

Research Article

On Stability of a Third Order of Accuracy Difference Scheme for Hyperbolic Nonlocal BVP with Self-Adjoint Operator

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A third order of accuracy absolutely stable difference schemes is presented for nonlocal boundary value hyperbolic problem of the differential equations in a Hilbert space H with self-adjoint positive definite operator A . Stability estimates for solution of the difference scheme are established. In practice, one-dimensional hyperbolic equation with nonlocal boundary conditions is considered.

1. Introduction

In modeling several phenomena of physics, biology, and ecology mathematically, there often arise problems with nonlocal boundary conditions (see [1–5] and the references given therein). Nonlocal boundary value problems have been a major research area in the case when it is impossible to determine the boundary conditions of the unknown function. Over the last few decades, the study of nonlocal boundary value problems is of substantial contemporary interest (see, e.g., [6–14] and the references given therein).

We consider the nonlocal boundary value problem

$$\begin{aligned} \frac{d^2u(t)}{dt^2} + Au(t) &= f(t), \quad 0 < t < 1, \\ u(0) &= \alpha u(1) + \varphi, \\ u'(0) &= \beta u'(1) + \psi, \end{aligned} \tag{1}$$

for hyperbolic equations in a Hilbert space H with self-adjoint positive definite linear operator A with domain $D(A)$.

A function $u(t)$ is called a solution of problem (1) if the following conditions are satisfied.

- (i) $u(t)$ is twice continuously differentiable on the segment $[0, 1]$. The derivatives at the endpoints of the

segment are understood as the appropriate unilateral derivatives.

- (ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0, 1]$ and the function $Au(t)$ is continuous on the segment $[0, 1]$.
- (iii) $u(t)$ satisfies the equations and the nonlocal boundary conditions (1).

Here, $\varphi(x)$, $\psi(x)$ ($x \in [0, 1]$) and $f(t, x)$ ($t, x \in [0, 1]$) are smooth functions.

In the study of numerical methods for solving PDEs, stability is an important research area (see [6–27]). Many scientists work on difference schemes for hyperbolic partial differential equations, in which stability was established under the assumption that the magnitudes of the grid steps τ and h with respect to the time and space variables are connected. This particularly means that $\tau \|A_h\| \rightarrow 0$ when $\tau \rightarrow 0$.

We are interested in studying high order of accuracy unconditionally stable difference schemes for hyperbolic PDEs.

In the present paper, third order of accuracy difference scheme generated by integer power of A for approximately solving nonlocal boundary value problem (1) is presented.

The stability estimates for solution of the difference scheme are established.

In [8], some results of this paper, without proof, were presented.

The well posedness of nonlocal boundary value problems for parabolic equations, elliptic equations, and equations of mixed types have been studied extensively by many scientists (see, e.g., [11–14, 19–32] and the references therein).

2. Third Order of Accuracy Difference Scheme Subject to Nonlocal Conditions

In this section, we obtain stability estimates for the solution of third order of accuracy difference scheme

$$\begin{aligned} & \tau^{-2} (u_{k+1} - 2u_k + u_{k-1}) + \frac{2}{3}Au_k + \frac{1}{6}A(u_{k+1} + u_{k-1}) \\ & + \frac{1}{12}\tau^2 A^2 u_{k+1} = f_k, \\ & f_k = \frac{2}{3}f(t_k) + \frac{1}{6}(f(t_{k+1}) + f(t_{k-1})) \\ & - \frac{1}{12}\tau^2 (-Af(t_{k+1}) + f''(t_{k+1})), \\ & t_k = k\tau, \quad 1 \leq k \leq N-1, \quad N\tau = 1, \\ & u_0 = \alpha u_N + \varphi, \\ & \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2 \right) \tau^{-1} (u_1 - u_0) + \frac{\tau}{2}Au_0 - \tau f_{1,1} \\ & = \beta \left(I - \frac{\tau^2 A}{12} \right) \\ & \times \left(\frac{7u_N - 8u_{N-1} + u_{N-2}}{6\tau} + \frac{\tau}{3}(f_N - Au_N) \right) \\ & + \left(I - \frac{\tau^2 A}{12} \right) \psi \end{aligned} \tag{2}$$

for numerical solution of nonlocal boundary value problem (1). Here,

$$f_{1,1} = f(0) + (-f(0) + \tau f'(0)) \frac{1}{2} - 2f'(0) \frac{\tau}{6}. \tag{3}$$

We study the stability of solutions of difference scheme (2) under the following assumption:

$$|\alpha| + 2|\beta| + 2|\alpha||\beta| < 1. \tag{4}$$

We give a lemma that will be needed in the sequel which was presented in [18]. First, let us present the following operators:

$$\begin{aligned} R &= \left(I - \frac{1}{3}\tau^2 A + i\tau A^{1/2} \sqrt{I + \frac{1}{72}\tau^4 A^2} \right) \\ &\times \left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2 \right)^{-1}, \end{aligned} \tag{5}$$

and its conjugate \tilde{R} ,

$$\begin{aligned} R_1 &= \left(-\frac{5\tau^4}{144}A^2 + \frac{7\tau^6}{216}A^3 - i\tau A^{1/2} \right. \\ &\times \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2 \right) \sqrt{I + \frac{1}{72}\tau^4 A^2} \right. \\ &\times \left(-i\tau A^{1/2} \left(\sqrt{I + \frac{1}{72}\tau^4 A^2} \right) \right. \\ &\left. \left. \times \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2 \right) \right)^{-1}, \right. \end{aligned} \tag{6}$$

and its conjugate \tilde{R}_1 ,

$$\begin{aligned} R_2 &= \left(I - \frac{\tau^2}{12}A \right) \left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2 \right) \\ &\times \left(-iA^{1/2} \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2 \right) \sqrt{I + \frac{1}{72}\tau^4 A^2} \right)^{-1}, \\ R_3 &= \left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2 \right) \\ &\times \left(\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2 \right) \left(-i\tau A^{1/2} \sqrt{I + \frac{1}{72}\tau^4 A^2} \right) \right)^{-1}, \\ R_4 &= \left(I + \frac{\tau^2}{3}A + \frac{\tau^4}{9}A^2 + \frac{\tau^6}{72}A^3 \right) \\ &\times \left(-iA^{1/2} \left(\sqrt{I + \frac{1}{72}\tau^4 A^2} \right) \left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2 \right) \right. \\ &\left. \times \left(I + \frac{\tau^2}{6}A \right) \right)^{-1}, \\ R_5 &= \left(-\frac{\tau^2}{2}A - \frac{\tau^4}{12}A^2 + i\tau A^{1/2} \sqrt{I + \frac{1}{72}\tau^4 A^2} \right) \\ &\times \left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2 \right)^{-1}, \end{aligned} \tag{7}$$

and its conjugate \tilde{R}_5 , and

$$\begin{aligned} R_6 &= \left(I - \frac{1}{3}\tau^2 A + i\tau A^{1/2} \sqrt{I + \frac{1}{72}\tau^4 A^2} \right) \\ &\times \left(\frac{\tau^2}{2}A + \frac{\tau^4}{12}A^2 - i\tau A^{1/2} \sqrt{I + \frac{1}{72}\tau^4 A^2} \right)^{-1}, \end{aligned} \tag{8}$$

and its conjugate \tilde{R}_6 .

We consider the following operators:

$$\begin{aligned} R_7 &= \frac{(7R - I)}{6\tau} \\ &= \left(I - \frac{5}{12}\tau^2 A + \frac{1}{72}\tau^4 A^2 + \frac{7}{6}i\tau A^{1/2} \sqrt{I + \frac{1}{72}\tau^4 A^2} \right) \\ &\quad \times \tau^{-1} \left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2 \right)^{-1}, \end{aligned} \quad (9)$$

and its conjugate \tilde{R}_7 ,

$$\begin{aligned} \tilde{R}_7 &= \frac{(7\tilde{R} - I)}{6\tau} \\ &= \left(I - \frac{5}{12}\tau^2 A + \frac{1}{72}\tau^4 A^2 - \frac{7}{6}i\tau A^{1/2} \sqrt{I + \frac{1}{72}\tau^4 A^2} \right) \\ &\quad \times \tau^{-1} \left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2 \right)^{-1}, \\ R_8 &= \left(\frac{7I - 2\tau^2 A}{6\tau} \right) \left(I + \frac{\tau^2 A}{3} + \frac{\tau^4 A^2}{9} + \frac{\tau^6 A^3}{72} \right) \\ &\quad \times \tau^{-1} \left(I + \frac{\tau^2 A}{6} \right)^{-1} \left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2 \right)^{-2}, \\ R_9 &= \left(I - \frac{5}{3}\tau^2 A + \frac{\tau^4 A^2}{9} \right) \left(I + \frac{\tau^2 A}{3} + \frac{\tau^4 A^2}{9} + \frac{\tau^6 A^3}{72} \right) \\ &\quad \times \tau^{-1} \left(I + \frac{\tau^2 A}{6} \right)^{-1} \left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2 \right)^{-3}, \\ R_{10} &= I + \left(\frac{5}{144}\tau^4 A^2 - \frac{9}{288}\tau^6 A^3 + \frac{9}{1728}\tau^8 A^4 \right) \\ &\quad \times \left(i\tau A^{1/2} \sqrt{I + \frac{1}{72}\tau^4 A^2} \left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2 \right) \right)^{-1}, \end{aligned} \quad (10)$$

and its conjugate \tilde{R}_{10} .

