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Research Article

Approximations of Antieigenvalue and Antieigenvalue-Type Quantities

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We will extend the definition of antieigenvalue of an operator to antieigenvalue-type quantities, in the first section of this paper, in such a way that the relations between antieigenvalue-type quantities and their corresponding Kantorovich-type inequalities are analogous to those of antieigenvalue and Kantorovich inequality. In the second section, we approximate several antieigenvalue-type quantities for arbitrary accretive operators. Each antieigenvalue-type quantity is approximated in terms of the same quantity for normal matrices. In particular, we show that for an arbitrary accretive operator, each antieigenvalue-type quantity is the limit of the same quantity for a sequence of finite-dimensional normal matrices.

1. Introduction

Since 1948, the Kantorovich and Kantorovich-type inequalities for positive bounded operators have had many applications in operator theory and other areas of mathematical sciences such as statistics. Let T be a positive operator on a Hilbert space H with $mI \leq T \leq IM$, then the Kantorovich inequality asserts that

$$(Tf, f)\left(T^{-1}f, f\right) \leq \frac{(m+M)^2}{4mM}, \tag{1.1}$$

for every unit vector f (see [1]). When λ_m and λ_M are the smallest and the largest eigenvalues of T, respectively, it can be easily verified that

$$\frac{(\lambda_m + \lambda_M)^2}{4\lambda_m\lambda_M} \leq \frac{(m+M)^2}{4mM}, \tag{1.2}$$

for every pair of nonnegative numbers m, M, with $m \le \lambda_m$ and $M \ge \lambda_M$. The expression

$$\frac{(\lambda_m + \lambda_M)^2}{4\lambda_m \lambda_M} \tag{1.3}$$

is called the Kantorovich constant and is denoted by $K(T)$.

Given an operator T on a Hilbert space H, the antieigenvalue of T, denoted by $\mu(T)$, is defined by Gustafson (see [2–5]) to be

$$\mu(T) = \inf_{Tf \neq 0} \frac{\operatorname{Re}(Tf, f)}{\|Tf\| \|f\|}. \tag{1.4}$$

Definition (1.4) is equivalent to

$$\mu(T) = \inf_{\substack{Tf \neq 0 \\ \|f\|=1}} \frac{\operatorname{Re}(Tf, f)}{\|Tf\|}. \tag{1.5}$$

A unit vector f for which the inf in (1.5) is attained is called an antieigenvector of T. For a positive operator T, we have

$$\mu(T) = \frac{2\sqrt{\lambda_m \lambda_M}}{\lambda_m + \lambda_M}. \tag{1.6}$$

Thus, for a positive operator, both the Kantorovich constant and $\mu(T)$ are expressed in terms of the smallest and the largest eigenvalues. It turns out that the former can be obtained from the latter.

Matrix optimization problems analogues to (1.4), where the quantity to be optimized involves inner products and norms, frequently occur in statistics. For example, in the analysis of statistical efficiency one has to compute quantities such as

$$\inf_{\mathbf{X}'\mathbf{X}=\mathbf{1}_p} \frac{1}{|X'TX||X'T^{-1}X|}, \tag{1.7}$$

$$\inf_{X'X=1_p} \frac{\left|(X'TX)^2\right|}{\left|X'T^2X - (X'TX)^2\right|}, \tag{1.8}$$

$$\inf_{X'X=1_p} \frac{\left|\left(X'T^{-1}X\right)^{-1}\right|}{\left|X'TX-\left(X'T^{-1}X\right)^{-1}\right|}, \tag{1.9}$$

$$\inf_{X'X=1_p} \frac{1}{\left|X'T^2X-\left(X'TX\right)^2\right|}, \tag{1.10}$$

$$\inf_{X'X=1_p} \frac{1}{\left|X'TX-\left(X'T^{-1}X\right)^{-1}\right|}, \tag{1.11}$$

