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Research Article

Vague Filters of Residuated Lattices

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Notions of vague filters, subpositive implicative vague filters, and Boolean vague filters of a residuated lattice are introduced and some related properties are investigated. The characterizations of (subpositive implicative, Boolean) vague filters is obtained. We prove that the set of all vague filters of a residuated lattice forms a complete lattice and we find its distributive sublattices. The relation among subpositive implicative vague filters and Boolean vague filters are obtained and it is proved that subpositive implicative vague filters are equivalent to Boolean vague filters.

1. Introduction

In the classical set, there are only two possibilities for any elements: in or not in the set. Hence the values of elements in a set are only one of 0 and 1. Therefore, this theory cannot handle the data with ambiguity and uncertainty.

Zadeh introduced fuzzy set theory in 1965 [1] to handle such ambiguity and uncertainty by generalizing the notion of membership in a set. In a fuzzy set A each element x is associated with a point-value $\mu_A(x)$ selected from the unit interval $[0,1]$, which is termed the grade of membership in the set. This membership degree contains the evidences for both supporting and opposing x.

A number of generalizations of Zadeh's fuzzy set theory are intuitionistic fuzzy theory, L-fuzzy theory, and vague theory. Gau and Buehrer proposed the concept of vague set in 1993 [2], by replacing the value of an element in a set with a subinterval of $[0,1]$. Namely, a true membership function $t_v(x)$ and a false-membership function $f_v(x)$ are used to describe the boundaries of membership degree. These two boundaries form a subinterval $[t_v(x), 1 - f_v(x)]$ of $[0,1]$. The vague set theory improves description of the objective real world, becoming a promising tool to deal with inexact, uncertain, or vague knowledge. Many researchers have applied this theory to many situations, such as fuzzy control, decision-making, knowledge discovery, and fault diagnosis. Recently in [3], Jun and Park introduced the notion of vague ideal in pseudo MV-algebras and Broumand Saeid [4] introduced the notion of vague BCK/BCI-algebras.

The concept of residuated lattices was introduced by Ward and Dilworth [5] as a generalization of the structure of the set of ideals of a ring. These algebras are a common structure among algebras associated with logical systems (see [6–9]). The residuated lattices have interesting algebraic and logical properties. The main example of residuated lattices related to logic is and BL-algebras. A basic logic algebra (BL-algebra for short) is an important class of logical algebras introduced by Hajek [10] in order to provide an algebraic proof of the completeness of "Basic Logic" (BL for short). Continuous t-norm based fuzzy logics have been widely applied to fuzzy mathematics, artificial intelligence, and other areas. The filter theory plays an important role in studying these logical algebras and many authors discussed the notion of filters of these logical algebras (see [11, 12]). From a logical point of view, a filter corresponds to a set of provable formulas.

In this paper, the concept of vague sets is applied to residuated lattices. In Section 2, some basic definitions and results are explained. In Section 3, we introduce the notion of vague filters of a residuated lattice and investigate some related properties. Also, we obtain the characterizations of vague filters. In Section 4, the smallest vague filters containing a given vague set are established and we obtain the distributive sublattice of the set of all vague filters of a residuated lattice. In Section 5, notions of vague filters, subpositive implicative vague filters, and Boolean vague filters of a residuated lattice are introduced and some related properties are obtained.

2. Preliminaries

We recall some definitions and theorems which will be needed in this paper.

Definition 1 (see [6]). A *residuated lattice* is an algebraic structure $(L, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1)$ such that

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice with the least element 0 and the greatest element 1,

(2) $(L, *, 1)$ is a monoid; that is, $*$ is associative and $x * 1 = 1 * x = x$,

(3) $x * y \leq z$ if and only if $x \leq y \rightarrow z$ if and only if $y \leq x \rightsquigarrow z$, for all $x, y, z \in L$.

In the rest of this paper, we denote the residuated lattice $(L, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1)$ by L, $\neg x = x \rightarrow 0$, and $\sim x = x \rightsquigarrow 0$.

Proposition 2 (see [6, 9]). *Let L be a residuated lattice. Then we have the following properties: for all* $x, y, z \in L$,

(1) $x \rightarrow x = x \rightsquigarrow x = 1$,

(2) $1 \rightarrow x = 1 \rightsquigarrow x = x$,

(3) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$,

(4) $x \leq y$ *if and only if* $x \rightarrow y = 1$ *if and only if* $x \rightsquigarrow y = 1$,

(5) *if* $x \leq y$, *then* $x * z \leq y * z$, $z * x \leq z * y$,

(6) *if* $x \leq y$, *then* $z \rightarrow x \leq z \rightarrow y$, $z \rightsquigarrow x \leq z \rightsquigarrow y$,

(7) *if* $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$, $y \rightsquigarrow z \leq x \rightsquigarrow z$,

(8) $x \leq (x \rightsquigarrow y) \rightarrow y$, $x \leq (x \rightarrow y) \rightsquigarrow y$,

(9) $y \rightarrow z \leq (x \rightarrow y) \rightsquigarrow (x \rightarrow z)$, $y \rightsquigarrow z \leq (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$,

(10) $x \rightarrow (y \rightsquigarrow (x * y)) = 1$, $y \rightsquigarrow (x \rightarrow (x * y)) = 1$.

Definition 3 (see [6, 9]). Let F be a nonempty subset of a residuated lattice L. F is called a *filter* if

(F1) $x, y \in F$ imply $x * y \in F$,

(F2) $x \leq y$ and $x \in F$ imply $y \in F$.

Theorem 4 (see [6, 9]). *A nonempty subset F of a residuated lattice L is a filter if and only if*

(F3) $1 \in F$ *and* $x, x \rightarrow y \in F$ *imply* $y \in F$, *or*

(F3′) $1 \in F$ *and* $x, x \rightsquigarrow y \in F$ *imply* $y \in F$.

Definition 5 (see [11]). Let F be a nonempty subset of a residuated lattice L. F is called a *subpositive implicative filter* of L if

(SPIF1) $1 \in F$,

(SPIF2) $((x \rightarrow y) * z) \rightsquigarrow ((y \rightsquigarrow x) \rightarrow x) \in F$ and $z \in F$ imply $(x \rightarrow y) \rightsquigarrow y \in F$ for any $x, y, z \in L$,

(SPIF3) $(z * (x \rightsquigarrow y)) \rightarrow ((y \rightarrow x) \rightsquigarrow x) \in F$ and $z \in F$ imply $(x \rightsquigarrow y) \rightarrow y \in F$ for any $x, y, z \in L$.

