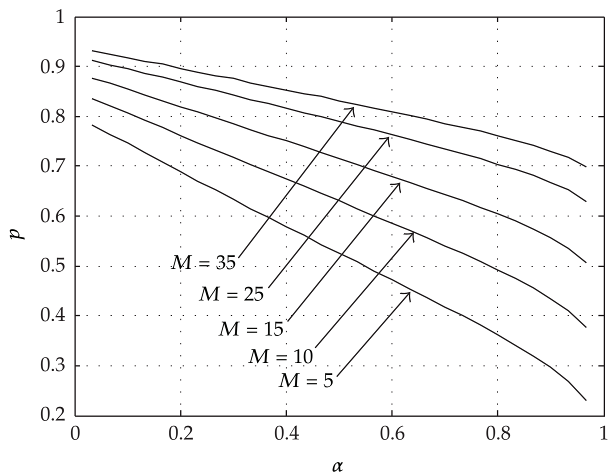

Figure 1: Plots of optimal Laguerre poles for FLD versus fractional-order $\alpha \in (0, 1)$.

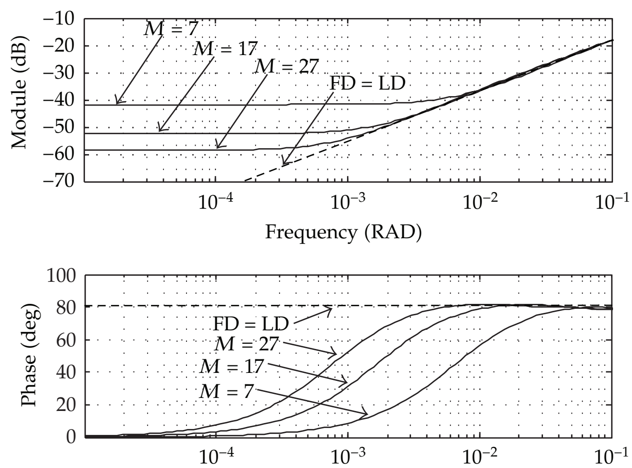

Figure 2: Bode plots for FLD model with $\alpha = 0.9$ and various values of M.

The above examples demonstrate that FLD is effective for $\alpha \in (0, 1)$. For $\alpha \in (1, 2)$, the FLD is not as effective as FFD in approximation of FD. The motivation of the work presented in the next section is searching for a "good" FLD-like model also for $\alpha > 1$.

5. Combined Fractional/Laguerre-Based Difference

To cope with the problem, we introduce a new difference, which is a combination of the "classical" FFD and our FLD.

Definition 5.1. Define the combined fractional/Laguerre-based difference (CFLD) as

$$\Delta^{\alpha}_{\text{CFLD}} x(t) = x(t) + X_{\text{CFLD}}(t) \quad t = 0, 1, \ldots, \tag{5.1}$$

Table 1: Approximation errors for FLD with various values of M and α.

	$M = 7$	$M = 17$	$M = 27$
$\alpha = 0.1$	$6.3485e - 5$	$1.2127e - 5$	$4.8239e - 6$
$\alpha = 0.5$	$5.4513e - 5$	$3.4898e - 6$	$7.4791e - 7$
$\alpha = 0.9$	$1.2874e - 6$	$2.7372e - 8$	$3.1065e - 9$
$\alpha = 0.98$	$5.7516e - 8$	$9.8064e - 10$	$9.6533e - 11$
$\alpha = 1.5$	$3.2417e - 6$	$9.9157e - 9$	$4.1481e - 10$
$\alpha = 1.8$	$7.1224e - 7$	$9.6608e - 10$	$2.5258e - 11$

Table 2: Bound $\overline{J}$ for FFD providing equivalent accuracies to FLD, with various values of α and M.

	$M = 7$	$M = 17$	$M = 27$
$\alpha = 0.1$	53	207	445
$\alpha = 0.5$	28	108	231
$\alpha = 0.9$	17	65	141
$\alpha = 0.98$	14	55	120
$\alpha = 1.5$	12	47	103
$\alpha = 1.8$	11	41	89

Table 3: Approximation errors for FLD with various values of M and α.