where T is a positive definite matrix and $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p]$ with $2p \le n$. Each $\mathbf{X}_k$ is a column vector of size n, and $\mathbf{X}'\mathbf{X} = \mathbf{1}_p$. Here, $\mathbf{1}_p$ denotes the $p \times p$ identity matrix, and $|Y|$ stands for the determinant of a matrix Y. Please see [6–12]. Notice that in the references just cited, the sup's of the reciprocal of expressions involved in (1.8), (1.9), (1.10), and (1.11) are sought. Nevertheless, since the quantities involved are always positive, those sup's are obtained by finding the reciprocals of the inf's found in (1.8), (1.9), (1.10) and (1.11), while the optimizing vectors remain the same. Note that in (1.7) through (1.11). one wishes to compute optimizing matrices X for quantities involved, whereas in (1.5) the objective is to find optimizing vectors f for the quantity involved. Also, please note that for any vector f, we have $(Tf, f) = X'TX$, where X is the matrix of rank one with $X = [X_1]$ and $X_1 = f$. Hence the optimizing vectors in (1.5) can also be considered as optimizing matrices for respective quantity. When we use the term "an antieigenvalue-type quantity" throughout this paper, we mean a real number obtained by computing the inf in expressions similar to those given previously. The terms "an antieigenvector-type f" or "an antieigenmatrix-type X" are used for a vector f or a matrix X for which the inf in the associated expression is attained. A large number of well-known operator inequalities for positive operators are indeed generalizations of Kantorovich inequality. The following inequality, called the Holder-McCarthy inequality is an example. Let T be a positive operator on a Hilbert space H satisfying $M \ge A \ge m > 0$. Also, let $F(t)$ be a real valued convex function on $[m, M]$ and let q be a real number, then the inequality,

$$(F(T)f, f) \le \frac{(mf(M) - Mf(m))}{(q-1)(M-m)} \left(\frac{(q-1)(f(M) - f(m))}{q(mf(M) - Mf(m))} \right)^q (Tf, f)^q, \tag{1.12}$$

which holds for every unit vector f under certain conditions, is called the Holder-McCarthy inequality (see [13, 14]). Many authors have established Kantorovich-type inequalities, such as (1.12), for a positive operator T by going through a two-step process which consists of computing upper bounds for suitable functions on intervals containing the spectrum of T and then applying the standard operational calculus to T (see [14]). These methods have limitations as they do not shed light on vectors or matrices for which inequalities become equalities. Also, they cannot be used to extend these inequalities from positive matrices to normal matrices. To extend these kinds of inequalities from positive operators to other types of operators in our previous papers, we have developed a number of effective techniques which have been useful in discovering new results.

In particular, a technique which we have frequently used is the conversion of a matrix/operator optimization problem to a convex programming problem. In this approach the problem is reduced to finding the minimum of a convex or concave function on the numerical range of a matrix/operator. This technique is not only straight forward but also sheds light on the question of when Kantorovich-type inequalities become equalities. For example, the proof given in [1] for inequality (1.1) does not shed light on vectors for which the inequality becomes equality. Likewise, in [14], the methods used to prove a number of Kantorovich-type inequalities do not provide information about vectors for which the respective inequalities become equalities. From the results we obtained later (see [15–17]) it is now evident that, for a positive definite matrix T, the equality in (1.1) holds for unit vectors f that satisfy the following properties. Assume that $\{\lambda_i\}_{i=1}^{n}$ is the set of distinct eigenvalues of T such that $\lambda_1 < \lambda_2 \cdots < \lambda_n$ and $E_i = E(\lambda_i)$ is the eigenspace corresponding to eigenvalue λ_i of T. If P_i is the orthogonal projection on E_i and $z_i = P_i f$, then

$$\begin{aligned} \|z_1\|^2 &= \frac{\lambda_n}{\sqrt{\lambda_1 + \lambda_n}}, \\ \|z_n\|^2 &= \frac{\lambda_1}{\sqrt{\lambda_1 + \lambda_n}}, \end{aligned} \tag{1.13}$$

and $\|z_i\| = 0$ if $i \neq 1$ and $i \neq n$. Furthermore, for such a unit vector f, we have

$$(Tf, f)\left(T^{-1}f, f\right) = \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n}. \tag{1.14}$$