Proposition 6 (see [11]). *Let F be a filter of L. F is a subpositive implicative filter if and only if*

(SPIF4) $(x \rightarrow y) \rightsquigarrow ((y \rightsquigarrow x) \rightarrow x) \in F$ *imply* $(x \rightarrow y) \rightsquigarrow y \in F$ *for any* $x, y \in L$,

(SPIF5) $(x \rightsquigarrow y) \rightarrow ((y \rightarrow x) \rightsquigarrow x) \in F$ *imply* $(x \rightsquigarrow y) \rightarrow y \in F$ *for any* $x, y \in L$.

Definition 7 (see [1]). Let X be a nonempty set. A *fuzzy set* in X is a mapping $\mu : X \rightarrow [0, 1]$.

Proposition 8 (see [12]). *Let L be a residuated lattice. A fuzzy set* μ *is called a fuzzy filter in L if it satisfies*

(fF1) $x \leq y$ *imply* $\mu(x) \leq \mu(y)$,

(fF2) $\min\{\mu(x), \mu(y)\} \leq \mu(x * y)$,

for all $x, y \in L$.

Definition 9 (see [13]). A *vague set* A in the universe of discourse X is characterized by two membership functions given by

(i) a true membership function $t_A : X \rightarrow [0, 1]$,

(ii) a false membership function $f_A : X \rightarrow [0, 1]$,

where $t_A(x)$ is a lower bound on the grade of membership of X derived from the "evidence for x," $f_A(x)$ is a lower bound on the negation of x derived from the "evidence against x," and $t_A(x) + f_A(x) \leq 1$.

Thus the grade of membership of x in the vague set A is bounded by a subinterval $[t_A(x), 1 - f_A(x)]$ of $[0, 1]$. This indicates that if the actual grade of membership of x is $\mu(x)$, then $t_A(x) \leq \mu(x) \leq 1 - f_A(x)$. The vague set A is written as $A = \{\langle x, [t_A(x), 1 - f_A(x)] \rangle : x \in X\}$, where the interval $[t_A(x), 1 - f_A(x)]$ is called the *vague value* of x in A, denoted by $V_A(x)$.

Definition 10 (see [13]). A vague set A of a set X is called

(1) the *zero vague set* of X if $t_A(x) = 0$ and $f_A(x) = 1$ for all $x \in X$,

(2) the *unit vague set* of X if $t_A(x) = 1$ and $f_A(x) = 0$ for all $x \in X$,

(3) the α-*vague set* of X if $t_A(x) = \alpha$ and $f_A(x) = 1 - \alpha$ for all $x \in X$, where $\alpha \in (0, 1)$.

Let $I[0, 1]$ denote the family of all closed subintervals of $[0, 1]$. Now we define the refined minimum (briefly, imin) and an order $\leq$ on elements $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ of $I[0,1]$ as

$$\text{imin}(I_1, I_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}],$$
$$I_1 \leq I_2 \quad \text{iff } a_1 \leq a_2,\ b_1 \leq b_2. \tag{1}$$

Similarly we can define $\geq$, $=$, and imax. Then the concept of imin and imax could be extended to define iinf and isup of infinite number of elements of $I[0,1]$. It is a known fact that $L = \{I[0,1], \text{iinf}, \text{isup}, \leq\}$ is a lattice with universal bounds zero vague set and unit vague set. For $\alpha, \beta \in [0,1]$ we now define (α, β)-cut and α-cut of a vague set.

Definition 11 (see [13]). Let A be a vague set of a universe X with the true-membership function t_A and false-membership function f_A. The (α, β)-cut of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by

$$A_{(\alpha,\beta)} = \{x \in X : V_A(x) \geq [\alpha, \beta]\}, \quad \text{where } \alpha \leq \beta. \tag{2}$$

Clearly $A_{(0,0)} = X$. The (α, β)-cuts are also called vague-cuts of the vague set A.

Definition 12 (see [13]). The α-cut of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$. Equivalently, we can define the α-cut as $A_\alpha = \{x \in X : t_A(x) \geq \alpha\}$.

3. Vague Filters of Residuated Lattices

In this section, we define the notion of vague filters of residuated lattices and obtain some related results.

Definition 13. A vague set A of a residuated lattice L is called a *vague filter* of L if the following conditions hold:

(VF1) if $x \leq y$, then $V_A(x) \leq V_A(y)$, for all $x, y \in L$,

(VF2) $\text{imin}\{V_A(x), V_A(y)\} \leq V_A(x * y)$, for all $x, y \in L$.

That is,

(1) $x \leq y$ implies $t_A(x) \leq t_A(y)$ and $1 - f_A(x) \leq 1 - f_A(y)$, for all $x, y \in L$,

(2) $\min\{t_A(x), t_A(y)\} \leq t_A(x * y)$ and $\min\{1 - f_A(x), 1 - f_A(y)\} \leq 1 - f_A(x * y)$, for all $x, y \in L$.

In the following proposition, we will show that the vague filters of a residuated lattice exist.

Proposition 14. *Unit vague set and α-vague set of a residuated lattice L are vague filters of L.*

Proof. Let A be a α-vague set of L. It is clear that A satisfies (VF1). For $x, y \in L$ we have

$$t_A(x * y) = \alpha = \min\{\alpha, \alpha\} = \min\{t_A(x), t_A(y)\},$$
$$1 - f_A(x * y) = \alpha = \min\{\alpha, \alpha\} = \min\{1 - f_A(x), 1 - f_A(y)\}. \tag{3}$$

Thus $\text{imin}\{V_A(x), V_A(y)\} \leq V_A(x * y)$, for all $x, y \in L$. Hence it is a vague filter of L. The proof of the other cases is similar. □

Lemma 15. *Let A be a vague set of a residuated lattice L such that it satisfies (VF1). Then the following are equivalent:*

(VF4) $\text{imin}\{V_A(x), V_A(x \rightarrow y)\} \leq V_A(y)$,

(VF5) $\text{imin}\{V_A(x), V_A(x \rightsquigarrow y)\} \leq V_A(y)$,

for all $x, y \in L$.