Lemma 1. *The following estimates hold:*

$$\begin{aligned} \|R\|_{H \rightarrow H} &\leq 1, & \|\tilde{R}\|_{H \rightarrow H} &\leq 1, \\ \|R_1\|_{H \rightarrow H} &\leq 1, & \|\tilde{R}_1\|_{H \rightarrow H} &\leq 1, \\ \|A^{1/2}R_2\|_{H \rightarrow H} &\leq 1, & \|\tau A^{1/2}R_3\|_{H \rightarrow H} &\leq 1, \\ \|A^{1/2}R_4\|_{H \rightarrow H} &\leq 1, & \|A^{-1/2}R_5\|_{H \rightarrow H} &\leq \tau, \\ \|A^{-1/2}\tilde{R}_5\|_{H \rightarrow H} &\leq \tau, & \|\tau A^{1/2}R_6\|_{H \rightarrow H} &\leq 1, \\ \|\tau A^{1/2}\tilde{R}_6\|_{H \rightarrow H} &\leq 1. \end{aligned} \quad (11)$$

Now let us give, without proof, the second lemma.

Lemma 2. *The following estimates hold:*

$$\begin{aligned} \|(I + i\tau A^{1/2})R\|_{H \rightarrow H} &\leq 2, \\ \|(I + i\tau A^{1/2})\tilde{R}\|_{H \rightarrow H} &\leq 2, \\ \|\tau R_7\|_{H \rightarrow H} &\leq 1, & \|\tau \tilde{R}_7\|_{H \rightarrow H} &\leq 1, \\ \left\| \frac{1}{3}\tau A^{1/2}R^2 \right\|_{H \rightarrow H} &\leq 1, & \left\| \frac{1}{3}\tau A^{1/2}\tilde{R}^2 \right\|_{H \rightarrow H} &\leq 1, \\ \|\tau R_8\|_{H \rightarrow H} &\leq \frac{7}{6}, & \|\tau R_9\|_{H \rightarrow H} &\leq 1, \\ \left\| R_{10}(I + i\tau A^{1/2})^{-1} \right\|_{H \rightarrow H} &\leq 2, \\ \left\| \tilde{R}_{10}(I + i\tau A^{1/2})^{-1} \right\|_{H \rightarrow H} &\leq 2. \end{aligned} \quad (12)$$

Throughout the section, for simplicity, we denote

$$\begin{aligned} B_\tau &= \beta \frac{1}{2} R_2 \left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-2} \\ &\quad + \beta \frac{1}{2} R_2 \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) R^{N-2} \\ &\quad - \alpha \frac{1}{2} [\tilde{R}_1 R^N - R_1 \tilde{R}^N] \\ &\quad + \alpha \beta \frac{1}{4} \tilde{R}_1 R_2 \left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) R^N \tilde{R}^{N-2} \\ &\quad + \alpha \beta \frac{1}{4} R_1 R_2 \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) \tilde{R}^N R^{N-2} \\ &\quad - \alpha \beta \frac{1}{4} \tilde{R}_1 R_2 \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) \tilde{R}^N R^{N-2} \\ &\quad - \alpha \beta \frac{1}{4} R_1 R_2 \left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) R^N \tilde{R}^{N-2}. \end{aligned} \quad (13)$$

Lemma 3. *Suppose that assumption (4) holds. Then, the operator $I - B_\tau$ has an inverse $T_\tau = (I - B_\tau)^{-1}$. From symmetry and positivity properties of operator A , the following estimate is satisfied:*

$$\|T_\tau\|_{H \rightarrow H} \leq \frac{1}{1 - |\alpha| - 2|\beta| - 2|\alpha||\beta|}. \quad (14)$$

Proof. Using the definitions of B_τ , R , \tilde{R} , estimates (11), and the following simple estimates,

$$\begin{aligned} \left\| \tau A^{1/2} \left(I - \frac{\tau^2}{12} A \right) \left(I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right)^{-1} \right\|_{H \rightarrow H} &\leq 12, \\ \left\| \tau A^{1/2} \left(I + \frac{1}{12} \tau^2 A + \frac{1}{144} \tau^4 A^2 \right)^{-1} \right\|_{H \rightarrow H} &\leq \frac{12\sqrt{11}}{12 + \sqrt{11}}, \end{aligned} \quad (15)$$

and the triangle inequality, we get

$$\begin{aligned}
B_\tau &\leq |\beta| \frac{1}{2} \|A^{1/2} R_2\|_{H \rightarrow H} \|\tilde{R}^{N-2}\|_{H \rightarrow H} \\
&\quad \times \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \\
&\quad + |\beta| \frac{1}{2} \|A^{1/2} R_2\|_{H \rightarrow H} \times \|\tilde{R}^{N-2}\|_{H \rightarrow H} \\
&\quad \times \left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \\
&\quad + \alpha \frac{1}{2} [\|\tilde{R}_1\|_{H \rightarrow H} \|R^N\|_{H \rightarrow H} + \|R_1\|_{H \rightarrow H} \|\tilde{R}^N\|_{H \rightarrow H}] \\
&\quad + |\alpha| |\beta| \frac{1}{4} \|\tilde{R}_1\|_{H \rightarrow H} \|A^{1/2} R_2\|_{H \rightarrow H} \\
&\quad \times \|R^N\|_{H \rightarrow H} \|\tilde{R}^{N-2}\|_{H \rightarrow H} \\
&\quad \times \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \\
&\quad + |\alpha| |\beta| \frac{1}{4} \|R_1\|_{H \rightarrow H} \|A^{1/2} R_2\|_{H \rightarrow H} \\
&\quad \times \|\tilde{R}^N\|_{H \rightarrow H} \|\tilde{R}^{N-2}\|_{H \rightarrow H} \\
&\quad \times \left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \\
&\quad + |\alpha| |\beta| \frac{1}{4} \|\tilde{R}_1\|_{H \rightarrow H} \|A^{1/2} R_2\|_{H \rightarrow H} \\
&\quad \times \|\tilde{R}^N\|_{H \rightarrow H} \|\tilde{R}^{N-2}\|_{H \rightarrow H} \\
&\quad \times \left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \\
&\quad + |\alpha| |\beta| \frac{1}{4} \|R_1\|_{H \rightarrow H} \|A^{1/2} R_2\|_{H \rightarrow H} \\
&\quad \times \|R^N\|_{H \rightarrow H} \|\tilde{R}^{N-2}\|_{H \rightarrow H} \\
&\quad \times \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \\
&\quad + |\alpha| |\beta| \frac{1}{4} \|R_1\|_{H \rightarrow H} \|A^{1/2} R_2\|_{H \rightarrow H} \\
&\quad \times \|R^N\|_{H \rightarrow H} \|\tilde{R}^{N-2}\|_{H \rightarrow H} \\
&\quad \times \left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \\
&\quad + |\alpha| |\beta| \frac{1}{4} \|R_1\|_{H \rightarrow H} \|A^{1/2} R_2\|_{H \rightarrow H} \\
&\quad \times \|R^N\|_{H \rightarrow H} \|\tilde{R}^{N-2}\|_{H \rightarrow H} \\
&\quad \times \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \\
&\quad + |\alpha| |\beta| \frac{1}{4} \|R_1\|_{H \rightarrow H} \|A^{1/2} R_2\|_{H \rightarrow H} \\
&\quad \times \|R^N\|_{H \rightarrow H} \|\tilde{R}^{N-2}\|_{H \rightarrow H} \\
&\quad \times \left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \\
&\quad + |\alpha| |\beta| \frac{1}{4} \|R_1\|_{H \rightarrow H} \|A^{1/2} R_2\|_{H \rightarrow H} \\
&\quad \times \|R^N\|_{H \rightarrow H} \|\tilde{R}^{N-2}\|_{H \rightarrow H} \\
&\quad \times \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \\
&\quad + |\alpha| |\beta| \frac{1}{4} \|R_1\|_{H \rightarrow H} \|A^{1/2} R_2\|_{H \rightarrow H} \\
&\quad \times \|R^N\|_{H \rightarrow H} \|\tilde{R}^{N-2}\|_{H \rightarrow H} \\
&\leq q,
\end{aligned} \tag{16}$$

where

$$q = |\alpha| + 2 |\beta| + 2 |\alpha| |\beta|. \tag{17}$$

Since $q < 1$, the operator $I - B_\tau$ has a bounded inverse and

$$\|(I - B_\tau)^{-1}\|_{H \rightarrow H} \leq \frac{1}{1 - q} = \frac{1}{1 - |\alpha| - 2 |\beta| - 2 |\alpha| |\beta|}. \tag{18}$$