Please note that by a change of variable, (1.14) is equivalent to

$$\frac{(Tf, f)}{\|Tf\|} = \frac{2\sqrt{\lambda_1\lambda_n}}{\lambda_1 + \lambda_n}. \tag{1.15}$$

Furthermore, in [15–17], we have applied convex optimization methods to extend Kantorovich-type inequalities and antieigenvalue-type quantities to other classes of operators. For instance, in [16], we proved that for an accretive normal matrix, antieigenvalue is expressed in terms of two identifiable eigenvalues.

This result was obtained by noticing the fact that

$$\mu^2(T) = \inf\left\{ \frac{x^2}{y} : x + iy \in W(S) \right\}, \tag{1.16}$$

where $S = \operatorname{Re} T + iT^*T$ and $W(S)$ denotes the numerical range of S. Since T is normal so is S. Also, by the spectral mapping theorem, if $\sigma(S)$ denotes the spectrum of S, then

$$\sigma(S) = \left\{ \beta_i + i|\lambda_i|^2 : \lambda_i = (\beta_i + \delta_i i)\epsilon\sigma(T) \right\}. \tag{1.17}$$

Hence, in [17], the problem of computing $\mu^2(T)$ was reduced to the problem of finding the minimum of the convex function $f(x, y) = x^2/y$ on the boundary of the convex set $W(S)$. It turns out that $\mu(T) = \beta_p/|\lambda_p|$ or

$$\mu(T) = \frac{2\sqrt{(\beta_q - \beta_p)\left(\beta_p|\lambda_q|^2 - \beta_q|\lambda_p|^2\right)}}{|\lambda_q|^2 - |\lambda_p|^2}, \tag{1.18}$$

where $\beta_p + i|\lambda_p|^2$ and $\beta_q + i|\lambda_q|^2$ are easily identifiable eigenvalues of S. In [17], we called them the first and the second critical eigenvalues of S, respectively. The corresponding quantities $\lambda_p = \beta_p + \delta_p i$ and $\lambda_q = \beta_q + \delta_q i$ are called the first and second critical eigenvalues of T, respectively. Furthermore, the components of antieigenvectors satisfy

$$\begin{aligned}
\|z_p\|^2 &= \frac{\beta_q|\lambda_q|^2 - 2\beta_p|\lambda_q|^2 + \beta_q|\lambda_p|^2}{\left(|\lambda_q|^2 - |\lambda_p|^2\right)(\beta_q - \beta_p)}, \\
\|z_q\|^2 &= \frac{\beta_p|\lambda_p|^2 - 2\beta_q|\lambda_p|^2 + \beta_p|\lambda_q|^2}{\left(|\lambda_q|^2 - |\lambda_p|^2\right)(\beta_q - \beta_p)},
\end{aligned} \tag{1.19}$$

and $\|z_i\| = 0$ if $i \neq p$ and $i \neq q$. An advantage to this technique is that we were able to inductively define and compute higher antieigenvalues $\mu_i(T)$ and their corresponding higher antieigenvectors for accretive normal matrices (see [17]). This technique can also be used to approximate antieigenvalue-type quantities for bounded arbitrary bounded accretive operators as we will show in next section.

2. Approximations of Antieigenvalue-Type Quantities

If T is not a finite dimensional normal matrix, (1.16) is still valid, but $W(S)$ is not a polygon any more. Thus, we cannot use our methods discussed in previous section for an arbitrary bounded accretive operator T. In this section, we will develop methods for approximating antieigenvalue and antieigenvalue-type quantities for an arbitrary bounded accretive operator T by counterpart quantities for finite dimensional matrices. Computing an antieigenvalue-type quantity for an operator T is reduced to computing the minimum of a convex or concave function $f(x, y)$ on $\partial W(S)$, the boundary of the numerical range of another operator S. To make such approximations, first we will approximate $W(S)$ with polygons from inside and outside. Then, we use techniques developed in [17] to compute the minimum of the convex or concave functions $f(x, y)$ on the polygons inside and outside $W(S)$.