Proof. Suppose that A satisfies (VF4). We have $x \leq (x \rightsquigarrow y) \rightarrow y$. By (VF1), $V_A(x) \leq V_A((x \rightsquigarrow y) \rightarrow y)$. We get that

$$\text{imin}\{V_A(x), V_A(x \rightsquigarrow y)\} \leq \text{imin}\{V_A((x \rightsquigarrow y) \longrightarrow y), V_A(x \rightsquigarrow y)\} \leq V_A(y). \tag{4}$$

Conversely, if A satisfies (VF5), then the inequality $x \leq (x \rightarrow y) \rightsquigarrow y$ analogously entails (VF4). □

Theorem 16. *Let A be a vague set of a residuated lattice L. Then A is a vague filter if and only if it satisfies (VF4) and*

(VF6) $V_A(x) \leq V_A(1)$, *for all $x \in L$.*

Proof. Let A be a vague filter of L. Since $x \leq 1$, then $V_A(x) \leq V_A(1)$, for all $x \in L$ by (VF1). We have $x * (x \rightarrow y) \leq y$. By (VF1) and (VF2) we get

$$\text{imin}\{V_A(x), V_A(x \longrightarrow y)\} \leq V_A(x * (x \longrightarrow y)) \leq V_A(y). \tag{5}$$

Hence (VF4) hold.

Conversely, let A satisfy (VF4) and (VF6). Since $x \leq y$, then $x \rightarrow y = 1$;

$$V_A(y) \geq \text{imin}\{V_A(x \longrightarrow y), V_A(x)\} = \text{imin}\{V_A(1), V_A(x)\} = V_A(x). \tag{6}$$

Hence V_A is order-preserving and (VF1) holds. By Proposition 2 part (10), $x \rightarrow (y \rightsquigarrow (x * y)) = 1$. By (VF4), we have

$$V_A(y \rightsquigarrow (x * y)) \geq \text{imin}\{V_A(x), V_A(x \longrightarrow (y \rightsquigarrow (x * y)))\} = V_A(x). \tag{7}$$

By Lemma 15,

$$V_A(x * y) \geq \text{imin}\{V_A(y \rightsquigarrow (x * y)), V_A(y)\} \geq \text{imin}\{V_A(x), V_A(y)\}. \tag{8}$$

Hence (VF2) holds. □

Theorem 17. *Let A be a vague set of a residuated lattice L. Then A is a vague filter if and only if it satisfies (VF5) and (VF6).*

Definition 18. A vague set A of a residuated lattice L is called a *vague lattice filter* of L if it satisfies (VF1) and

(VLF2) $\operatorname{imin}\{V_A(x), V_A(y)\} \leq V_A(x \wedge y)$, for all $x, y \in L$.

Proposition 19. *Let A be a vague filter of a residuated lattice L; then A is a vague lattice filter.*

In the following example, we will show that the converse of Proposition 19 may not be true in general.

Example 20. Let $\{0, a, b, c, 1\}$ such that $0 < a < b$, $c < 1$, where b and c are incomparable. Define the operations $*$, $\to$, and $\rightsquigarrow$ as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	a	b	a	b
c	0	0	0	c	c
1	0	a	b	c	1

$\longrightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

(9)

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	c	1	c	1
c	b	b	b	1	1
1	0	a	b	c	1.

Then L is a residuated lattice. Let A be the vague set defined as follows:

$$A = \{\langle 0, [0.1, 0.2]\rangle, \langle a, [0.3, 0.4]\rangle, \langle b, [0.3, 0.4]\rangle, \langle c, [0.3, 0.8]\rangle, \langle 0, [0.5, 0.9]\rangle\}. \tag{10}$$

It is routine to verify that A is a vague lattice filter, but it is not a vague filter of L.

Let B be the vague set defined as follows:

$$B = \{\langle 0, [0.2, 0.5]\rangle, \langle a, [0.2, 0.5]\rangle, \langle b, [0.2, 0.5]\rangle, \langle c, [0.4, 0.7]\rangle, \langle 0, [0.7, 0.7]\rangle\}. \tag{11}$$

Then B is a vague filter of L.

In the following theorems, we will study the relationship between filters and vague filters of a residuated lattice.

Theorem 21. *Let A be a vague set of a residuated lattice L. Then A is a vague filter of L if and only if, for all $\alpha, \beta \in [0, 1]$, the set $A_{(\alpha,\beta)}$ is either empty or a filter of L, where $\alpha \leq \beta$.*

Proof. Suppose that A is a vague filter of L and $\alpha, \beta \in [0, 1]$. Let $A_{(\alpha,\beta)} \neq \emptyset$. We will show that $A_{(\alpha,\beta)}$ is a filter of L.

(1) Suppose that $x \leq y$ and $x \in A_{(\alpha,\beta)}$. By (VF1), $V_A(y) \geq V_A(x) \geq [\alpha, \beta]$. Therefor $y \in A_{(\alpha,\beta)}$.

(2) Suppose that $x, y \in A_{(\alpha,\beta)}$. Then $V_A(x), V_A(y) \geq [\alpha, \beta]$. By (VF2), we have $V_A(x * y) \geq \operatorname{imin}\{V_A(x), V_A(y)\} \geq [\alpha, \beta]$. Hence $x * y \in A_{(\alpha,\beta)}$ and then $A_{(\alpha,\beta)}$ is a filter of L.

Conversely, suppose that, for all $\alpha, \beta \in [0, 1]$, the sets $A_{(\alpha,\beta)}$ is either empty or a filter of L. Let $t_A(x) = \alpha_1$, $t_A(y) = \alpha_2$, $1 - f_A(x) = \beta_1$, and $1 - f_A(y) = \beta_2$. Put $\alpha = \min\{\alpha_1, \alpha_2\}$ and $\beta = \min\{1 - \beta_1, 1 - \beta_2\}$. Then $t_A(x), t_A(y) \geq \alpha$ and $1 - f_A(x), 1 - f_A(y) \geq \beta$. Hence $V_A(y), V_A(x) \geq [\alpha, \beta]$; that is, $x, y \in A_{(\alpha,\beta)}$. So $A_{(\alpha,\beta)} \neq \emptyset$ and, by assumption, $A_{(\alpha,\beta)}$ is a filter of L. We obtain that $x * y \in A_{(\alpha,\beta)}$; that is,

$$\begin{aligned} V_A(x * y) &\geq [\alpha, \beta] \\ &= [\min\{\alpha_1, \alpha_2\}, \min\{1 - \beta_1, 1 - \beta_2\}] \\ &= \operatorname{imin}\{[t_A(x), 1 - f_A(x)]\,[t_A(y), 1 - f_A(y)]\} \\ &= \operatorname{imin}\{V_A(x), V_A(y)\}. \end{aligned} \tag{12}$$

Hence (VF2) holds.