	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 1.5$
$M = 1$	$3.4705e - 3$	$3.3670e - 4$	$2.0469e - 2$
$M = 2$	$3.4705e - 3$	$3.3670e - 4$	$4.5878e - 4$
$M = 3$	$4.9393e - 4$	$2.6149e - 5$	$4.5878e - 4$
$M = 4$	$4.9393e - 4$	$2.6149e - 5$	$2.5059e - 5$
$M = 5$	$1.3914e - 4$	$4.6882e - 6$	$2.5059e - 5$
$M = 6$	$1.3914e - 4$	$4.6882e - 6$	$3.2417e - 6$
$M = 7$	$5.4513e - 5$	$1.2874e - 6$	$3.2417e - 6$
$M = 8$	$5.4513e - 5$	$1.2874e - 6$	$6.623e - 7$
$M = 9$	$2.5931e - 5$	$4.5774e - 7$	$6.6237e - 7$
$M = 10$	$2.5931e - 5$	$4.5774e - 7$	$1.8047e - 7$

where

$$X_{\mathrm{CFLD}}(t) = \sum_{i=1}^{\overline{J}} P_i(\alpha)x(t)q^{-i} + \sum_{j=1}^{\infty} c_j L_j\left(q^{-1}\right)q^{-\overline{J}}x(t), \tag{5.2}$$

and the first component at the right-hand side of (5.2) constituting the FFD share in the CFLD, the second one being the ($\overline{J}$-delayed) LD share, with $P_j(\alpha)$, $j = 1, \ldots, \overline{J}$, as in (2.2) and (2.3), $L_j(q^{-1})$ and c_j, $j = 1, 2, \ldots$, as in (3.3) and (3.4), respectively.

Now, we have another fundamental result in perfect modeling of FD via CFLD.

Theorem 5.2. *Let the Grünwald-Letnikov fractional difference (FD) be defined as in* (2.1) *through* (2.3)*, Laguerre-based difference (LD) is as in Definition 4.1 and combined fractional/Laguerre-based* *difference (CFLD). Then CFLD is equivalent to FD in that* $\Delta^{\alpha}_{CFLD}x(t) \equiv \Delta^{\alpha}x(t)$ *(or* $X_{CFLD}(t) \equiv X_{FD}(t)$*) if and only if*

$$c_j = \sum_{i=0}^{j-1}\binom{j-1}{i}\frac{k^{2i}(-p)^{j-1-i}}{i!}\left.\frac{d^iD_1(z)}{dz^i}\right|_{z=p} \qquad j = 1,2,\ldots \tag{5.3}$$

with $k = \sqrt{1-p^2}$, $p \in (-1,1) \setminus \{0\}$ *being the dominant Laguerre pole and*

$$D_1(z) = k\frac{(1-z)^{\alpha} - 1 - \sum_{j=1}^{\overline{J}} P_j(\alpha)z^j}{z^{\overline{J}+1}}. \tag{5.4}$$

Proof. See Appendix D. □

The first two coefficients c_j, $j = 1,2$, in (5.2) calculated as in (5.3) and (5.4) are exampled in Appendix E.

Remark 5.3. Note that regardless of an actual value of p we have FD ≡ LD ≡ CFLD, in the sense that $X_{\text{FD}}(t) \equiv X_{\text{LD}}(t) \equiv X_{\text{CFLD}}(t)$, $t = 0,1,\ldots$.

The CFLD model as in (5.1) can also be presented in the form:

$$\Delta^{\alpha}_{\text{CFLD}}x(t) = x(t) + \sum_{j=1}^{\infty}\underline{c}_j f_j\left(q^{-1}\right)x(t), \tag{5.5}$$

where $\underline{c}_j$ and $f_j(q^{-1})$, $j = 1,2,\ldots$, are as follows:

$$\underline{c}_j = \begin{cases} P_j(\alpha) & j = 1,\ldots,\overline{J} \\ c_{j-\overline{J}} & j = \overline{J}+1,\ldots, \end{cases} \tag{5.6}$$

$$f_j\left(q^{-1}\right) = \begin{cases} q^{-j} & j = 1,\ldots,\overline{J} \\ L_{j-\overline{J}}(q^{-1})q^{-\overline{J}} & j = \overline{J}+1,\ldots \end{cases} \tag{5.7}$$

with $c_{j-\overline{J}}$, $j = \overline{J}+1,\ldots$, calculated from (5.3).