Theorem 2.1. *Assume that $f(x,y)$ is a convex or concave function on $W(S)$, the numerical range of an operator S. Then, for each positive integer k, the real part of the rotations $e^{i\theta}S$ of S induces polygons G_k contained in $W(S)$ and polygons H_k which contain $W(S)$ such that*

$$\inf_{(x,y)\in\partial W(S)} f(x,y) \geq \inf_{(x,y)\in\partial H_k} f(x,y), \tag{2.1}$$

$$\inf_{(x,y)\in\partial W(S)} f(x,y) \leq \inf_{(x,y)\in\partial G_k} f(x,y), \tag{2.2}$$

$$\inf_{(x,y)\in\partial W(S)} f(x,y) = \lim_{k\Rightarrow\infty}\left[\inf_{(x,y)\in\partial H_k} f(x,y)\right], \tag{2.3}$$

$$\inf_{(x,y)\in\partial W(S)} f(x,y) = \lim_{k\Rightarrow\infty}\left[\inf_{(x,y)\in\partial G_k} f(x,y)\right] \tag{2.4}$$

∂G_k and ∂H_k denote the boundaries of G_k and H_k, respectively.

Proof. Following the notations in [18], let $0 \leq \theta < 2\pi$, and λ_θ is the largest eigenvalue of the positive operator $\mathrm{Re}(e^{i\theta}S)$. If f_θ is a unit eigenvector associated with λ_θ, then the complex number (Sf_θ, f_θ) which is denoted by p_θ belongs to $\partial W(S)$. Furthermore, the parametric equation of the line of support of $W(S)$ at (Sf_θ, f_θ) is

$$e^{-i\theta}(\lambda_\theta + ti), \quad 0 \leq t < \infty. \tag{2.5}$$

Let Θ denote a set of "mesh" points $\Theta = \{\theta_1, \theta_2, \ldots, \theta_k\}$, where $0 \leq \theta_1 < \theta_2 < \cdots \theta_k < 2\pi$. Let $P_1 = P_{\theta_1}$, $P_2 = p_{\theta_2}, \ldots, P_k = p_{\theta_k}$, then the polygon whose vertices are $P_1, P_2, \ldots, P_k$ is contained in $W(S)$. This polygon is denoted by $W_{\mathrm{in}}(S,\Theta)$ in [18], but we denote it by G_k in this paper for simplicity in notations. Let Q_i be the intersection of the lines

$$e^{-i\theta_i}(\lambda_{\theta_i} + ti), \quad 0 \leq t < \infty, \tag{2.6}$$

β and

$$e^{-i\theta_{i+1}}(\lambda_{\theta_{i+1}} + ti), \quad 0 \leq t < \infty, \tag{2.7}$$

which are the lines of support of $W(S)$ at points p_{θ_i} and $p_{\theta_{i+1}}$, respectively, where $k+1$ is identified with 1. Then we have

$$Q_i = e^{-i\theta_i}\left(\lambda_{\theta_i} + \frac{\lambda_{\theta_i}\cos\delta_i - \lambda_{\theta_{i+1}}}{\sin\delta_i} i\right), \tag{2.8}$$

where $\delta_i = \theta_{i+1} - \theta_i$. The polygon whose vertices are Q_i, $1 \leq i \leq k-1$ contains $W(S)$. This polygon is denoted by $W_{\mathrm{out}}(S,\Theta)$ in [18], but we denote it by H_k here. Hence, for each k, we have