Suppose that $x \leq y$, $t_A(x) = \alpha$, and $1 - f_A(x) = \beta$. Hence $x \in A_{(\alpha,\beta)}$. By assumption $A_{(\alpha,\beta)}$ is a filter. We get that $y \in A_{(\alpha,\beta)}$; that is, $V_A(y) \geq [\alpha, \beta] = [t_A(x), 1 - f_A(x)] = V_A(x)$. Hence A is a vague filter of L. □

The filters like $A_{(\alpha,\beta)}$ are also called *vague-cuts filters* of L.

Corollary 22. *Let A be a vague filter of a residuated lattice L. Then for all $\alpha \in [0, 1]$, the set A_α is either empty or a filter of L.*

Corollary 23. *Any filter F of a residuated lattice L is a vague-cut filter of some vague filter of L.*

Proof. For all $x \in L$, define

$$1 - f_A(x) = t_A(x) = \begin{cases} 1 & \text{if } x \in F \\ \alpha & \text{otherwise.} \end{cases} \tag{13}$$

Hence we have $V_A(x) = [1, 1]$, if $x \in F$ and $V_A(x) = [\alpha, \alpha]$, otherwise, where $\alpha \in (0, 1)$. It is clear that $F = A_{(1,1)}$. Let $x, y \in L$. If $x, y \in F$, then $x * y \in F$. We have $V_A(x * y) = [1, 1] = \operatorname{imin}\{V_A(x), V_A(y)\}$. If one of x or y does not belong to F, then one of $V_A(x)$ and $V_A(y)$ is equal to $[\alpha, \alpha]$. Therefore $V_A(x * y) \geq [\alpha, \alpha] = \operatorname{imin}\{V_A(x), V_A(y)\}$. Suppose that $x \leq y$. If $x \in F$, then $y \in F$. Hence $V_A(x) = V_A(y)$. If $x \notin F$, then $V_A(x) = [\alpha, \alpha]$. We have $V_A(x) \leq V_A(y)$. Hence A is a vague filter of L. □

In the following proposition, we will show that vague filters of a residuated lattice are a generalization of fuzzy filters.

Proposition 24. *Let A be a vague set of a residuated lattice L. Then A is a vague filter of L if and only if the fuzzy sets t_A and $1 - f_A$ are fuzzy filters in L.*

4. Lattice of Vague Filters

Let A and B be vague sets of a residuated lattice L. If $V_A(x) \leq V_B(x)$, for all $x \in L$, then we say B containing A and denote it by $A \preceq B$.

The unite vague set of a residuated lattice L is a vague filter of L containing any vague filter of L. For a vague set, we can define the intersection of two vague sets A and B of a residuated lattice L by $V_{A\cap B}(x) = \text{imin}\{V_A(x), V_B(x)\}$, for all $x \in L$. It is easy to prove that the intersection of any family of vague filters of L is a vague filter of L. Hence we can define a vague filter generated by a vague set as follows.

Definition 25. Let A be a vague set of a residuated lattice L. A vague filter B is called a *vague filter generated* by A, if it satisfies

(i) $A \preceq B$,

(ii) $A \preceq C$ that implies $B \preceq C$, for all vague filters C of L.

The vague filter generated by A is denoted by $\langle A\rangle$.

Example 26. Consider the vague lattice filter A in Example 20 which is not a vague filter of L. Then the vague filter generated by A is

$$\langle A\rangle = \{\langle 0, [0.3, 0.4]\rangle, \langle a, [0.3, 0.4]\rangle, \langle b, [0.3, 0.4]\rangle, \langle c, [0.3, 0.8]\rangle, \langle 0, [0.5, 0.9]\rangle\}. \tag{14}$$

Theorem 27. *Let A be a vague set of a residuated lattice L. Then the vague value of x in $\langle A\rangle$ is*

$$V_{\langle A\rangle}(x) = \text{isup}\{\text{imin}\{V_A(a_1), V_A(a_2), \dots, V_A(a_n)\} : x \geq a_1 * \cdots * a_n\}, \tag{15}$$

for all $x \in L$.

Proof. First, we will prove that $\langle A\rangle$ is a vague set of L:

$$\begin{aligned} t_{\langle A\rangle}(x) &= \sup\{\min\{t_A(a_1), t_A(a_2), \dots, t_A(a_n)\} : x \geq a_1 * \cdots * a_n\} \\ &\leq \sup\{\min\{1 - f_A(a_1), 1 - f_A(a_2), \dots, 1 - f_A(a_n)\} : x \geq a_1 * \cdots * a_n\} \\ &= \sup\{1 - \max\{f_A(a_1), f_A(a_2), \dots, f_A(a_n)\} : x \geq a_1 * \cdots * a_n\} \\ &= 1 - \inf\{\max\{f_A(a_1), f_A(a_2), \dots, f_A(a_n)\} : x \geq a_1 * \cdots * a_n\}, \\ &= 1 - f_{\langle A\rangle}(x). \end{aligned} \tag{16}$$

Now, we will show that $\langle A\rangle$ is a vague filter of L.

(VF1) Suppose that $x, y \in L$ is arbitrary. For every $\varepsilon > 0$, we can take $a_1, \dots, a_n, b_1, \dots, b_m \in L$ such that $x \geq a_1 * \cdots * a_n$, $y \geq b_1 * \cdots * b_m$, and

$$\begin{aligned} t_{\langle A\rangle}(x) - \varepsilon &< \min\{t_A(a_1), t_A(a_2), \dots, t_A(a_n)\}, \\ t_{\langle A\rangle}(y) - \varepsilon &< \min\{t_A(b_1), t_A(b_2), \dots, t_A(b_m)\}. \end{aligned} \tag{17}$$

By Proposition 2 part (4), we have $x * y \geq a_1 * \cdots * a_n * b_1 * \cdots * b_m$ and then

$$\begin{aligned} t_{\langle A\rangle}(x * y) &\geq \min\{t_A(a_1), \dots, t_A(a_n), t_A(b_1), \dots, t_A(b_m)\} \\ &= \min\{\min\{t_A(a_1), \dots, t_A(a_n)\}, \min\{t_A(b_1), \dots, t_A(b_m)\}\} \\ &> \min\{(t_{\langle A\rangle}(x) - \varepsilon), (t_{\langle A\rangle}(y) - \varepsilon)\} \\ &= \min\{t_{\langle A\rangle}(x), t_{\langle A\rangle}(y)\} - \varepsilon. \end{aligned} \tag{18}$$

Since $\varepsilon > 0$ is arbitrary, we obtain that $t_{\langle A\rangle}(x * y) \geq \min\{t_{\langle A\rangle}(x), t_{\langle A\rangle}(y)\}$.