An interesting CFLD orthonormality result can now be obtained.

Theorem 5.4. *Consider the CFLD as in* (5.5) *with the filters* $f_j(q^{-1})$, $j = 1,2,\ldots$, *as in* (5.7). *The filters* $f_j(q^{-1})$, $j = 1,2,\ldots$, *are orthonormal basis functions.*

Proof. See Appendix F. □

Remark 5.5. Like in the FD, possible accounting for the sampling period T in LD and CFLD models when transferring from a continuous-time derivative to the discrete-time difference results in dividing the right-hand side of (4.1) and (5.1) by T^{α}, respectively.

Like in the FD/LD, the infinite length expansion incorporated in CFLD leads to a computational explosion. Therefore, in analogy to FLD, we introduce a finite-length approximation to CFLD called finite (combined) fractional/Laguerre-based difference (FFLD).

5.1. Finite Approximation of CFLD

The idea behind combining FFD and FLD comes from *a priori* knowledge about the natures of (1) FFD versus FD in the initial (or high-frequency) part of the model [22] and (2) FLD versus classical FIR in the remaining (or medium/low-frequency) part. In fact, FFD $\equiv$ FD for $t < \overline{J}$ so the "only" problem is to find a "good" $\overline{J}$ and, on the other hand, the number M of Laguerre filters is essentially lower than a number of FIR components (and FD is a "sort of" IIR, in particular in the medium/low-frequency part).

Step by step, we arrive at the most practically important model of FD, being the truncated or finite CFLD.

Definition 5.6. Let the combined fractional/Laguerre-based difference (CFLD) be defined as in Definition 5.1. Then the finite (combined) fractional/Laguerre-based difference (FFLD) is defined as

$$\Delta^{\alpha}_{\mathrm{FFLD}} x\left(t, \overline{J}, M\right) = x(t) + \sum_{i=1}^{\overline{J}} P_i(\alpha) x(t) q^{-i} + \sum_{j=1}^{M} c_j L_j\left(q^{-1}\right) q^{-\overline{J}} x(t), \quad t = 0, 1, \ldots, \tag{5.8}$$

where M is the number of Laguerre filters used in the model.

Again, the bound M in FFLD leads to an approximation error in FFLD modeling. Immediately, based on Theorem 5.4, an approximation error for the FFLD model can be calculated like for the FLD one (compare (4.8)):

$$\left\|\varepsilon_{\mathrm{FFLD}}\left(t, \overline{J}, M\right)\right\|^2 = \|g(t)\|^2 - \sum_{j=1}^{\overline{J}+M} \underline{c}_j^2 = \sum_{j=1}^{\infty} P_j^2(\alpha) - \sum_{j=1}^{\overline{J}+M} \underline{c}_j^2 = \sum_{j=\overline{J}+M+1}^{\infty} \underline{c}_j^2 \tag{5.9}$$

with $\underline{c}_j$ as in (5.6).

Remark 5.7. It is essential that, like for FLD, the approximation error for FFLD can be made arbitrarily small by selection of sufficiently high $M \geq M_0$, which is the well-known feature of OBF. However, the power of FFLD is that, owing to the FFD contribution, the value of M_0 can be much lower than that for FLD.

5.2. Selection of Dominant Laguerre Pole

Here, an optimal Laguerre pole is selected by minimization of the approximation error (5.9) in a similar way as in (4.9). Figures 3 and 4 present the optimal Laguerre pole p as a function of the order α for $\overline{J} = 10$ and various values of M, and for $M = 15$ and various values of $\overline{J}$, respectively.

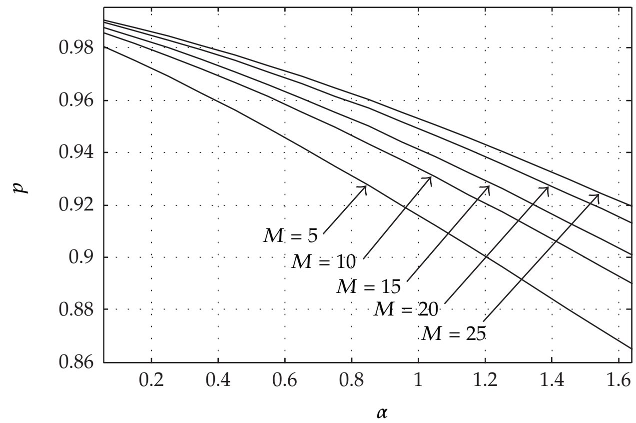

Figure 3: Optimal Laguerre poles for FFLD model as a function of α, with $\overline{J} = 10$ and various values of M.