$$G_k \subseteq W(S) \subseteq H_k. \tag{2.9}$$

Please see Figure 1. □

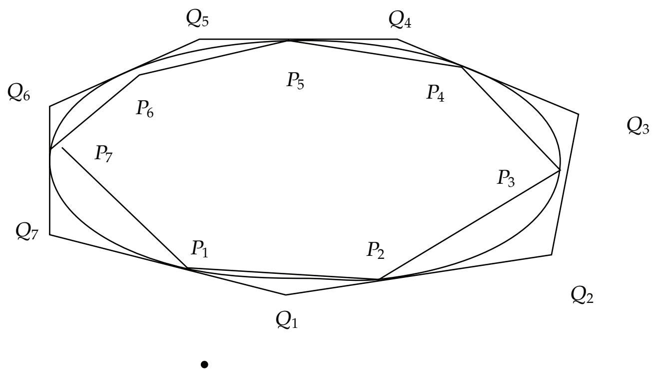

Figure 1

Therefore, inf

$$f(x,y) \le \inf_{(x,y)\in\partial W(S)} f(x,y) \le \inf_{(x,y)\in\partial G_k} f(x,y). \tag{2.10}$$

As a measure of the approximation given by (2.1) and (2.2) we will adopt a normalized difference between the values:

$$\inf_{(x,y)\in\partial G_k} f(x,y), \qquad \inf_{(x,y)\in\partial H_k} f(x,y), \tag{2.11}$$

that is

$$\Delta(f,k) = \frac{\inf_{(x,y)\in\partial G_k} f(x,y) - \inf_{(x,y)\in\partial H_k} f(x,y)}{\inf_{(x,y)\in\partial G_k} f(x,y)}. \tag{2.12}$$

Once the vertices of ∂G_k and ∂H_k are determined,

$$\inf_{(x,y)\in\partial G_k} f(x,y), \qquad \inf_{(x,y)\in\partial H_k} f(x,y) \tag{2.13}$$

are computable by methods we used in [17], where $W(S)$ was a convex polygon. For the convex or concave functions $f(x,y)$ arising in antieigenvalue-type problems, the minimums on ∂G_k and ∂H_k will occur either at the upper-left or upper-right portion of ∂G_k and ∂H_k. As our detailed analysis in [17] shows, the minimum of the convex functions whose level cures appear on the left side of ∂H_k is attained at either the first or the second critical vertex of ∂H_k or on the line segment connecting these two vertices. The same can be said about the minimum of such functions on ∂G_k. In Figure 1 above, Q_6 and Q_5 are the first and second critical vertices of ∂H_k, respectively. Similarly, P_7 and P_6 are the first and the second critical vertices of ∂G_k, respectively. In [17], an algebraic algorithm for determining the first and the second critical vertices of a polygon is developed based on the slopes of lines connecting vertices of a polygon. This eliminates the need for computing the values of the function $f(x,y)$ at all vertices of ∂G_k and H_k. It also eliminates the need for computing and comparing the minimums of $f(x,y)$ on all edges of ∂G_k and ∂H_k. Thus, to compute the minimum of $f(x,y)$ on ∂H_k, for example, we only need to evaluate $f(x,y)$ for the components of the first and the second critical vertices and use Lagrange multipliers to compute the minimum of $f(x,y)$ on the line segment connecting these two vertices.

Example 2.2. The Holder-McCarthy inequality for positive operators given by (1.12) can be also written as

$$\frac{(Tf,f)^q}{(F(T)f,f)} \geq \frac{(q-1)(M-m)}{(mf(M)-Mf(m))}\left(\frac{q(mf(M)-Mf(m))}{(q-1)(f(M)-f(m))}\right)^q. \tag{2.14}$$

Therefore, for a positive operator, one can define a new antieigenvalue-type quantity by

$$\mu_{(F,q)}(T) = \inf_{(F(T)f,f)\neq 0}\frac{(Tf,f)^q}{(F(T)f,f)}. \tag{2.15}$$

If we can compute the minimizing unit vectors f for (2.15), then obviously for these vectors (1.12) becomes equality. $\mu_{(F,q)}(T)$ is the antieigenvalue-type quantity associated with the Holder-McCarthy inequality, which is a Kantorovich-type inequality. The minimizing unit vectors for (2.15) are antieigenvector-type vectors associated with $\mu_{(F,q)}(T)$. It is easily seen that the standard antieigenvalue $\mu(T)$ is a special case of this antieigenvalue-type quantity.