Similarly, we can show that $1 - f_{\langle A\rangle}(x * y) \geq \min\{1 - f_{\langle A\rangle}(x), 1 - f_{\langle A\rangle}(y)\}$. Hence $V_A(x * y) \geq \text{imin}\{V_A(x), V_A(y)\}$, for all $x, y \in L$.

(VF2) Suppose that $x \leq y$ where $x, y \in L$. Let $a_1, \dots, a_n \in L$ such that $a_1 * \cdots * a_n \leq x$. Then $a_1 * \cdots * a_n \leq y$. And we get that

$$V_A(y) \geq \text{imin}\{V_A(a_1), V_A(a_2), \dots, V_A(a_n)\}. \tag{19}$$

Since $a_1, \dots, a_n \in L$, where $x \geq a_1 * \cdots * a_n$ is arbitrary, then

$$\begin{aligned} V_{\langle A\rangle}(y) &\geq \text{isup}\{\text{imin}\{V_A(a_1), V_A(a_2), \dots, V_A(a_n)\} : x \geq a_1 * \cdots * a_n\} \\ &= V_{\langle A\rangle}(x). \end{aligned} \tag{20}$$

Therefore $\langle A\rangle$ is a vague filter of L by Theorem 16. It is clear that $A \preceq \langle A\rangle$. Now, let C be a vague filter of L such that $A \preceq C$. Then we have

$$\begin{aligned} V_{\langle A\rangle}(x) &= \text{isup}\{\text{imin}\{V_A(a_1), \dots, V_A(a_n)\} : x \geq a_1 * \cdots * a_n\} \\ &\leq \text{isup}\{\text{imin}\{V_C(a_1), \dots, V_C(a_n)\} : x \geq a_1 * \cdots * a_n\} \\ &= V_C(x), \end{aligned} \tag{21}$$

for all $x \in L$. Hence $\langle A\rangle \preceq C$ and then $\langle A\rangle$ is a vague filter generated by A. □

We denote the set of all vague filters of a residuated lattice L by VF(L).

Corollary 28. $(VF(L), \cap, \vee)$ *is a complete lattice.*

Proof. Clearly, if $\{A_i\}_{i\in\Gamma}$ is a family of vague filters of a residuated lattice L, then the infimum of this family is $\bigcap_{i\in\Gamma} A_i$ and the supremum is $\bigvee_{i\in\Gamma} A_i = \langle \bigcup_{i\in\Gamma} A_i \rangle$. It is easy to prove that $(\mathrm{IFF}(L), \cap, \vee)$ is a complete lattice. □

Let A and B be vague sets of a residuated lattice L. Then $A * B$ is defined as follows:

$$V_{(A*B)}(z) = \mathrm{isup}\{\mathrm{imin}\{V_A(x), V_B(y)\} : z \geq x * y\}, \quad (22)$$

for all $z \in L$.

Theorem 29. *Let A and B be vague sets of a residuated lattice L. Then $A * B$ is a vague set of L.*

Proof. Since A and B are vague sets of L, we have $t_A(x) \leq 1 - f_A(x)$ and $t_B(y) \leq 1 - f_B(y)$ for all $x, y \in L$. Thus

$$\begin{aligned} t_{A*B}(z) &= \sup\{\min\{t_A(x), t_A(y)\} : z \geq x * y\} \\ &\leq \sup\{\min\{1 - f_A(x), 1 - f_A(y)\} : z \geq x * y\} \\ &\leq 1 - f_{A*B}(z). \end{aligned} \quad (23)$$

Hence $t_{A*B}(z) + f_{A*B}(z) \leq 1$ and then $A * B$ is a vague set of L. □

Theorem 30. *Let A and B be vague filters of a residuated lattice L. Then $A * B$ is a vague filter of L.*

Proof. By Theorem 29, $A * B$ is a vague set of L. We will prove that $A * B$ is a vague filter of L.

(VF2) Let $z_1 \leq z_2$ and $x * y \leq z_1$. We get that $\mathrm{imin}\{V_A(x), V_B(y)\} \leq V_{A*B}(z_2)$. Since $x * y \leq z_1$ is arbitrary, we have

$$\begin{aligned} V_{A*B}(z_1) &= \mathrm{isup}\{\mathrm{imin}\{V_A(x), V_B(y)\} : z_1 \geq x * y\} \\ &\leq V_{A*B}(z_2). \end{aligned} \quad (24)$$

(VF1) Let $x, y \in L$. Then

$$\begin{aligned} &\mathrm{imin}\{V_{A*B}(x), V_{A*B}(y)\} \\ &\quad = \mathrm{imin}\{\mathrm{isup}\{\mathrm{imin}\{V_A(a), V_B(b)\} : x \geq a * b\}, \\ &\qquad \mathrm{isup}\{\mathrm{imin}\{V_A(c), V_B(d)\} : y \geq c * d\}\} \\ &\quad = \mathrm{isup}\{\mathrm{isup}\{\mathrm{imin}\{V_A(a * c), V_B(c * d)\} : \\ &\qquad x \geq a * b\} : y \geq c * d\} \\ &\quad \leq \mathrm{isup}\{\mathrm{imin}\{V_A(a * c), V_B(c * d)\} : \\ &\qquad x * y \geq (a * b) * (c * d)\} \\ &\quad \leq \mathrm{isup}\{\mathrm{imin}\{V_A(e), V_B(f)\} : x * y \geq e * f\} \\ &\quad = V_{A*B}(x * y). \end{aligned} \quad (25)$$

By Definition 13, $A * B$ is a vague filter of L. □

Theorem 31. *Let A and B be vague filters of a residuated lattice L such that $V_A(1) = V_B(1)$. Then $A * B$ is the least upper bound of A and B.*

Proof. By Definition 13, we have $V_A(1) = V_B(1) \geq V_A(y)$ for all $y \in L$. Let $z \in L$. Since $z = 1 * z$, we get that

$$\begin{aligned} V_{A*B}(z) &= \mathrm{isup}\{\mathrm{imin}\{V_A(x), V_B(y)\} : z \geq x * y\} \\ &\geq \mathrm{imin}\{V_A(z), V_B(1)\} \geq V_A(z). \end{aligned} \quad (26)$$

Hence $A \preceq A * B$. Similarly, we can show that $B \preceq A * B$. Suppose that C is the vague filter of L containing A and B. Then

$$\begin{aligned} &V_{A*B}(z) \\ &\quad = \mathrm{isup}\{\mathrm{imin}\{V_A(x), V_B(y)\} : z \geq x * y\} \\ &\quad \leq \mathrm{isup}\{\mathrm{imin}\{V_C(x), V_C(y)\} : z \geq x * y\} \\ &\quad \leq \mathrm{isup}\{\mathrm{imin}\{V_C(x * y)\} : z \geq x * y\} = \mu_C(z). \end{aligned} \quad (27)$$

Hence $A * B \preceq C$; that is, $A * B$ is the least upper bound of A and B. □

Lemma 32. *Let C be a vague filter of a residuated lattice L and $z \in L$. Then*

(1) $V_C(z) = \mathrm{isup}\{V_C(x * y) : z \geq x * y\}$, *for all* $x, y, z \in L$;

(2) $V_C(x * y) = \mathrm{imin}\{V_C(x), V_C(y)\}$, *for* $x, y \in L$.