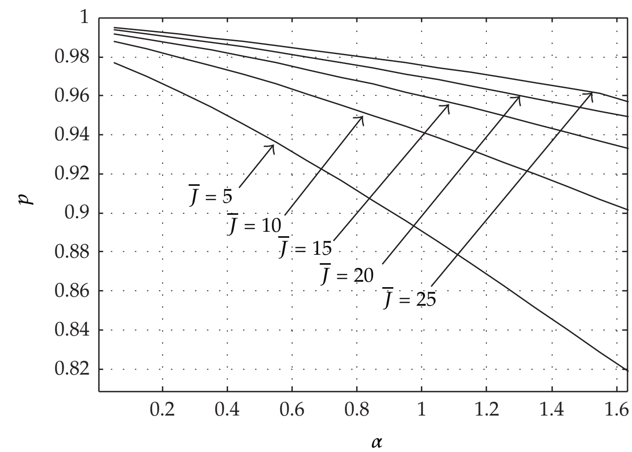

Figure 4: Optimal Laguerre poles for FFLD model as a function of α, with $M = 15$ and various values of $\overline{J}$.

As in the case of the FLD model, on the basis of a number of simulations, an (heuristic) approximation of an optimal Laguerre pole in the FFLD model, as a function $\alpha \in (0, 2)$, M and $\overline{J}$, is presented in Appendix B.

Example 5.8. Consider the fractional difference FD and its FFLD model with $p = p_{\text{opt}}$. Table 4 presents the approximation error $\|\varepsilon(t, M)\|^2$ for the FFLD model with $\overline{J} = 10$ and various values of M and α. Table 5 shows values of $\overline{J}$ in the FFD model that are accuracy-equivalent to the FFLD with specified M and $\overline{J}$.

It can be seen from Tables 4 and 5 that the FFLD model is much more effective than FFD in modeling of FD in that FFD needs a huge number of $\overline{J}$ to provide equivalent approximation accuracy to FFLD. Figure 5 presents Bode plots for the FFLD model versus FD = LD = CFLD, with $\alpha = 0.9$, $\overline{J} = 10$, and various values of M.

Example 5.9. Consider the fractional difference FD with $\alpha = 0.9$ and its FFD versus FFLD models, with $\overline{J}_{\text{FFD}}$ and $\overline{J}_{\text{FFLD}}$, respectively, and $p = p_{\text{opt}}$ and $M = 27$ for FFLD. Table 6 presents

Table 4: Approximation errors for FFLD model with $\overline{J} = 10$ and various values of M and α.

	$M = 7$	$M = 17$	$M = 27$
$\alpha = 0.1$	$3.2760e-6$	$7.0817e-7$	$2.9710e-7$
$\alpha = 0.5$	$4.2422e-7$	$3.7965e-8$	$9.5936e-9$
$\alpha = 0.9$	$1.2167e-9$	$4.7492e-11$	$7.2540e-12$
$\alpha = 0.98$	$2.6855e-11$	$8.7066e-13$	$9.8936e-14$
$\alpha = 1.5$	$1.8660e-10$	$2.2660e-12$	$1.3051e-13$

Table 5: Number of elements $\overline{J}$ in FFD providing equivalent approximation accuracy to FFLD, with $\overline{J} = 10$ and various values of α and M.

	$M = 7$	$M = 17$	$M = 27$
$\alpha = 0.1$	612	2132	4195
$\alpha = 0.5$	307	1024	2032
$\alpha = 0.9$	197	625	1218
$\alpha = 0.98$	184	577	1122
$\alpha = 1.5$	126	375	718

the same approximation errors for both models under various values of $\overline{J}_{\mathrm{FFD}}$ and $\overline{J}_{\mathrm{FFLD}}$. It can be seen from Table 6 that increasing $\overline{J}_{\mathrm{FFLD}}$ by 5 in the FFLD model is equivalent to increasing $\overline{J}_{\mathrm{FFD}}$ by some 500 in the FFD model. However, increasing $\overline{J}$ by 5 in both FFD and FFLD models results in roughly the same increase in the computational burden. So, in FFLD modeling we have some 100 times better computational efficiency.