Lemma 3 is proved. \square

Now, let us obtain formula for the solution of problem (2). Using the results of [18], one can obtain the following formula:

$$\begin{aligned}
&u_0 = \mu, \\
&u_1 = \left(I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right)^{-1} \\
&\quad \times \left(\left(I - \frac{5}{12} \tau^2 A + \frac{\tau^4}{144} A^2 \right) \mu \right. \\
&\quad \left. + \tau \left(I - \frac{\tau^2}{12} A \right) \omega + \tau^2 f_{1,1} \right), \\
&u_k = \frac{1}{2} [\tilde{R}_{10} R^k - R_{10} \tilde{R}^k] \mu + \frac{1}{2} [\tilde{R}^k - R^k] R_2 \omega \\
&\quad + \frac{1}{2} [\tilde{R}^k - R^k] R_3 \tau^2 f_{1,1} + \frac{1}{2} R_4 \sum_{s=1}^{k-1} [\tilde{R}^{k-s} - R^{k-s}] f_s \tau^2 \\
&\tag{19}
\end{aligned}$$

for the solution of difference scheme

$$\begin{aligned}
&\tau^{-2} (u_{k+1} - 2u_k + u_{k-1}) + \frac{2}{3} A u_k + \frac{1}{6} A (u_{k+1} + u_{k-1}) \\
&\quad + \frac{1}{12} \tau^2 A^2 u_{k+1} = f_k, \\
&f_k = \frac{2}{3} f(t_k) + \frac{1}{6} (f(t_{k+1}) + f(t_{k-1})) \\
&\quad - \frac{1}{12} \tau^2 (-A f(t_{k+1}) + f''(t_{k+1})), \\
&t_k = k\tau, \quad 1 \leq k \leq N-1, \quad N\tau = 1,
\end{aligned} \tag{20}$$

$$\begin{aligned}
&u_0 = \mu, \\
&\left(I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right) \tau^{-1} (u_1 - u_0) + \frac{\tau}{2} A u_0 - \tau f_{1,1} \\
&\quad = \left(I - \frac{\tau^2}{12} A \right) \omega.
\end{aligned}$$

Applying formula (19) and nonlocal boundary conditions

$$\begin{aligned}
&u_0 = \alpha u_N + \varphi, \\
&\omega = \beta \left(\frac{7u_N - 8u_{N-1} + u_{N-2}}{6\tau} + \frac{\tau}{3} (f_N - A u_N) \right) + \psi,
\end{aligned} \tag{21}$$

one can write

$$\begin{aligned} \mu &= \alpha \left\{ \frac{1}{2} [\tilde{R}_{10} R^N - R_{10} \tilde{R}^N] \mu + \frac{1}{2} [\tilde{R}^N - R^N] R_2 \omega \right. \\ &\quad + \frac{1}{2} [\tilde{R}^N - R^N] R_3 \tau^2 f_{1,1} \\ &\quad \left. + \frac{1}{2} R_4 \sum_{s=1}^{N-1} [\tilde{R}^{N-s} - R^{N-s}] f_s \tau^2 \right\} + \varphi, \\ \omega &= \beta \left\{ \frac{1}{2} \left[\left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) \tilde{R}_{10} R^{N-2} \right. \right. \\ &\quad \left. - \left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) R_{10} \tilde{R}^{N-2} \right] \mu \\ &\quad + \frac{1}{2} \left[\left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-2} \right. \\ &\quad \left. - \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) R^{N-2} \right] R_2 \omega \\ &\quad + \frac{1}{2} \left[\left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-2} \right. \\ &\quad \left. - \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) R^{N-2} \right] R_3 \tau^2 f_{1,1} \\ &\quad + \frac{\tau}{3} f_N + \frac{1}{2} R_8 f_{N-1} \tau^2 + \frac{1}{2} R_9 f_{N-2} \tau^2 \\ &\quad + \frac{1}{2} R_4 \tau \sum_{s=1}^{N-3} \left[\left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-2-s} \right. \\ &\quad \left. - \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) R^{N-2-s} \right] f_s \tau \Big\} + \psi. \end{aligned} \quad (22)$$

Using formulas in (22), we obtain

$$\begin{aligned} \mu &= T_\tau \left\{ \left[\alpha \left(\frac{1}{2} (\tilde{R}^N - R^N) R_3 \tau^2 f_{1,1} \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{1}{2} R_4 \tau \sum_{s=1}^{N-1} (\tilde{R}^{N-s} - R^{N-s}) f_s \tau \right) + \varphi \right] \\ &\quad \times \left[I - \frac{1}{2} \left(\left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-2} \right. \right. \\ &\quad \left. \left. - \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) R^{N-2} \right) R_2 \right] + \left[\alpha \frac{1}{2} (\tilde{R}^N - R^N) R_2 \right] \\ &\quad \times \left[\beta \frac{1}{2} \left\{ \left(\left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-2} \right. \right. \right. \\ &\quad \left. \left. \left. - \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) R^{N-2} \right) \times R_3 \tau^2 f_{1,1} + \frac{2\tau}{3} f_N + R_8 f_{N-1} \tau^2 \right. \right. \\ &\quad \left. \left. + R_9 f_{N-2} \tau^2 + R_4 \tau \sum_{s=1}^{N-3} \left[\left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-2-s} \right. \right. \right. \\ &\quad \left. \left. \left. - \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) \times R^{N-2-s} \right] f_s \tau \right\} + \psi \right], \end{aligned}$$

$$\begin{aligned} \omega &= T_\tau \left\{ \left[I - \alpha \frac{1}{2} \left(\tilde{R}_{10} R^N - R_{10} \tilde{R}^N \right) \right] \right. \\ &\quad \times \left[\beta \frac{1}{2} \left\{ \left(\left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-2} \right. \right. \right. \\ &\quad \left. \left. \left. - \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) R^{N-2} \right) \times R_3 \tau^2 f_{1,1} + \frac{2\tau}{3} f_N + R_8 f_{N-1} \tau^2 \right. \right. \\ &\quad \left. \left. + R_9 f_{N-2} \tau^2 + R_4 \tau \times \sum_{s=1}^{N-3} \left(\left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-2-s} \right. \right. \right. \\ &\quad \left. \left. \left. - \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) \times R^{N-2-s} \right) f_s \tau \right\} + \psi \right] \\ &\quad + \frac{1}{2} \left[\left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) \tilde{R}_{10} R^{N-2} + \left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) R_{10} \tilde{R}^{N-2} \right] \\ &\quad \times \left[\alpha \left(\frac{1}{2} (\tilde{R}^N - R^N) R_3 \tau^2 f_{1,1} \right. \right. \\ &\quad \left. \left. + \frac{1}{2} R_4 \tau \sum_{s=1}^{N-1} (\tilde{R}^{N-s} - R^{N-s}) f_s \tau \right) + \varphi \right]. \end{aligned} \quad (23)$$

So, formulas (19) and (23) give a solution of problem (2).

Unfortunately, the estimates for $\max_{1 \leq k \leq N} \|u_k\|_H$, $\max_{1 \leq k \leq N} \|A^{1/2} u_k\|_H$, and $\max_{1 \leq k \leq N} \|Au_k\|_H$ cannot be obtained under the conditions

$$\begin{aligned} \max_{1 \leq k \leq N} \|u_k\|_H &\leq M \left\{ \sum_{s=1}^{N-1} \|A^{-1/2} f_s\|_H \tau + \|\varphi\|_H + \|A^{-1/2} \psi\|_H \right. \\ &\quad \left. + \tau \|A^{-1/2} f_{1,1}\|_H \right\}, \\ \max_{1 \leq k \leq N} \|A^{1/2} u_k\|_H &\leq M \left\{ \sum_{s=1}^{N-1} \|f_s\|_H \tau + \|A^{1/2} \varphi\|_H + \|\psi\|_H + \tau \|f_{1,1}\|_H \right\}, \\ \max_{1 \leq k \leq N} \|Au_k\|_H &\leq M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H \right. \\ &\quad \left. + \|A^{1/2} \psi\|_H + \tau \|A^{1/2} f_{1,1}\|_H \right\}. \end{aligned} \quad (24)$$

Nevertheless, we have the following theorem.

Theorem 4. Suppose that assumption (4) holds and $\varphi \in D(A^{3/2})$, $\psi \in D(A^{1/2})$. Then, for solution of difference scheme (2), the following stability estimates hold:

$$\begin{aligned}
& \max_{1 \leq k \leq N} \|u_k\|_H \\
& \leq M \left\{ \sum_{s=1}^{N-1} \|A^{-1/2} f_s\|_H \tau + \|(I + i\tau A^{1/2}) \varphi\|_H \right. \\
& \quad \left. + \|A^{-1/2} \psi\|_H + \tau \|A^{-1/2} f_{1,1}\|_H \right\}, \\
& \max_{1 \leq k \leq N} \|A^{1/2} u_k\|_H \\
& \leq M \left\{ \sum_{s=1}^{N-1} \|f_s\|_H \tau + \|A^{1/2} (I + i\tau A^{1/2}) \varphi\|_H \right. \\
& \quad \left. + \|\psi\|_H + \tau \|f_{1,1}\|_H \right\}, \\
& \max_{1 \leq k \leq N} \|A u_k\|_H \\
& \leq M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A (I + i\tau A^{1/2}) \varphi\|_H \right. \\
& \quad \left. + \|A^{1/2} \psi\|_H + \tau \|A^{1/2} f_{1,1}\|_H \right\}, \\
& \quad (25) \\
& \times \left[1 + \frac{1}{2} \left(\left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} \right. \right. \right. \\
& \quad \left. \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \left. \times \left\| \tilde{R}^{N-2} \right\|_{H \rightarrow H} \right. \\
& \quad \left. + \left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \right. \right. \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \left. \times \left\| R^{N-2} \right\|_{H \rightarrow H} \right. \\
& \quad \left. \times \left\| A^{1/2} R_2 \right\|_{H \rightarrow H} \right] \\
& + |\alpha| \frac{1}{2} \left(\left((I + i\tau A^{1/2}) \tilde{R}^N \right) \right. \\
& \quad \left. + \left\| (I + i\tau A^{1/2}) R^N \right\|_{H \rightarrow H} \right) \\
& \times \left\| A^{1/2} R_2 \right\|_{H \rightarrow H} \\
& \times \left[|\beta| \frac{1}{2} \left\{ \left(\left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \right. \right. \\
& \quad \left. \left. \left. \times \left\| \tilde{R}^{N-2} \right\|_{H \rightarrow H} \right. \right. \right. \\
& \quad \left. \left. \left. + \left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right. \right. \right. \\
& \quad \left. \left. \left. \times \left\| R^{N-2} \right\|_{H \rightarrow H} \right. \right. \right. \\
& \quad \left. \left. \left. + \left(\|\tau A^{1/2} R_3\|_{H \rightarrow H} \tau \|A^{-1/2} f_{1,1}\|_H \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. + \frac{2\tau}{3} \|A^{-1/2} f_N\|_H \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. + \|\tau R_8\|_{H \rightarrow H} \|A^{-1/2} f_{N-1}\|_H \tau \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. + \|\tau R_9\|_{H \rightarrow H} \|A^{-1/2} f_{N-2}\|_H \tau \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. + \|\tau A^{1/2} R_4\|_{H \rightarrow H} \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \times \sum_{s=1}^{N-3} \left(\left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \right. \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \left. \left. \times \left\| \tau^{-1} A^{-1/2} \tilde{R}_5\right\|_{H \rightarrow H} \right. \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \left. \left. \times \left\| \tilde{R}^{N-2-s} \right\|_{H \rightarrow H} \right) \right. \right. \right. \right. \right.
\end{aligned}$$