Proof. (1) Since $x * y \leq z$, then $V_C(x * y) \leq V_C(z)$ by (VF2). Hence $V_C(z) \geq \mathrm{isup}\{V_C(x * y) : z \geq x * y\}$. Also, we have $1 * z = z$. We obtain that

$$V_C(z) = V_C(1 * z) \leq \mathrm{isup}\{V_C(x * y) : z \geq x * y\}. \quad (28)$$

(2) It follows from Proposition 2 part (2) and Definition 13. □

Let $\mathrm{VFD}(L)$ be the set of all vague filters A and B of L such that $V_A(1) = V_B(1)$.

Theorem 33. $(VFD(L), \cap, *)$ *is a distributive lattice.*

Proof. Let $A, B, C \in \mathrm{VFD}(L)$ be such that $A \le C$. We will show that $(A * B) \cap C = (A * C) \cap (B * C)$. By Lemma 32,

$$\begin{aligned} &V_{(A*B)\cap C}(z) \\ &\quad = \operatorname{imin}\{(V_{A*B})(z), V_C(z)\} \\ &\quad = \operatorname{imin}\{\operatorname{isup}\{\operatorname{imin}\{V_A(x), V_B(y)\} : z \ge x * y\}, V_C(z)\} \\ &\quad = \operatorname{isup}\{\operatorname{imin}\{V_A(x), V_B(y), V_C(z)\} : z \ge x * y\} \\ &\quad = \operatorname{isup}\{\operatorname{imin}\{V_A(x), V_B(y), V_C(x * y)\} : z \ge x * y\} \\ &\quad = \operatorname{isup}\{\operatorname{imin}\{V_A(x), V_B(y), V_C(x), V_C(y)\} : z \ge x * y\} \\ &\quad = \operatorname{isup}\{\operatorname{imin}\{V_A(x), V_C(x)\}, \\ &\qquad\qquad \operatorname{imin}\{V_B(y), V_C(y)\} : z \ge x * y\} \\ &\quad = \operatorname{isup}\{\operatorname{imin}\{V_{A\cap C}(x), V_{B\cap C}(y)\} : z \ge x * y\} \\ &\quad = V_{(A\cap C)*(B\cap C)}(z). \end{aligned} \tag{29}$$

□

5. Subpositive Implicative Vague Filters

Definition 34. Let A be a vague subset of a residuated lattice L. A is called a *subpositive implicative vague filter* of L if

(VF6) $V_A(x) \le V_A(1)$, for all $x \in L$,

(SVF1) $\operatorname{imin}\{V_A(((x \to y) * z) \rightsquigarrow ((y \rightsquigarrow x) \to x)), V_A(z)\} \le V_A((x \to y) \rightsquigarrow y)$ for any $x, y, z \in L$,

(SVF2) $\operatorname{imin}\{V_A((z * (x \rightsquigarrow y)) \to ((y \to x) \rightsquigarrow x)), V_A(z)\} \le V_A((x \rightsquigarrow y) \to y)$ for any $x, y, z \in L$.

Proposition 35. *Let A be a vague filter of a residuated lattice L. Then A is a subpositive implicative vague filter of L if and only if it satisfies*

(SVF3) $V_A((x \to y) \rightsquigarrow ((y \rightsquigarrow x) \to x)) \le V_A((x \to y) \rightsquigarrow y)$ *for any* $x, y \in L$,

(SVF4) $V_A((x \rightsquigarrow y) \to ((y \to x) \rightsquigarrow x)) \le V_A((x \rightsquigarrow y) \to y)$ *for any* $x, y \in L$.

Proposition 36. *Let A be a subpositive implicative vague filter of a residuated lattice L. Then A is a vague filter of L.*

Proof. We have $\operatorname{imin}\{V_A(x \rightsquigarrow y), V_A(x)\} = \operatorname{imin}\{V_A((y \to y) * x) \rightsquigarrow ((y \rightsquigarrow y) \to y), V_A(x)\} \le V_A((y \to y) \rightsquigarrow y) = V_A(y)$. By (VS1) and Theorem 17, A is a vague filter of a residuated lattice L. □

The relation between subpositive implicative vague filter and its level subset is as follows.

Theorem 37. *Let A be a vague set of a residuated lattice L. Then A is a subpositive implicative vague filter of L if and only if for all $\alpha, \beta \in [0,1]$, the set $A_{(\alpha,\beta)}$ is either empty or a subpositive implicative filter of L, where $\alpha \le \beta$.*

Proof. Suppose that A is a subpositive implicative filter vague set of L and $\alpha, \beta \in [0,1]$. Let $A_{(\alpha,\beta)} \ne \emptyset$. Since A is a vague filter, then $A_{(\alpha,\beta)}$ is a filter of L by Theorem 21. Now, let $(x \to y) \rightsquigarrow ((y \rightsquigarrow x) \to x) \in A_{(\alpha,\beta)}$. Then $V_A((x \to y) \rightsquigarrow ((y \rightsquigarrow x) \to x)) \ge [\alpha, \beta]$. By (SVF3), $V_A((x \to y) \rightsquigarrow y) \ge [\alpha, \beta]$; that is, $(x \to y) \rightsquigarrow y \in A_{(\alpha,\beta)}$. Similarly, we can show that $A_{(\alpha,\beta)}$ satisfies (SVF4). Hence $A_{(\alpha,\beta)}$ is a subpositive implicative filter of L.