It is worth emphasizing that the approximation error is so low for FFLD that the normalization factor incorporated into FFD [22] may be not necessary for FFLD. Now, FFLD can be competitive to another powerful adaptive (normalized) finite fractional difference (AFFD) [22, 32], an intriguing issue to be a subject of a comparative research study.

6. Conclusion

This paper has offered a series of original results in modeling of Grünwald-Letnikov discrete-time fractional-difference (FD) using Laguerre filters. Firstly, a new quality has been presented, namely, the Laguerre-based difference (LD), which has been proven to be equivalent, under specified conditions, to the FD. For implementation reasons, a finite LD (FLD) approximator has been introduced as an analogue to the "classical" finite FD (FFD), and the two have been shown to perform in a similar way.

Another new quality, is that combined fractional/Laguerre-based difference (CFLD) has also been shown equivalent, under specified conditions, to the FD. Interestingly, a finite-length approximator to CFLD, called finite (combined) FLD, or FFLD, has been demonstrated in simulations to constitute an excellent model of FD, both in terms of the accuracy and computational efficiency. This is due to the fact that FFLD constitutes an expert combination of the high-frequency FFD component and medium/low-frequency FLD part, both efficiently balanced using the bounds $\overline{J}$ and M, respectively. Additionally, simple approximate derivations for optimal Laguerre poles are supplemented. Summing up, FFLD is recommended as a high-performance approximator to FD. Future research in the area

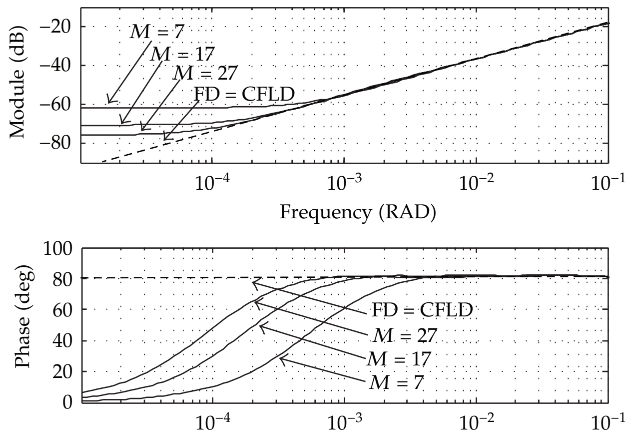

Figure 5: Bode plots for the FFLD model with $\alpha = 0.9$ and various values of M.

Table 6: Approximation errors for FFLD versus FFD model with $\alpha = 0.9$, $M = 27$, and various values of $\overline{J}$.

$\overline{J}_{\text{FFLD}}$	5	10	15	20
$\|\|\varepsilon\|\|^2$	$4.3055e-11$	$7.2540e-12$	$2.3650e-12$	$1.0252e-12$
$\overline{J}_{\text{FFD}}$	646	1218	1802	2390

will concentrate on a comparison of FFLD and AFFD models of FD and their application in fractional-order predictive control.

Appendices

A. Proof of Theorem 4.2

FD defined in (2.1) through (2.3) or in (2.5) can be presented as

$$\begin{aligned}\Delta^{\alpha}x(t) &= x(t) + X_{\text{FD}}(t)\\ &= x(t) + G\left(q^{-1}\right)x(t),\end{aligned} \tag{A.1}$$

where $G(z)$ is in form of (3.1) with

$$G(z) = \sum_{j=1}^{\infty} P_j(\alpha)z^{-j} \tag{A.2}$$

and $P_j(\alpha)$ defined as in (2.2) and (2.3).