where M does not depend on τ , φ , ψ , $f_{1,1}(x)$, and $f_s(x)$, $1 \leq s \leq N-1$.

Proof. Using formulas in (23) and estimates (11), (12), and (14), we obtain

$$\begin{aligned}
& \|(I + i\tau A^{1/2}) \mu\|_H \leq \|T_\tau\|_{H \rightarrow H} \\
& \times \left\{ \left[|\alpha| \left(\frac{1}{2} \left(\left((I + i\tau A^{1/2}) \tilde{R}^N \right) \right. \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \left. + \left\| (I + i\tau A^{1/2}) R^N \right\|_{H \rightarrow H} \right. \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \left. \times \left\| \tau A^{1/2} R_3 \right\|_{H \rightarrow H} \tau \|A^{-1/2} f_{1,1}\|_H \right. \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \left. + \frac{1}{2} \left\| \tau A^{1/2} R_4 \right\|_{H \rightarrow H} \right. \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \left. \times \sum_{s=1}^{N-1} \left(\left((I + i\tau A^{1/2}) \tilde{R}^{N-s} \right) \right. \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \left. + \left\| (I + i\tau A^{1/2}) R^{N-s} \right\|_{H \rightarrow H} \right. \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \left. \times \left\| A^{-1/2} f_s \right\|_{H \rightarrow H} \tau \right) + \left((I + i\tau A^{1/2}) \varphi \right) \varphi \right\|_H \right] \right. \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. \left. \left. \times \left\| \tilde{R}^{N-2-s} \right\|_{H \rightarrow H} \right) \right. \right. \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\|\tau R_7\|_{H \rightarrow H} \right. \\
& \times \left. \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \\
& \times \left. \|R^{N-2-s}\|_{H \rightarrow H} \right) \\
& \times \left. \|A^{-1/2} f_s\|_H \tau \right\} + \left. \|A^{-1/2} \psi\|_H \right\} \\
& \leq M \left\{ \sum_{s=1}^{k-1} \left\| A^{-1/2} f_s \right\|_H \tau + \left\| (I + i\tau A^{1/2}) \varphi \right\|_H \right. \\
& \quad \left. + \left\| A^{-1/2} \psi \right\|_H + \tau \left\| A^{-1/2} f_{1,1} \right\|_H \right\}, \\
\|A^{-1/2} \omega\|_H & \leq \|T_\tau\|_{H \rightarrow H} \\
& \times \left[\left[1 + |\alpha| \frac{1}{2} \left(\|(I + i\tau A^{1/2})^{-1} \tilde{R}_{10}\|_{H \rightarrow H} \right. \right. \right. \\
& \quad \times \left. \|(I + i\tau A^{1/2}) R^N\|_{H \rightarrow H} \right. \\
& \quad + \left. \|(I + i\tau A^{1/2})^{-1} R_{10}\|_{H \rightarrow H} \right. \\
& \quad \times \left. \|(I + i\tau A^{1/2}) \tilde{R}^N\|_{H \rightarrow H} \right] \\
& \times \left[|\beta| \frac{1}{2} \left\{ \left(\left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} \right. \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \right. \right. \\
& \quad \times \left. \left. \left. \left. \|\tilde{R}^{N-2}\|_{H \rightarrow H} \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \right. \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right. \right. \right. \\
& \quad \times \left. \left. \left. \left. \|R^{N-2}\|_{H \rightarrow H} \right) \right. \right. \right. \\
& \times \left. \left. \left. \left. \left(\|\tau A^{1/2} R_3\|_{H \rightarrow H} \tau \left\| A^{-1/2} f_{1,1} \right\|_H \right. \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left\| \tau A^{1/2} R_4 \right\|_{H \rightarrow H} \right. \right. \right. \right. \\
& \quad \times \sum_{s=1}^{N-1} \left(\left(\|(I + i\tau A^{1/2}) \tilde{R}^{N-s}\|_{H \rightarrow H} \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left\| (I + i\tau A^{1/2}) R^{N-s} \right\|_{H \rightarrow H} \right) \right. \right. \right. \\
& \quad \times \left. \left. \left. \left. \left(\|A^{-1/2} f_s\|_H \tau \right) + \left((I + i\tau A^{1/2}) \varphi \right)_H \right) \right\} \right. \\
& \leq M \left\{ \sum_{s=1}^{k-1} \left\| A^{-1/2} f_s \right\|_H \tau + \left\| (I + i\tau A^{1/2}) \varphi \right\|_H \right. \\
& \quad + \left. \left\| A^{-1/2} \psi \right\|_H + \tau \left\| A^{-1/2} f_{1,1} \right\|_H \right\}. \tag{26}
\end{aligned}$$

Applying $A^{1/2}$ to formulas in (23), we get

$$\begin{aligned}
\|A^{1/2} (I + i\tau A^{1/2}) \mu\|_H & \leq \|T_\tau\|_{H \rightarrow H} \\
& \times \left\{ \left[|\alpha| \left(\frac{1}{2} \left(\|(I + i\tau A^{1/2}) \tilde{R}^N\|_{H \rightarrow H} \right. \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left\| (I + i\tau A^{1/2}) R^N \right\|_{H \rightarrow H} \right) \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left\| A^{-1/2} f_N \right\|_H \tau \right) \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left(\|\tau R_8\|_{H \rightarrow H} \|A^{-1/2} f_{N-1}\|_H \tau \right. \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left(\|\tau R_9\|_{H \rightarrow H} \|A^{-1/2} f_{N-2}\|_H \tau \right. \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left(\|\tau A^{1/2} R_4\|_{H \rightarrow H} \right. \right. \right. \right. \\
& \quad \times \sum_{s=1}^{N-3} \left(\left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \right. \right. \right. \\
& \quad \times \left. \left. \left. \left. \left(\|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left(\|\tau A^{1/2} R_3\|_{H \rightarrow H} \tau \left\| A^{-1/2} f_{1,1} \right\|_H \right. \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left(\|\tau A^{1/2} R_4\|_{H \rightarrow H} \right. \right. \right. \right. \\
& \quad \times \sum_{s=1}^{N-1} \left(\left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \right. \right. \right. \\
& \quad \times \left. \left. \left. \left. \left(\|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right. \right. \right. \\
& \quad + \left. \left. \left. \left. \left(\|\tau A^{1/2} f_N\|_H \tau \right) + \left((I + i\tau A^{1/2}) \varphi \right)_H \right) \right\} \right. \right. \right. \\
& \leq M \left\{ \sum_{s=1}^{k-1} \left\| A^{-1/2} f_s \right\|_H \tau + \left\| (I + i\tau A^{1/2}) \varphi \right\|_H \right. \\
& \quad + \left. \left\| A^{-1/2} \psi \right\|_H + \tau \left\| A^{-1/2} f_{1,1} \right\|_H \right\}.
\end{aligned}$$