Conversely, let $t_A((x \to y) \rightsquigarrow ((y \rightsquigarrow x) \to x)) = \alpha$ and $1 - f_A((x \to y) \rightsquigarrow ((y \rightsquigarrow x) \to x)) = \beta$. Thus $V_A((x \to y) \rightsquigarrow ((y \rightsquigarrow x) \to x)) \ge [\alpha, \beta]$. We get that $(x \to y) \rightsquigarrow ((y \rightsquigarrow x) \to x) \in A_{(\alpha,\beta)}$. So $A_{(\alpha,\beta)} \ne \emptyset$. Thus $(x \to y) \rightsquigarrow y \in A_{(\alpha,\beta)}$; that is, $V_A((x \to y) \rightsquigarrow y) \ge [\alpha, \beta] = V_A((x \to y) \rightsquigarrow ((y \rightsquigarrow x) \to x))$. Similarly, we can prove that A satisfies (SVF4). □

In the following theorems, we obtain some characterizations of subpositive implicative vague filter.

Theorem 38. *Let A be a vague filter of a residuated lattice L. Then A is a subpositive implicative vague filter of L if and only if it satisfies*

(SVF5) $\operatorname{imin}\{\mathrm{V_A}((\mathrm{y} \to \mathrm{z}) \rightsquigarrow (\mathrm{x} \to \mathrm{y})), \mathrm{V_A}(\mathrm{x})\} \le \mathrm{V_A}(\mathrm{y})$ *for any* $x, y, z \in L$,

(SVF6) $\operatorname{imin}\{\mathrm{V_A}((\mathrm{y} \rightsquigarrow \mathrm{z}) \to (\mathrm{x} \rightsquigarrow \mathrm{y})), \mathrm{V_A}(\mathrm{x})\} \le \mathrm{V_A}(\mathrm{y})$ *for any* $x, y, z \in L$.

Proof. Let A be a subpositive implicative vague filter of L. By Proposition 2 part (3), we have $\operatorname{imin}\{V_A((y \to z) \rightsquigarrow (x \to y)), V_A(x)\} = \operatorname{imin}\{V_A(x \to ((y \to z) \rightsquigarrow y)), V_A(x)\} \le V_A((y \to z) \rightsquigarrow y)$. By Proposition 2 part (7), $(y \to z) \rightsquigarrow y \le z \rightsquigarrow y \le (y \to z) \rightsquigarrow ((z \rightsquigarrow y) \to y)$. We get that $V_A((y \to z) \rightsquigarrow y) \le V_A(z \rightsquigarrow y) \le V_A((y \to z) \rightsquigarrow ((z \rightsquigarrow y) \to y)) \le V_A((z \rightsquigarrow y) \to y)$. Hence $\operatorname{imin}\{V_A((y \to z) \rightsquigarrow y), V_A(z \rightsquigarrow y)\} \le \operatorname{imin}\{V_A((z \rightsquigarrow y) \to y), V_A(z \rightsquigarrow y)\} \le V_A(y)$. Thus it satisfies (SVF5). Similarly, we can prove (SVF6). Conversely, let A satisfy (SVF5) and (SVF6); $V_A((x \to y) \rightsquigarrow ((y \rightsquigarrow x) \to x)) = \operatorname{imin}\{V_A([((x \to y) \rightsquigarrow y) \to 1] \rightsquigarrow [(x \to y) \rightsquigarrow ((y \rightsquigarrow x) \to x) \to ((x \to y) \rightsquigarrow y)]), V_A(1)\} \le V_A((x \to y) \rightsquigarrow y)$. Similarly, we can prove that (SVF5) holds. Hence A is a vague subpositive implicative filter. □

Theorem 39. *Let A be a vague filter of a residuated lattice L. Then A is a subpositive implicative vague filter of L if and only if it satisfies*

(SVF7) $V_A((x \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x)) \le V_A((y \rightsquigarrow x) \to x)$ *for any* $x, y \in L$;

(SVF8) $V_A((x \rightsquigarrow y) \to ((y \rightsquigarrow x) \to x)) \le V_A((y \to x) \rightsquigarrow x)$ *for any* $x, y \in L$.

Proof. Let (SVF7) and (SVF8) hold. We will show that A satisfies (SVF5) and (SVF6). We have $\operatorname{imin}\{V_A((y \to z) \rightsquigarrow (x \to y)), V_A(x)\} = \operatorname{imin}\{V_A(x \to ((y \to z) \rightsquigarrow y)), V_A(x)\} \le V_A((y \to z) \rightsquigarrow y) \le V_A((y \to z) \rightsquigarrow$

$((z \to y) \rightsquigarrow y)) \leq V_A((z \rightsquigarrow y) \to y))$. Also, we have $V_A((y \to z) \rightsquigarrow y) \leq V_A(z \rightsquigarrow y)$. Since A is a vague filter, $V_A((y \to z) \rightsquigarrow y) \leq \operatorname{imin}\{V_A(z \rightsquigarrow y), V_A((z \rightsquigarrow y) \to y)\} \leq V_A(y)$. Hence A satisfies (SVF5). Analogously, we can prove that A satisfies (SVF6). Hence A is a subpositive implicative vague filter of L by Theorem 38. Conversely, let A be a subpositive implicative vague filter of L. We have

$$\begin{aligned}
&[((y \rightsquigarrow x) \longrightarrow x) \longrightarrow 1] \\
&\quad \rightsquigarrow [((x \longrightarrow y) \rightsquigarrow ((y \longrightarrow x) \rightsquigarrow x)) \\
&\qquad \longrightarrow ((y \rightsquigarrow x) \longrightarrow x)] \\
&= [(x \longrightarrow y) \rightsquigarrow ((y \longrightarrow x) \rightsquigarrow x)] \\
&\quad \longrightarrow [((y \longrightarrow x) \rightsquigarrow x) \rightsquigarrow ((y \longrightarrow x) \rightsquigarrow x)] \\
&= (x \longrightarrow y) \rightsquigarrow ((y \longrightarrow x) \rightsquigarrow x).
\end{aligned} \tag{30}$$

We get that $V_A((x \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x)) = \operatorname{imin}\{V_A((x \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x)), V_A([((y \rightsquigarrow x) \to x) \to 1] \rightsquigarrow [((x \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x)) \to ((y \rightsquigarrow x) \to x)])\} \leq V_A((y \rightsquigarrow x) \to x)$ by (SVF5). Similarly, we can prove that (SVF8) holds. □

Theorem 40. *Let A be a vague filter of a residuated lattice L. Then A is a subpositive implicative vague filter of L if and only if it satisfies*

(SVF9) $V_A((x \to y) \rightsquigarrow x) \leq V_A(x)$ *for any* $x, y \in L$,

(SVF10) $V_A((x \rightsquigarrow y) \to x) \leq V_A(x)$ *for any* $x, y \in L$.