Note that using the generalized Newton binomial

$$(a+b)^{\alpha} = \sum_{j=0}^{\infty}\binom{\alpha}{j}a^{\alpha-j}b^{j} \tag{A.3}$$

and accounting for the fact that $\sum_{j=0}^{\infty} P_j(\alpha)z^{-j}$ is the binomial expansion for $a = 1$ and $b = -z^{-1}$, and $P_0(\alpha) = 1$, we can write (A.2) as

$$G(z) = \left(1 - z^{-1}\right)^{\alpha} - 1. \tag{A.4}$$

It has been shown in Section 3 that (A.2) can be described by (3.2), where $L_j(z)$, $j = 1, 2, \ldots$, are the Laguerre filters presented in (3.3). The coefficients c_j, $j = 1, 2, \ldots$, can be obtained using formula (3.4) [28]:

$$\begin{aligned} c_j &= \frac{1}{2\pi i}\oint_{\gamma} G^{*}(z)L_j(z)\frac{dz}{z} \\ &= \frac{1}{2\pi i}\oint_{\gamma} G\left(z^{-1}\right)L_j(z)\frac{dz}{z}, \end{aligned} \tag{A.5}$$

where $G^{*}(z) = G(z^{-1}) = (1 - z)^{\alpha} - 1$ is the complex conjugate of $G(z)$. Note that $G(z)$ and $L_j(z)$, $j = 1, 2, \ldots$, are analytic in γ. Using the Cauchy integral formula for $j = 1$, we have

$$c_1 = \frac{k}{2\pi i}\oint_{\gamma} \frac{G(z^{-1})}{z - p}\frac{dz}{z} = k\frac{G(z^{-1})}{z}\Bigg|_{z=p} = C_1(z)|_{z=p}, \tag{A.6}$$

where $C_1(z) = k(G(z^{-1})/z)$.

For $j = 2, 3, \ldots$, we have

$$\begin{aligned} c_j &= \frac{k}{2\pi i}\oint_{\gamma} \frac{G(z^{-1})(1 - pz)^{j-1}}{(z - p)^{j}}\frac{dz}{z} \\ &= \frac{1}{2\pi i}\oint_{\gamma} \frac{C_1(z)}{z - p}\left(\frac{k^2}{z - p} - p\right)^{j-1} dz, \end{aligned} \tag{A.7}$$

where $k = \sqrt{1 - p^2}$. Now, expanding the element $((k^2/(z - p)) - p)^{j-1}$ via the binomial theorem, we arrive at

$$\begin{aligned} c_j &= \frac{1}{2\pi i}\oint_{\gamma} \frac{C_1(z)}{z - p}\left(\sum_{i=0}^{j-1}\binom{j-1}{i}\frac{k^{2i}(-p)^{j-1-i}}{(z - p)^{i}}\right)dz \\ &= \sum_{i=0}^{j-1}\binom{j-1}{i}k^{2i}(-p)^{j-1-i}\frac{1}{2\pi i}\oint_{\gamma} \frac{C_1(z)}{(z - p)^{i+1}}dz, \end{aligned} \tag{A.8}$$

where $\binom{j-1}{i}$, $i = 1, \ldots, j - 1$, are the binomial coefficients. Finally, on the basis of Cauchy integral formula, again, we obtain (4.2).

B. Exemplary Coefficients c_j in (4.2)

Exemplary coefficients c_j, $j = 1, 2, 3$ as in (4.2) are as follows:

$$\begin{aligned}
c_1 &= k\frac{(1-p)^{\alpha}-1}{p},\\
c_2 &= -k\frac{(1-p)^{\alpha}-1}{p}\left(p+\frac{k^2}{p}\right)-k^3\frac{\alpha(1-p)^{\alpha-1}}{p},\\
c_3 &= k\frac{(1-p)^{\alpha}-1}{p}\left(p^2+2k^2+\frac{k^4}{p^2}\right)\\
&\quad+k^3\frac{\alpha(1-p)^{\alpha-1}}{p}\left(2p+\frac{k^2}{p}\right)+\frac{k^5}{2}\frac{\alpha(\alpha-1)(1-p)^{\alpha-2}}{p}.
\end{aligned} \tag{B.1}$$

C. Approximated Laguerre Pole for FLD

An approximation of the optimal Laguerre pole p_{opt} for the FLD model is given by the following heuristic function:

$$p_{\text{opt}} \cong a_0 + a_1\alpha + a_2 e^{a_3\alpha}, \tag{C.1}$$

where

$$\begin{aligned}
a_0 &= 0.96949842 - 0.50616819 e^{-0.48357222\sqrt{M}},\\
a_1 &= -0.064593783 - 0.96336530 e^{-0.32042783\sqrt{M}},\\
a_2 &= -0.33695592 e^{-0.023882929M-0.45318016},\\
a_3 &= -3.5706767 - 30.316352 e^{-1.2122077\sqrt{M}}.
\end{aligned} \tag{C.2}$$

Note that the approximation function can be used for $\alpha \in (0.001, 0.999)$ and $M \in (5, 100)$.