Example 2.3. There are a number of ways that we can extend the definition of $\mu_{(F,q)}(T)$ to an arbitrary operator where F is an analytic function. One way is to extend the definition of $\mu_{(F,q)}(T)$ by

$$\mu_{(F,q)}(T) = \inf_{(|F(T)|f,f)\neq 0}\frac{\mathrm{Re}\,(Tf,f)^q}{(|F(T)|f,f)}, \tag{2.16}$$

where F is a complex-valued analytic function defined on the spectrum of T. The problem then becomes

$$\mu_{(F,q)}(T) = \inf\left\{\frac{x^q}{y} : x+iy \in W(S)\right\}, \tag{2.17}$$

where $S = (\mathrm{Re}\,T)^q + |F(T)|i$. For an arbitrary operator T, the set $W(S)$ is not in general a polygon but a bounded convex subset of the complex plane. Nevertheless, we can approximate $W(S)$ with polygons from inside and outside and thus obtain an approximation for $\mu_{(F,q)}(T)$ by looking at the minimum of the function $f(x,y) = x^q/y$ on those inside and outside polygons.

Example 2.4. In [19], in the study of statistical efficiency, we computed the value of a number of antieigenvalue-type quantities. Each antieigenvalue-type quantity there is itself the product of several simpler antieigenvalue-type quantities [23, Theorem 3]. One example is

$$\delta(T) = \inf_{\mathbf{X}'\mathbf{X}=\mathbf{1}_P} \frac{1}{|X'TX||X'T^{-1}X|} = \prod_{i=1}^{p} \frac{4\lambda_i\lambda_{n-i+1}}{(\lambda_i + \lambda_{n-i+1})^2}. \tag{2.18}$$

To compute the first of these simpler antieigenvalue-type quantities, one has

$$\delta_1(T) = \frac{4\lambda_1\lambda_n}{(\lambda_1 + \lambda_n)^2}. \tag{2.19}$$

To find this quantity, we converted the problem to finding the minimum of the function $f(x, y) = 1/xy$ on the convex set $W(S)$, where $S = T + iT^{-1}$. If T is not a positive β operator on a finite dimensional space, $W(S)$ is not a polygon. We can, however, approximate $W(S)$ with polygons from inside and outside and thus obtain an approximation for

$$\delta_1(T) = \inf_{\substack{Tf \neq 0 \\ \|f\|=1}} \frac{1}{(Tx, x)(T^{-1}x, x)}, \tag{2.20}$$

an antieigenvalue-type quantity, by looking at the minimum of the function $f(x, y) = 1/xy$ on those inside and outside polygons. We can compute other simpler antieigenvalue-type quantities involved in the previous product by same way.

Theorem 2.5. *For any bounded accretive operator T, there is a sequence of finite-dimensional normal matrices $\{T_k\}$ such that*

$$\begin{gathered} \mu(T) \leq \mu(T_k), \quad k = 1, 2, 3, \ldots \\ \mu(T) = \lim_{k \Rightarrow \infty} \mu(T_k). \end{gathered} \tag{2.21}$$