$$\begin{aligned}
& \times \| \tau A^{1/2} R_3 \|_{H \rightarrow H} \| f_{1,1} \|_H \\
& + \frac{1}{2} \| \tau A^{1/2} R_4 \|_{H \rightarrow H} \\
& \times \sum_{s=1}^{N-1} \left(\| (I + i\tau A^{1/2}) \tilde{R}^{N-s} \|_{H \rightarrow H} \right. \\
& \quad \left. + \| (I + i\tau A^{1/2}) R^{N-s} \|_{H \rightarrow H} \right) \\
& \times \| f_s \|_{H \rightarrow H} \tau \Bigg) + \left. \left[\| f_s \|_H \tau \right] + \left. \left[\psi \right] \right\} \\
& \leq M \left\{ \sum_{s=1}^{k-1} \| f_s \|_H \tau + \left\| A^{1/2} (I + i\tau A^{1/2}) \varphi \right\|_H \right. \\
& \quad \left. + \left\| \psi \right\|_H + \tau \| f_{1,1} \|_H \right\}, \\
& \| \omega \|_H \leq \| T_\tau \|_{H \rightarrow H} \\
& \times \left\{ \left[1 + |\alpha| \frac{1}{2} \left(\| (I + i\tau A^{1/2})^{-1} \tilde{R}_{10} \|_{H \rightarrow H} \right. \right. \right. \\
& \quad \left. \left. \times \| (I + i\tau A^{1/2}) R^N \|_{H \rightarrow H} \right. \right. \\
& \quad \left. \left. + \left\| (I + i\tau A^{1/2})^{-1} R_{10} \right\|_{H \rightarrow H} \right. \right. \\
& \quad \left. \left. \times \| (I + i\tau A^{1/2}) \tilde{R}^N \|_{H \rightarrow H} \right] \right. \\
& \times \left[|\beta| \frac{1}{2} \left\{ \left(\| \tau \tilde{R}_7 \|_{H \rightarrow H} \| \tau^{-1} A^{-1/2} \tilde{R}_5 \|_{H \rightarrow H} \right. \right. \right. \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \left. \left. \times \| \tilde{R}^{N-2} \|_{H \rightarrow H} + \left(\| \tau R_7 \|_{H \rightarrow H} \| \tau^{-1} A^{-1/2} R_5 \|_{H \rightarrow H} \right. \right. \right. \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right\} \right. \\
& \quad \left. \left. \times \| A^{1/2} R_2 \|_{H \rightarrow H} \right] \right. \\
& \quad \left. + |\alpha| \frac{1}{2} \left(\| (I + i\tau A^{1/2}) \tilde{R}^N \|_{H \rightarrow H} \right. \right. \\
& \quad \left. \left. + \| (I + i\tau A^{1/2}) R^N \|_{H \rightarrow H} \right) \times \| A^{1/2} R_2 \|_{H \rightarrow H} \right. \\
& \times \left[|\beta| \frac{1}{2} \left\{ \left(\| \tau \tilde{R}_7 \|_{H \rightarrow H} \| \tau^{-1} A^{-1/2} \tilde{R}_5 \|_{H \rightarrow H} \right. \right. \right. \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \left. \left. \times \| \tilde{R}^{N-2} \|_{H \rightarrow H} \right. \right. \\
& \quad \left. \left. + \left(\| \tau R_7 \|_{H \rightarrow H} \| \tau^{-1} A^{-1/2} R_5 \|_{H \rightarrow H} \right. \right. \right. \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right\} \right. \\
& \quad \left. \left. \times \| R^{N-2} \|_{H \rightarrow H} \right] \right. \\
& \times \| \tau A^{1/2} R_3 \|_{H \rightarrow H} \| f_{1,1} \|_H \tau + \frac{2\tau}{3} \| f_N \|_H \\
& + \| \tau R_8 \|_{H \rightarrow H} \| f_{N-1} \|_H \tau + \| \tau R_9 \|_{H \rightarrow H} \\
& \times \| f_{N-2} \|_H \tau + \| \tau A^{1/2} R_4 \|_{H \rightarrow H} \\
& \times \sum_{s=1}^{N-3} \left(\| \tau \tilde{R}_7 \|_{H \rightarrow H} \| \tau^{-1} A^{-1/2} \tilde{R}_5 \|_{H \rightarrow H} \right. \\
& \quad \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \\
& \times \| \tilde{R}^{N-2-s} \|_{H \rightarrow H} \\
& + \left(\| \tau R_7 \|_{H \rightarrow H} \| \tau^{-1} A^{-1/2} R_5 \|_{H \rightarrow H} \right. \\
& \quad \left. \times \| \tilde{R}^{N-2-s} \|_{H \rightarrow H} + \left(\| \tau R_7 \|_{H \rightarrow H} \right. \right. \\
& \quad \left. \left. \times \| \tau^{-1} A^{-1/2} R_5 \|_{H \rightarrow H} + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left[\|R^{N-2-s}\|_{H \rightarrow H} \right] \left[\|f_s\|_H \tau \right] + \|\psi\|_H \\
& + \frac{1}{2} \left[\left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \right. \right. \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \left\| (I + i\tau A^{1/2})^{-1} \tilde{R}_{10} \right\|_{H \rightarrow H} \|R^{N-2}\|_{H \rightarrow H} \\
& \quad + \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} \right. \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \left\| (I + i\tau A^{1/2})^{-1} R_{10} \right\|_{H \rightarrow H} \left\| \tilde{R}^{N-2} \right\|_{H \rightarrow H} \\
& \times \left[|\alpha| \left(\frac{1}{2} \left(\left\| (I + i\tau A^{1/2}) \tilde{R}^N \right\|_{H \rightarrow H} \right. \right. \right. \\
& \quad \left. \left. + \left\| (I + i\tau A^{1/2}) R^N \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \left\| \tau A^{1/2} R_3 \right\|_{H \rightarrow H} \tau \|f_{1,1}\|_H \\
& \quad + \frac{1}{2} \left\| \tau A^{1/2} R_4 \right\|_{H \rightarrow H} \\
& \quad \times \sum_{s=1}^{N-1} \left(\left\| (I + i\tau A^{1/2}) \tilde{R}^{N-s} \right\|_{H \rightarrow H} \right. \\
& \quad \left. \left. + \left\| (I + i\tau A^{1/2}) R^{N-s} \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \left\| f_s \right\|_H \tau \left. \right] + \left\| A^{1/2} (I + i\tau A^{1/2}) \varphi \right\|_H \Bigg\} \\
& \leq M \left\{ \sum_{s=1}^{k-1} \left[\left\| f_s \right\|_H \tau + \left\| A^{1/2} (I + i\tau A^{1/2}) \varphi \right\|_H \right. \right. \\
& \quad \left. \left. + \|\psi\|_H + \tau \|f_{1,1}\|_H \right] \right\}. \tag{27}
\end{aligned}$$

$$\begin{aligned}
\omega &= T_\tau \left\{ \left[I - \alpha \frac{1}{2} (\tilde{R}_1 R^N - R_1 \tilde{R}^N) \right] \right. \\
&\quad \times \left[\beta \left\{ \frac{1}{2} \left(\left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-2} \right. \right. \right. \\
&\quad \left. \left. \left. - \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) R^{N-2} \right) \right. \right. \\
&\quad \times R_3 \tau^2 f_{1,1} + \frac{\tau}{3} f_N + \frac{1}{2} R_8 f_{N-1} \tau^2 \\
&\quad + \frac{1}{2} R_9 f_{N-2} \tau^2 \\
&\quad \left. \left. \left. + R_4 \frac{1}{2} \tau^2 \left(\sum_{s=2}^{N-3} \left(R_6 \left(\frac{R_7 R_5}{\tau} - \frac{\tau A}{3} R^2 \right) R^{N-2-s} \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. - \tilde{R}_6 \left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-2-s} \right) \right. \right. \\
&\quad \times (f_s - f_{s-1}) \\
&\quad + \left(\tilde{R}_6 \left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \right. \\
&\quad \left. \left. - R_6 \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) \right) f_{N-3} \right. \\
&\quad \left. \left. \left. - \left(\tilde{R}_6 \left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-3} \right. \right. \right. \\
&\quad \left. \left. \left. - R_6 \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) R^{N-3} \right) \right. \right. \\
&\quad \left. \left. \left. \times f_1 \right) \right] \right\} + \psi \Bigg\}, \tag{28}
\end{aligned}$$

Now, applying Abel's formula to (23), we obtain the following formulas:

$$\begin{aligned}
\mu &= T_\tau \left\{ \left[\alpha \left(\frac{1}{2} (\tilde{R}^N - R^N) R_3 \tau^2 f_{1,1} \right. \right. \right. \\
&\quad + \frac{1}{2} R_4 \tau^2 \left(\sum_{s=2}^{N-1} (R_6 R^{N-s} - \tilde{R}_6 \tilde{R}^{N-s}) \right. \\
&\quad \times (f_s - f_{s-1}) + (\tilde{R}_6 - R_6) f_{N-1} \\
&\quad \left. \left. \left. - (\tilde{R}_6 \tilde{R}^{N-1} - R_6 R^{N-1}) f_1 \right) \right] \right) + \varphi \Bigg\}
\end{aligned}$$

$$\begin{aligned}
& \times (f_s - f_{s-1}) \\
& + \left(\tilde{R}_6 \left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \right. \\
& \quad - R_6 \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) \Big) f_{N-3} \\
& \quad - \left(\tilde{R}_6 \left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) \tilde{R}^{N-3} \right. \\
& \quad \left. - R_6 \left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) \right. \\
& \quad \left. \times R^{N-3} \right) f_1 \Bigg) + \psi \\
& + \frac{1}{2} \left[\left(R_7 R_5 - \frac{\tau A}{3} R^2 \right) \tilde{R}_1 R^{N-2} \right. \\
& \quad \left. - \left(\tilde{R}_7 \tilde{R}_5 - \frac{\tau A}{3} \tilde{R}^2 \right) R_1 \tilde{R}^{N-2} \right] \\
& \times \left[\alpha \left(\frac{1}{2} (\tilde{R}^N - R^N) R_3 \tau^2 f_{1,1} \right. \right. \\
& \quad \left. + \frac{1}{2} R_4 \tau^2 \left(\sum_{s=2}^{N-1} (R_6 R^{N-s} - \tilde{R}_6 \tilde{R}^{N-s}) \right. \right. \\
& \quad \left. \times (f_s - f_{s-1}) \right. \\
& \quad \left. + (\tilde{R}_6 - R_6) f_{N-1} \right. \\
& \quad \left. - (\tilde{R}_6 \tilde{R}^{N-1} - R_6 R^{N-1}) \right. \\
& \quad \left. \times f_1 \right) \Bigg) + \varphi \Bigg] \Bigg] . \tag{29}
\end{aligned}$$