D. Proof of Theorem 5.2

The GL fractional difference defined in (2.1) to (2.3) can be presented in the following IIR model:

$$\Delta^{\alpha}x(t) = x(t) + \sum_{j=1}^{\bar{J}} P_j(\alpha)x(t)q^{-j} + G_2\left(q^{-1}\right)x(t), \tag{D.1}$$

where $G_2(q^{-1})$ is the filter is transfer function of the form:

$$G_2\left(q^{-1}\right) = \sum_{j=\overline{J}+1}^{\infty} P_j(\alpha) q^{-j} = q^{-\overline{J}} \sum_{j=\overline{J}+1}^{\infty} P_j(\alpha) q^{-j+\overline{J}} = q^{-\overline{J}} \underline{G}_2\left(q^{-1}\right). \tag{D.2}$$

Using the generalized Newton binomial (compare Proof of Theorem 4.2), (D.2) can be presented as follows:

$$G_2\left(q^{-1}\right) = \left(1-q^{-1}\right)^{\alpha} - 1 - \sum_{j=1}^{\overline{J}} P_j(\alpha) q^{-j}. \tag{D.3}$$

Getting back to Section 3, again, $\underline{G}_2(q^{-1})$ can be presented in the form of (3.2) so that

$$G_2\left(q^{-1}\right) = q^{-\overline{J}} \underline{G}_2\left(q^{-1}\right) = \sum_{j=1}^{\infty} c_j L_j\left(q^{-1}\right) q^{-\overline{J}}, \tag{D.4}$$

where $L_j(q^{-1})$, $j = 1, 2, \ldots$, is as in (3.3). The coefficients c_j, $j = 1, 2, \ldots$, are obtained from the scalar product:

$$\begin{aligned} c_j &= \left\langle G_2\left(q^{-1}\right), L_j\left(q^{-1}\right) q^{-\overline{J}} \right\rangle = \left\langle G_2(z), L_j(z) z^{-\overline{J}} \right\rangle \\ &= \frac{1}{2\pi i} \oint_{\gamma} G_2^*(z) L_j(z) \frac{dz}{z^{\overline{J}+1}}. \end{aligned} \tag{D.5}$$

Following the proof of Theorem 4.2, (A.6) is substituted by

$$c_1 = \frac{k}{2\pi i} \oint_{\gamma} \frac{G_2(z^{-1})}{z-p} \frac{dz}{z^{\overline{J}+1}} = k \left. \frac{G_2(z^{-1})}{z^{\overline{J}+1}} \right|_{z=p} = D_1(z)|_{z=p} \tag{D.6}$$

with $G_2(z^{-1}) = G_2^*(z) = (1-z)^{\alpha} - 1 - \sum_{j=1}^{\overline{J}} P_j(\alpha) z^j$ and $D_1(z)$ (substituted for $C_1(z)$) as in (5.4).

E. Exemplary Coefficients c_j in (5.2)

The first two coefficients c_j, $j = 1, 2$, in (5.2) calculated as in (5.3) and (5.4) are as follows:

$$\begin{aligned} c_1 &= k\frac{(1-p)^{\alpha} - 1 - \sum_{j=1}^{\overline{J}} P_j(\alpha)p^j}{p^{\overline{J}+1}} \\ c_2 &= -k\frac{(1-p)^{\alpha} - 1 - \sum_{j=1}^{\overline{J}} P_j(\alpha)p^j}{p^{\overline{J}+1}}\left(p + \frac{\left(\overline{J}+1\right)k^2}{p}\right) \\ &\quad - k^3\frac{\alpha(1-p)^{\alpha-1} - \sum_{j=1}^{\overline{J}} jP_j(\alpha)p^{j-1}}{p^{\overline{J}+1}}. \end{aligned} \tag{E.1}$$