Proof. Let A be a subpositive implicative vague filter of L. Then A satisfies (SVF5) and (SVF6):

$$\begin{aligned}
&V_A((x \longrightarrow y) \rightsquigarrow x) \\
&\quad = \operatorname{imin}\{V_A(1 \longrightarrow ((x \longrightarrow y) \rightsquigarrow x)), V_A(1)\} \\
&\quad = \operatorname{imin}\{V_A((x \longrightarrow y) \rightsquigarrow (1 \longrightarrow x)), V_A(1)\} \\
&\quad \leq V_A(x).
\end{aligned} \tag{31}$$

Similarly, we can prove that (SVF10) holds. Conversely, let A satisfy (SVF9) and (SVF10). We will show that A satisfies (SVF5) and (SVF6); $\operatorname{imin}\{V_A((y \to z) \rightsquigarrow (x \to y)), V_A(x)\} = \operatorname{imin}\{V_A((x \to (y \to z) \rightsquigarrow y)), V_A(x)\} \leq V_A((y \to z) \rightsquigarrow y) \leq V_A(y)$. Hence A satisfies (SVF5). Similarly, we can prove (SVF7) holds. Therefore A is a subpositive implicative vague filter of L by Theorem 38. □

Theorem 41. *Let A be a vague filter of a residuated lattice L. Then A is a subpositive implicative vague filter of L if and only if it satisfies*

(SVF11) $V_A(\neg x \rightsquigarrow x) \leq V_A(x)$ *for any* $x \in L$,

(SVF12) $V_A(\sim x \to x) \leq V_A(x)$ *for any* $x \in L$.

Proof. Let A be a subpositive implicative vague filter of L. Then (SVF11) and (SVF12) follow by (SVF9) and (SVF10). Conversely, since $0 \leq y$, then $x \to 0 \leq x \to y$. Hence $(x \to y) \rightsquigarrow x \leq \neg x \rightsquigarrow x$ by Proposition 2 part (6). Therefore $V_A((x \to y) \rightsquigarrow x) \leq V_A(\neg x \rightsquigarrow x) \leq V_A(x)$. Similarly, we can prove that (SVF10) holds. Hence A is a subpositive implicative vague filter of L by Theorem 40. □

The extension theorem of subpositive implicative vague filters is obtained from the following theorem.

Theorem 42. *Let A and B be two vague filters of a residuated lattice L which satisfies $A \leq B$ and $V_A(1) = V_B(1)$. If A is a subpositive implicative vague filter of L, so B is.*

Proof. Suppose that $u = (x \to y) \rightsquigarrow x$. Then

$$\begin{aligned}
1 &= u \longrightarrow u = u \longrightarrow ((x \longrightarrow y) \rightsquigarrow x) \\
&= (x \longrightarrow y) \rightsquigarrow (u \longrightarrow x) = 1.
\end{aligned} \tag{32}$$

We have $(x \to y) \rightsquigarrow (u \to x) \leq ((u \to x) \to y) \rightsquigarrow (u \to x)$. Therefore $V_A(((u \to x) \to y) \rightsquigarrow (u \to x)) = V_A(1)$. By (SVF10), $V_A(1) = V_A(((u \to x) \to y) \rightsquigarrow (u \to x)) \leq V_A(u \to x)$. Hence $V_A(u \to x) = V_A(1) = V_B(1)$. Since $A \leq B$, then $V_B(u \to x) \geq V_A(u \to x) = V_B(1)$. We get that $V_B((x \to y) \rightsquigarrow x) = V_B(u) = \operatorname{imin}\{V_B(u \to x), V_B(u)\} \leq V_B(x)$. Similarly, we can show that G satisfies (SVF11). Hence G is a vague subpositive implicative filter of L, by Theorem 40. □

Definition 43. Let A be a vague filter of a residuated lattice L. A is called a *vague Boolean filter* of L if

(BVF1) $V_A(x \vee \neg x) = V_A(1)$, for all $x \in L$,

(BVF2) $V_A(x \vee \sim x) = V_A(1)$, for all $x \in L$.

Similar to Theorem 40, we have the following theorem with respect to Boolean vague filters.

Theorem 44. *Let A be a vague set of a residuated lattice L. Then A is a Boolean vague filter of L if and only if for all $\alpha, \beta \in [0,1]$; the set $A_{(\alpha,\beta)}$ is either empty or a Boolean filter of L, where $\alpha \leq \beta$.*

Theorem 45. *Let A be a vague filter of a residuated lattice L. Then A is a subpositive implicative vague filter of L if and if only if A is a Boolean vague filter.*

Proof. Let A be a subpositive implicative vague filter. We have

$$\begin{aligned}
V_A(1) &= V_A(\neg(x \vee \neg x) \rightsquigarrow (x \vee \neg x)) \\
&= V_A((\neg x \wedge \neg\neg x) \rightsquigarrow (x \vee \neg x)) \leq V_A(x \vee \neg x).
\end{aligned} \tag{33}$$

Hence $V_A(x \vee \neg x) = V_A(1)$. Similarly, we can prove that A satisfies (BVF2). Hence A is a Boolean vague filter.

Conversely, let $V_A(x \vee \neg x) = V_A(1)$. We have $V_A(x \vee \neg x) = V_A((x \rightsquigarrow x) \vee (\neg x \rightsquigarrow x)) V_A((x \vee \neg x) \rightsquigarrow x) = \operatorname{imin}\{V_A((x \vee \neg x) \rightsquigarrow x), V_A(1)\} = \operatorname{imin}\{V_A((x \vee \neg x) \rightsquigarrow x), V_A(x \vee \neg x)\} \leq V_A(x)$. Similarly, we can show that A satisfies (SVF12). Hence A is a subpositive implicative vague filter by Theorem 41. □

6. Conclusion

In this paper, we introduced the notions of vague filters, subpositive implicative vague filters, and Boolean vague filters of a residuated lattice and investigated some related properties. We studied the relationship between filters and vague filters of a residuated lattice and showed that vague filters of a residuated lattice is a generalization of fuzzy filters. We proved that the set of all vague filters of a residuated lattice forms a complete lattice and obtained some characterizations of subpositive implicative vague filter. The extension theorem of subpositive implicative vague filters was obtained. Finally, it was proved that subpositive implicative vague filters are equivalent to Boolean vague filters.

Conflict of Interests

There is no conflict of interests regarding the publication of this paper.

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