Next, let us obtain the estimates for $\|A(I + i\tau A^{1/2})\mu\|_H$ and $\|A^{1/2}\omega\|_H$. First, applying A to formula (28) and using estimates (11), (12), and (14) and the triangle inequality, one can obtain

$$\begin{aligned}
& \|A(I + i\tau A^{1/2})\mu\|_H \leq \|T_\tau\|_{H \rightarrow H} \\
& \times \left[\left[|\alpha| \left(\frac{1}{2} (\|(I + i\tau A^{1/2})\tilde{R}^N\|_{H \rightarrow H} \right. \right. \right. \\
& \quad \left. + \|(I + i\tau A^{1/2})R^N\|_{H \rightarrow H} \right) \\
& \quad \times \|T A^{1/2} R_3\|_{H \rightarrow H} \|A^{1/2} f_{1,1}\|_H \tau \\
& \quad + \frac{1}{2} \|T A^{1/2} R_4\|_{H \rightarrow H} \\
& \quad \times \left(\sum_{s=2}^{N-1} (\|T A^{1/2} R_6\|_{H \rightarrow H} \right. \\
& \quad \left. \times \|(I + i\tau A^{1/2})R^{N-s}\|_{H \rightarrow H} \right. \\
& \quad \left. + \|\tau A^{1/2} \tilde{R}_6\|_{H \rightarrow H} \right. \\
& \quad \times \|(I + i\tau A^{1/2})\tilde{R}^{N-s}\|_{H \rightarrow H} \Big) \\
& \quad \times \|T A^{1/2} R_6\|_{H \rightarrow H} \\
& \quad \times \|(I + i\tau A^{1/2})\tilde{R}^{N-s}\|_{H \rightarrow H} \Big) \\
& \quad \times \|(I + i\tau A^{1/2})\tilde{R}^{N-1}\|_{H \rightarrow H} \\
& \quad + \|\tau A^{1/2} R_6\|_{H \rightarrow H} \\
& \quad \times \|(I + i\tau A^{1/2})R^{N-1}\|_{H \rightarrow H} \\
& \quad \times \|f_1\|_H \Big) \Big) + \|A(I + i\tau A^{1/2})\varphi\|_H \Big] \\
& \times \left[1 + \frac{1}{2} \left((\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} \right. \right. \\
& \quad \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \|\tilde{R}^{N-2}\|_{H \rightarrow H} \right. \\
& \quad \left. + \left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \right. \right. \\
& \quad \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \\
& \quad \times \|R^{N-2}\|_{H \rightarrow H} \Big) \|A^{1/2} R_2\|_{H \rightarrow H} \Big] \\
& + |\alpha| \frac{1}{2} \left((\|(I + i\tau A^{1/2})\tilde{R}^N\|_{H \rightarrow H} \right. \\
& \quad \left. + \|(I + i\tau A^{1/2})R^N\|_{H \rightarrow H} \right) \|A^{1/2} R_2\|_{H \rightarrow H} \\
& \times \left[|\beta| \frac{1}{2} \left\{ \left((\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} \right. \right. \right. \\
& \quad \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \|\tilde{R}^{N-2}\|_{H \rightarrow H} \right. \\
& \quad \left. + \left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \right. \right. \\
& \quad \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \\
& \quad \times \|R^{N-2}\|_{H \rightarrow H} \Big) \times \|\tau A^{1/2} R_3\|_{H \rightarrow H} \\
& \quad \times \tau \|A^{1/2} f_{1,1}\|_H + \frac{2\tau}{3} \|A^{1/2} f_N\|_H \\
& \quad + \|\tau R_8\|_{H \rightarrow H} \|A^{1/2} f_{N-1}\|_H \tau \\
& \quad + \|\tau R_9\|_{H \rightarrow H} \|A^{1/2} f_{N-2}\|_H \tau
\end{aligned}$$

$$\begin{aligned}
& + \|\tau A^{1/2} R_4\|_{H \rightarrow H} \\
& \times \left(\sum_{s=2}^{N-3} \left(\|\tau A^{1/2} R_6\|_{H \rightarrow H} \right. \right. \\
& \quad \times \left(\|\tau R_7\|_{H \rightarrow H} \right. \\
& \quad \times \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \|R^{N-2-s}\|_{H \rightarrow H} \\
& \quad + \|\tau A^{1/2} \tilde{R}_6\|_{H \rightarrow H} \\
& \quad \times \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \right. \\
& \quad \times \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \|\tilde{R}^{N-2-s}\|_{H \rightarrow H}) \|f_s - f_{s-1}\|_H \\
& + \left(\|\tau A^{1/2} \tilde{R}_6\|_{H \rightarrow H} \right. \\
& \quad \times \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \right. \\
& \quad \times \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad + \|\tau A^{1/2} R_6\|_{H \rightarrow H} \\
& \quad \times \left(\|\tau R_7\|_{H \rightarrow H} \right. \\
& \quad \times \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \|f_{N-3}\|_H \\
& + \left(\|\tau A^{1/2} \tilde{R}_6\|_{H \rightarrow H} \right. \\
& \quad \times \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \right. \\
& \quad \times \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \|\tilde{R}^{N-3}\|_{H \rightarrow H} + \|\tau A^{1/2} R_6\|_{H \rightarrow H} \\
& \quad \times \left(\|\tau R_7\|_{H \rightarrow H} \right. \\
& \quad \times \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \left(\sum_{s=2}^{N-3} \left(\|\tau A^{1/2} R_6\|_{H \rightarrow H} \right. \right. \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right) \\
& \quad \times \left(\|\tau R^{N-3}\|_{H \rightarrow H} \right) \|f_1\|_H \Big) \Big) \Big\} \\
& \leq M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \left\| A(I + i\tau A^{1/2}) \varphi \right\|_H \right. \\
& \quad \left. + \left\| A^{1/2} \psi \right\|_H + \tau \left\| A^{1/2} f_{1,1} \right\|_H \right\}. \tag{30}
\end{aligned}$$

Second, applying $A^{1/2}$ to formula (29) and using estimates (11), (12), and (14) and the triangle inequality, we get

$$\begin{aligned}
\|A^{1/2} \omega\|_H & \leq \|T_\tau\|_{H \rightarrow H} \\
& \times \left\{ \left[1 + |\alpha| \frac{1}{2} \left(\left\| (I + i\tau A^{1/2})^{-1} \tilde{R}_{10} \right\|_{H \rightarrow H} \right. \right. \right. \\
& \quad \times \left\| (I + i\tau A^{1/2}) R^N \right\|_{H \rightarrow H} \\
& \quad + \left\| (I + i\tau A^{1/2})^{-1} R_{10} \right\|_{H \rightarrow H} \\
& \quad \left. \left. \left. + \left\| (I + i\tau A^{1/2}) \tilde{R}^N \right\|_{H \rightarrow H} \right) \right] \right. \\
& \times \left[|\beta| \frac{1}{2} \left\{ \left(\left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \right. \right. \right. \right. \\
& \quad \times \left\| \tau^{-1} A^{-1/2} \tilde{R}_5 \right\|_{H \rightarrow H} \\
& \quad + \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \right. \\
& \quad \times \left\| \tilde{R}^{N-2} \right\|_{H \rightarrow H} \\
& \quad + \left(\|\tau R_7\|_{H \rightarrow H} \right. \right. \\
& \quad \times \left\| \tau^{-1} A^{-1/2} R_5 \right\|_{H \rightarrow H} \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \left\| R^{N-2} \right\|_{H \rightarrow H} \\
& \times \left\| \tau A^{1/2} R_3 \right\|_{H \rightarrow H} \tau \left\| A^{1/2} f_{1,1} \right\|_H \\
& + \frac{2}{3} \tau \left\| A^{1/2} f_N \right\|_H \\
& + \left\| \tau R_8 \right\|_{H \rightarrow H} \left\| A^{1/2} f_{N-1} \right\|_H \tau \\
& + \left\| \tau R_9 \right\|_{H \rightarrow H} \left\| A^{1/2} f_{N-2} \right\|_H \tau \\
& + \left\| \tau A^{1/2} R_4 \right\|_{H \rightarrow H} \tau \\
& \times \left(\sum_{s=2}^{N-3} \left(\|\tau A^{1/2} R_6\|_{H \rightarrow H} \right. \right. \\
& \quad \left. \left. + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right) \\
& \times \left(\|\tau R^{N-3}\|_{H \rightarrow H} \right) \|f_1\|_H
\end{aligned}$$