F. Proof of Theorem 5.4

The functions $f_j(q^{-1})$, $j = 1, 2, \ldots$, are orthonormal to each other if and only if

$$\langle f_i(z), f_j(z)\rangle = \begin{cases} 1 & \forall i = j \quad i, j \in I_+ \\ 0 & \forall i \neq j \quad i, j \in I_+, \end{cases} \tag{F.1}$$

where I_+ denotes the positive integers. Observe that $f_i(z)$, $i = 1, \ldots, \overline{J}$, is just FIR, that is, the special case of the Laguerre filters as in (3.3) with the dominant Laguerre pole $p = 0$. Since the Laguerre filters are orthonormal basis functions, we have $\langle f_i(z), f_i(z)\rangle = 1$ for each $i = 1, 2, \ldots$ (independently of a value of p) and $\langle f_i(z), f_j(z)\rangle = 0$, $i = 1, 2, \ldots$, for $i, j \leq \overline{J}$ $(p = 0)$ or $i, j > \overline{J}$ $(p \in (-1, 1))$. In the last step, we prove that $\langle f_i(z), f_j(z)\rangle = \langle f_j(z), f_i(z)\rangle = 0$ for each $j \leq \overline{J}$ and $i > \overline{J}$. In this case, we have

$$\langle f_i(z), f_j(z)\rangle = \langle l_i(t), l_j(t)\rangle = \langle l_j(t), l_i(t)\rangle = \sum_{t=1}^{\infty} l_i(t)l_j(t), \tag{F.2}$$

where $l_i(t) = f_i(q^{-1})\delta(t)$ and $l_j(t) = f_j(q^{-1})\delta(t)$. Accounting that $l_i(t) = 0$ for each $t = 1, \ldots, \overline{J}$, $l_j(t) = 0$ for each $t = 1, \ldots, j-1, j+1, \ldots$, and $j \leq \overline{J}$, we obtain $\sum_{t=1}^{\infty} l_i(t)l_j(t) = 0$, which completes the proof.

G. Approximated Laguerre Pole for FFLD

An approximation of the optimal Laguerre pole p_{opt} for the FFLD model is given by the following heuristic function:

$$p_{\text{opt}} \cong a_0 + a_1\alpha + a_2\alpha^2, \tag{G.1}$$

Table 7: Values of the parameters in approximation (G.1).

Name	Value	Name	Value	Name	Value
a_0^1	13.821178	a_1^1	−0.15915827	a_2^1	−0.026978880
a_0^2	−12.774481	a_1^2	3.5023767	a_2^2	−0.3859619
a_0^3	−4.0530011	a_1^3	−0.22109952	a_2^3	0.55931846
a_0^4	−0.81675544	a_1^4	0.50001831	a_2^4	−4.4609453
a_0^5	−4.5738909	a_1^5	−1.4689257	a_2^5	−0.0023129853
a_0^6	−0.14651483	a_1^6	−0.32557155	a_2^6	−1.1093519
				a_2^7	4.4590784

where

$$
\begin{aligned}
a_0 &= \left(a_0^1 + a_0^2 \exp\left[a_0^3 M^{a_0^4}\right]\right) \exp\left[a_0^5 \overline{J}^{a_0^6}\right], \\
a_1 &= \left(a_1^1 + a_1^2 \exp\left[a_1^3 M^{a_1^4}\right]\right) \exp\left[a_1^5 \overline{J}^{a_1^6}\right], \\
a_2 &= a_2^1 \exp\left[a_2^2 \overline{J}^{a_2^3}\right] + a_2^4 \exp\left[a_2^5 M^{a_2^6}\right] + a_2^7
\end{aligned}
\tag{G.2}
$$

with values of the parameters presented in Table 7. The function (G.1) can be used for $\alpha \in (0.01, 1.99)$.

Abbreviations

FD: Fractional difference
FFD: Finite fractional difference
LD: Laguerre-based difference
FLD: Finite Laguerre-based difference
CFLD: Combined fractional/Laguerre-based difference
FFLD: Finite (combined) fractional/Laguerre-based difference
IIR: Infinite impulse response
FIR: Finite impulse response
OBF: Orthonormal basis functions.

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