$$\begin{aligned}
& \times \left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \right. \\
& \quad + \left. \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \\
& \times \|R^{N-2-s}\|_{H \rightarrow H} + \|\tau A^{1/2} \tilde{R}_6\|_{H \rightarrow H} \\
& \times \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} \right. \\
& \quad + \left. \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \\
& \times \left. \left\| \tilde{R}^{N-2-s} \right\|_{H \rightarrow H} \right) \|f_s - f_{s-1}\|_H \\
& + \left(\|\tau A^{1/2} \tilde{R}_6\|_{H \rightarrow H} \right. \\
& \quad \times \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \right. \\
& \quad \times \left. \left\| \tau^{-1} A^{-1/2} \tilde{R}_5 \right\|_{H \rightarrow H} \right. \\
& \quad + \left. \left. \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad + \left. \left. \left\| \tau A^{1/2} R_6 \right\|_{H \rightarrow H} \right. \\
& \quad \times \left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \right. \\
& \quad + \left. \left. \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right. \\
& \times \|f_{N-3}\|_H \\
& + \left(\|\tau A^{1/2} \tilde{R}_6\|_{H \rightarrow H} \right. \\
& \quad \times \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} \right. \\
& \quad + \left. \left. \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \left. \left. \left\| \tilde{R}^{N-3} \right\|_{H \rightarrow H} \right. \\
& \quad + \left. \left. \left\| \tau A^{1/2} R_6 \right\|_{H \rightarrow H} \right. \\
& \quad \times \left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \right. \\
& \quad + \left. \left. \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \left. \left. \left\| R^{N-3} \right\|_{H \rightarrow H} \right) \|f_1\|_H \right\} \\
& + \|A^{1/2} \psi\|_H \\
& + \frac{1}{2} \left[\left(\|\tau R_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} R_5\|_{H \rightarrow H} \right. \right. \\
& \quad + \left. \left. \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \left. \left. \left\| \tilde{R}_{10}(I + i\tau A^{1/2})^{-1} \right\|_{H \rightarrow H} \right. \\
& \quad \times \left. \left. \left\| (I + i\tau A^{1/2}) R^{N-2} \right\|_{H \rightarrow H} \right. \\
& \quad + \left(\|\tau \tilde{R}_7\|_{H \rightarrow H} \|\tau^{-1} A^{-1/2} \tilde{R}_5\|_{H \rightarrow H} \right. \\
& \quad + \left. \left. \left\| \frac{1}{3} \tau A^{1/2} \tilde{R}^2 \right\|_{H \rightarrow H} \right) \right. \\
& \quad \times \left. \left. \left\| R_{10}(I + i\tau A^{1/2})^{-1} \right\|_{H \rightarrow H} \right. \\
& \quad \times \left. \left. \left\| (I + i\tau A^{1/2}) \tilde{R}^{N-2} \right\|_{H \rightarrow H} \right] \\
& \times \left[|\alpha| \frac{1}{2} \left(\|\tilde{R}^N\|_{H \rightarrow H} + \|R^N\|_{H \rightarrow H} \right) \right. \\
& \quad \times \left. \left\| \tau A^{1/2} R_3 \right\|_{H \rightarrow H} \tau \|A^{1/2} f_{1,1}\|_H \right. \\
& \quad + \left. \left. \left\| \tau A^{1/2} R_4 \right\|_{H \rightarrow H} \right. \\
& \quad \times \left(\sum_{s=2}^{N-1} \left(\|\tau A^{1/2} R_6\|_{H \rightarrow H} \|R^{N-s}\|_{H \rightarrow H} \right. \right. \\
& \quad + \left. \left. \left\| \tau A^{1/2} \tilde{R}_6 \right\|_{H \rightarrow H} \|\tilde{R}^{N-s}\|_{H \rightarrow H} \right) \right. \\
& \quad \times \left. \left. \left\| f_s - f_{s-1} \right\|_H \right. \\
& \quad + \left(\|\tau A^{1/2} \tilde{R}_6\|_{H \rightarrow H} + \|\tau A^{1/2} R_6\|_{H \rightarrow H} \right) \\
& \quad \times \left. \left. \left\| f_{N-1} \right\|_{H \rightarrow H} \right. \\
& \quad + \left(\|\tau A^{1/2} \tilde{R}_6\|_{H \rightarrow H} \|\tilde{R}^{N-1}\|_{H \rightarrow H} \right. \\
& \quad + \left. \left. \left\| \tau A^{1/2} R_6 \right\|_{H \rightarrow H} \|R^{N-1}\|_{H \rightarrow H} \right) \\
& \quad \times \|f_1\|_H \right) \\
& \quad + \left(\|(I + i\tau A^{1/2})^{-1}\|_{H \rightarrow H} \|A(I + i\tau A^{1/2})\varphi\|_H \right) \} \\
& \leq M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A(I + i\tau A^{1/2})\varphi\|_H \right. \\
& \quad + \left. \left\| A^{1/2} \psi \right\|_H + \tau \|A^{1/2} f_{1,1}\|_H \right\}.
\end{aligned} \tag{31}$$

$$\begin{aligned}
& + \frac{1}{2} \left(\|A^{1/2} R_2\|_{H \rightarrow H} \|\tilde{R}^k\|_{H \rightarrow H} \right. \\
& \quad \left. + \|A^{1/2} R_2\|_{H \rightarrow H} \|R^k\|_{H \rightarrow H} \right) \\
& \times \|A^{-1/2} \omega\|_H \\
& + \frac{1}{2} \left(\|\tau A^{1/2} R_3\|_{H \rightarrow H} \|\tilde{R}^k\|_{H \rightarrow H} \right. \\
& \quad \left. + \|\tau A^{1/2} R_3\|_{H \rightarrow H} \|R^k\|_{H \rightarrow H} \right) \tau \|A^{-1/2} f_{1,1}\|_H \\
& + \frac{1}{2} \left\| \tau A^{1/2} R_4 \right\|_{H \rightarrow H} \sum_{s=1}^{k-1} \left[\|\tilde{R}^{k-s}\|_{H \rightarrow H} + \|R^{k-s}\|_{H \rightarrow H} \right] \\
& \times \|A^{-1/2} f_s\|_H \tau \\
& \leq M \left\{ \sum_{s=1}^{N-1} \left[\|A^{-1/2} f_s\|_H \tau + \|(I + i\tau A^{1/2}) \varphi\|_H \right. \right. \\
& \quad \left. \left. + \|A^{-1/2} \psi\|_H + \tau \|A^{-1/2} f_{1,1}\|_H \right] \right\}
\end{aligned} \tag{32}$$

for any $k \geq 2$. Applying $A^{1/2}$ to (19), we get

$$\begin{aligned}
& \|A^{1/2} u_k\|_H \\
& \leq \frac{1}{2} \left(\left\| \tilde{R}_{10} (I + i\tau A^{1/2})^{-1} \right\|_{H \rightarrow H} \|\tilde{R}^k\|_{H \rightarrow H} \right. \\
& \quad \left. + \left\| R_{10} (I + i\tau A^{1/2})^{-1} \right\|_{H \rightarrow H} \|\tilde{R}^k\|_{H \rightarrow H} \right) \\
& \times \|A^{1/2} (I + i\tau A^{1/2}) \mu\|_H \\
& + \frac{1}{2} \left(\|A^{1/2} R_2\|_{H \rightarrow H} \|\tilde{R}^k\|_{H \rightarrow H} \right. \\
& \quad \left. + \|A^{1/2} R_2\|_{H \rightarrow H} \|R^k\|_{H \rightarrow H} \right) \times \|\omega\|_H \\
& + \frac{1}{2} \left(\|\tau A^{1/2} R_3\|_{H \rightarrow H} \|\tilde{R}^k\|_{H \rightarrow H} \right. \\
& \quad \left. + \|\tau A^{1/2} R_3\|_{H \rightarrow H} \|R^k\|_{H \rightarrow H} \right) \times \tau \|f_{1,1}\|_H \\
& + \frac{1}{2} \left\| \tau A^{1/2} R_4 \right\|_{H \rightarrow H} \sum_{s=1}^{k-1} \left[\|\tilde{R}^{k-s}\|_{H \rightarrow H} + \|R^{k-s}\|_{H \rightarrow H} \right] \\
& \times \|f_s\|_H \tau \leq M \left\{ \sum_{s=1}^{N-1} \left[\|f_s\|_H \tau + \left\| A^{1/2} (I + i\tau A^{1/2}) \varphi \right\|_H \right. \right. \\
& \quad \left. \left. + \|\psi\|_H + \tau \|f_{1,1}\|_H \right] \right\}
\end{aligned} \tag{33}$$

for $k \geq 2$. Now, applying Abel's formula to (19), we have

$$\begin{aligned}
u_k & = \frac{1}{2} [\tilde{R}_1 R^k - R_1 \tilde{R}^k] \mu + \frac{1}{2} [\tilde{R}^k - R^k] R_2 \omega \\
& \quad + \frac{1}{2} [\tilde{R}^k - R^k] R_3 \tau^2 f_{1,1} \\
& \quad + \tau^2 R_4 \frac{1}{2} \left(\sum_{s=2}^{k-1} [R_6 R^{k-s} - \tilde{R}_6 \tilde{R}^{k-s}] (f_s - f_{s-1}) \right. \\
& \quad \left. + (\tilde{R}_6 - R_6) f_{k-1} \right. \\
& \quad \left. - [\tilde{R}_6 \tilde{R}^{k-1} - R_6 R^{k-1}] f_1 \right), \quad 2 \leq k \leq N.
\end{aligned} \tag{34}$$

Applying A to formula (34) and using estimates (11) and (12) and the triangle inequality, we obtain

$$\begin{aligned}
\|Au_k\|_H & \leq \frac{1}{2} \left(\left\| \tilde{R}_{10} (I + i\tau A^{1/2})^{-1} \right\|_{H \rightarrow H} \|\tilde{R}^k\|_{H \rightarrow H} \right. \\
& \quad \left. + \left\| R_{10} (I + i\tau A^{1/2})^{-1} \right\|_{H \rightarrow H} \|\tilde{R}^k\|_{H \rightarrow H} \right) \\
& \times \|A (I + i\tau A^{1/2}) \mu\|_H \\
& + \frac{1}{2} \left(\|A^{1/2} R_2\|_{H \rightarrow H} \|\tilde{R}^k\|_{H \rightarrow H} \right. \\
& \quad \left. + \|A^{1/2} R_2\|_{H \rightarrow H} \|R^k\|_{H \rightarrow H} \right) \\
& \times \|A^{1/2} \omega\|_H + \frac{1}{2} \left(\|\tau A^{-1/2} R_3\|_{H \rightarrow H} \|\tilde{R}^k\|_{H \rightarrow H} \right. \\
& \quad \left. + \|\tau A^{-1/2} R_3\|_{H \rightarrow H} \|R^k\|_{H \rightarrow H} \right) \\
& \times \tau \|A^{1/2} f_{1,1}\|_H + \frac{1}{2} \|\tau A^{1/2} R_4\|_{H \rightarrow H} \\
& \times \left(\sum_{s=2}^{k-1} \left[\|\tau A^{1/2} R_6\|_{H \rightarrow H} \|R^{k-s}\|_{H \rightarrow H} \right. \right. \\
& \quad \left. \left. + \|\tau A^{1/2} \tilde{R}_6\|_{H \rightarrow H} \|\tilde{R}^{k-s}\|_{H \rightarrow H} \right] \right. \\
& \quad \left. \times \|f_s - f_{s-1}\|_H \right. \\
& \quad \left. + (\|\tau A^{1/2} \tilde{R}_6\|_{H \rightarrow H} + \|\tau A^{1/2} R_6\|_{H \rightarrow H}) \|f_{k-1}\|_H \right. \\
& \quad \left. + [\|\tau A^{1/2} \tilde{R}_6\|_{H \rightarrow H} \|\tilde{R}^{k-1}\|_{H \rightarrow H} \right. \\
& \quad \left. + \|\tau A^{1/2} R_6\|_{H \rightarrow H} \|R^{k-1}\|_{H \rightarrow H}] \|f_1\|_H \right)
\end{aligned}$$

$$\leq M \left\{ \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A(I + i\tau A^{1/2})\varphi\|_H \right. \\ \left. + \|A^{1/2}\psi\|_H + \tau \|A^{1/2}f_{1,1}\|_H \right\} \quad (35)$$

for $k \geq 2$. Theorem 4 is proved. \square

Note that the stability estimates obtained previously permit us to get the convergence estimate of difference scheme (2) under the smoothness property of solution (1). Actually, under the condition $u(t) \in C([0, 1], H)$, we can obtain the third order of accuracy for the error of difference scheme (2). Since $u^{(6)}(t) = -A^3u(t) + A^2f(t) - Af''(t) + f^{(4)}(t)$, this condition is satisfied under the given data $\varphi \in D(A^3)$, $\psi \in D(A^{5/2})$, $f'(t) \in D(A^2)$, and $f(0) \in D(A^3)$.

Now, let us give application of this abstract result for nonlocal boundary value problem

$$\begin{aligned} u_{tt} - (a(x)u_x)_x + \delta u &= f(t, x), \quad 0 < t < 1, \quad 0 < x < 1, \\ u(0, x) &= \alpha u(1, x) + \varphi(x), \quad 0 \leq x \leq 1, \\ u_t(0, x) &= \beta u_t(1, x) + \psi(x), \quad 0 \leq x \leq 1, \\ u(t, 0) &= u(t, 1), \quad u_x(t, 0) = u_x(t, 1), \quad 0 \leq t \leq 1 \end{aligned} \quad (36)$$

for hyperbolic equation. Problem (36) has a unique smooth solution $u(t, x)$, $\delta > 0$ and the smooth functions $a(x) \geq a > 0$ ($a(0) = a(1)$, $x \in (0, 1)$), $\varphi(x)$, $\psi(x)$ ($x \in [0, 1]$), and $f(t, x)$ ($t, x \in [0, 1]$). This allows us to reduce mixed problem (36) to nonlocal boundary value problem (1) in a Hilbert space $H = L_2[0, 1]$ with a self-adjoint positive definite operator A^x defined by (36).

The discretization of problem (36) is carried out in two steps. In the first step, let us define the grid space

$$[0, 1]_h = \{x : x_r = rh, 0 \leq r \leq K, Kh = 1\}. \quad (37)$$

We introduce Hilbert space $L_{2h} = L_2([0, 1]_h)$, $W_{2h}^1 = W_{2h}^1([0, 1]_h)$, and $W_{2h}^2 = W_{2h}^2([0, 1]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_r\}_1^{K-1}$ defined on $[0, 1]_h$, and we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \{-(a(x)\varphi_x)_{x,r} + \delta \varphi_r\}_1^{K-1}, \quad (38)$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_r\}_0^K$ satisfying the conditions $\varphi_0 = \varphi_K$, $\varphi_1 - \varphi_0 = \varphi_K - \varphi_{K-1}$.

With the help of A_h^x , we arrive at the nonlocal boundary value problem

$$\begin{aligned} \frac{d^2v^h}{dt^2}(t, x) + A_h^x v^h(t, x) &= f^h(t, x), \\ 0 < t < 1, \quad x \in [0, 1]_h, \\ v^h(0, x) &= \alpha v^h(1, x) + \varphi^h(x), \quad x \in [0, 1]_h, \\ v_t^h(0, x) &= \beta v_t^h(1, x) + \psi^h(x), \quad x \in [0, 1]_h \end{aligned} \quad (39)$$

for a system of ordinary differential equations.

In the second step, we replace problem (2) with difference scheme (40)

$$\begin{aligned} &\tau^{-2} (u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)) + \frac{2}{3} A_h^x u_k^h(x) \\ &+ \frac{1}{6} A_h^x (u_{k+1}^h(x) + u_{k-1}^h(x)) \\ &+ \frac{1}{12} \tau^2 (A_h^x)^2 u_{k+1}^h(x) = f_k^h(x), \\ f_k^h(x) &= \frac{2}{3} f^h(t_k, x) \\ &+ \frac{1}{6} (f^h(t_{k+1}, x) + f^h(t_{k-1}, x)) \\ &- \frac{1}{12} \tau^2 (-Af^h(t_{k+1}, x) + f_{tt}^h(t_{k+1}, x)), \quad x \in [0, 1]_h, \\ t_k &= k\tau, \quad N\tau = 1, \quad 1 \leq k \leq N-1, \\ u_0^h(x) &= \alpha u_N^h(x) + \varphi^h(x), \quad x \in [0, 1]_h, \\ \left(I + \frac{\tau^2}{12} (A_h^x) + \frac{\tau^4}{144} (A_h^x)^2 \right) \tau^{-1} (u_1^h(x) - u_0^h(x)) \\ &+ \frac{\tau}{2} (A_h^x) \varphi^h(x) - \tau f_{1,1}^h(x) \\ &= \beta \left(I - \frac{\tau^2}{12} (A_h^x) \right) \\ &\times \left(\frac{1}{6\tau} (7u_N^h(x) - 8u_{N-1}^h(x) + u_{N-2}^h(x)) \right. \\ &\left. + \frac{\tau}{3} (f_N^h(x) - Au_N^h(x)) \right) \\ &+ \left(I - \frac{\tau^2}{12} (A_h^x) \right) \psi^h(x), \quad x \in [0, 1]_h, \\ f_{1,1}^h(x) &= \frac{1}{2} f^h(0, x) + \frac{\tau}{6} f_t^h(0, x). \end{aligned} \quad (40)$$

Theorem 5. Let τ and h be sufficiently small numbers. Then, the solution of difference scheme (40) satisfies the following stability estimates:

$$\begin{aligned} &\max_{0 \leq k \leq N} \|u_k^h\|_{L_{2h}} + \max_{0 \leq k \leq N} \|u_k^h\|_{W_{2h}^1} \\ &\leq M_1 \left[\max_{1 \leq k \leq N-1} \|f_k^h\|_{L_{2h}} + \|\psi^h\|_{L_{2h}} \right. \\ &\quad \left. + \|\varphi^h\|_{W_{2h}^1} + \tau \|\varphi^h\|_{W_{2h}^2} + \tau \|f_{1,1}^h\|_{L_{2h}} \right], \\ &\max_{1 \leq k \leq N-1} \|\tau^{-2} (u_{k+1}^h - 2u_k^h + u_{k-1}^h)\|_{L_{2h}} + \max_{0 \leq k \leq N} \|u_k^h\|_{W_{2h}^2} \quad (41) \\ &\leq M_1 \left[\|f_1^h\|_{L_{2h}} + \max_{2 \leq k \leq N-1} \|\tau^{-1} (f_k^h - f_{k-1}^h)\|_{L_{2h}} \right. \\ &\quad \left. + \|\psi^h\|_{W_{2h}^1} + \|\varphi^h\|_{W_{2h}^2} \right. \\ &\quad \left. + \tau \|\varphi^h\|_{W_{2h}^3} + \tau \|f_{1,1}^h\|_{W_{2h}^1} \right]. \end{aligned}$$

Here, M_1 does not depend on τ , h , $\varphi^h(x)$, $\psi^h(x)$, $f_{1,1}^h(x)$, and $f_k^h(x)$, $1 \leq k < N$.

The proof of Theorem 5 is based on the proof of abstract Theorem 4 and the symmetry property of operator A_h^x defined by (38).

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