Proof. Recall that for any operator T, we have

$$\mu^2(T) = \inf\left\{ \frac{x^2}{y} : x + iy \in W(S) \right\}, \tag{2.22}$$

where

$$S = \text{Re } T + iT^*T. \tag{2.23}$$

Using the notations in Theorem 2.1, for each k, there is an accretive normal operator S_k with $W(S_k) = G_k$. We can define S_k to be the diagonal matrix whose eigenvalues are $P_1, P_2, \ldots, P_k$. By the spectral mapping theorem, there exists an accretive normal matrix T_k such that

$$S_k = \text{Re } T_k + iT_k^*T_k. \tag{2.24}$$

To see this, let the complex representation of the vertex P_j, $1 \le j \le k$, of G_k be

$$P_j = x_j + iy_j, \tag{2.25}$$

then T_k can be taken to be the diagonal matrix whose eigenvalues are

$$x_j + i\sqrt{y_j - x_j^2}. \tag{2.26}$$

Note that since P_j is on the boundary of $W(S)$, we have $y_j \ge x_j^2$. Since we have

$$\mu^2(T) = \inf_{(x,y)\in\partial W(S)} \frac{x^2}{y} = \lim_{k\Rightarrow\infty}\left[\inf_{(x,y)\in\partial G_k} \frac{x^2}{y}\right], \tag{2.27}$$

we have

$$\mu^2(T) = \lim_{k\Rightarrow\infty}\left[\inf_{(x,y)\in\partial W(S_k)} \frac{x^2}{y}\right]. \tag{2.28}$$

However,

$$\lim_{k\Rightarrow\infty}\left[\inf_{(x,y)\in\partial W(S_k)} \frac{x^2}{y}\right] = \lim_{k\Rightarrow\infty}\mu^2(T_k). \tag{2.29}$$

This implies,

$$\mu^2(T) = \lim_{k\Rightarrow\infty}\mu^2(T_k). \tag{2.30}$$

Since $\mu(T)$ is positive a $\mu(T_k)$ is positive for each k, we have

$$\mu(T) = \lim_{k\Rightarrow\infty}\mu(T_k). \tag{2.31}$$

□

Theorem 2.6. *For any bounded accretive operator T there is a sequence of normal matrices $\{T_k\}$ such that*

$$\mu_{(F,q)}(T) \le \mu_{(F,q)}(T_k), \quad k = 1, 2, 3, \ldots \tag{2.32}$$

for each k and

$$\mu_{(F,q)}(T) = \lim_{k\Rightarrow\infty}\mu_{(F,q)}(T_k). \tag{2.33}$$

Proof. Recall that for any operator T we have

$$\mu^2_{(F,q)}(T) = \inf\left\{ \frac{x^q}{y} : x + iy \in W(S) \right\}, \tag{2.34}$$

where

$$S = (\mathrm{Re}\ T)^q + |F(T)|i. \tag{2.35}$$

Using the notations in Theorem 2.1, for each k there is an accretive normal matrix S_k with $W(S_k) = G_k$. We can define S_k to be the diagonal matrix whose eigenvalues are $P_1, P_2, \ldots, P_k$. By the spectral mapping theorem, there exists a normal matrix T_k such that

$$S_k = (\mathrm{Re}\ T_k)^q + |F(T_k)|i. \tag{2.36}$$

To see this, let the complex representation of the vertex P_j, $1 \leq j \leq k$, of G_k be

$$P_j = x_j + iy_j, \tag{2.37}$$

then by the spectral mapping theorem T_k can be taken to be any diagonal matrix with eigenvalues $u_j + iv_j$ where $(u_j \quad v_j)$ are any solution to the system

$$\begin{gathered} u_j = x_j^{1/q}, \\ \left[\mathrm{Re}\ F(u_j + iv_j)\right]^2 + \left[\mathrm{Im}\ F(u_j + iv_j)\right]^2 = y_j^2. \end{gathered} \tag{2.38}$$

Since we have

$$\mu^2_{(F,q)} = \inf_{(x,y)\in\partial W(S)} \frac{x^q}{y} = \lim_{k\Rightarrow\infty}\left[\inf_{(x,y)\in\partial G_k} \frac{x^q}{y}\right], \tag{2.39}$$

□

we have

$$\mu^2_{(F,q)} = \lim_{k\Rightarrow\infty}\left[\inf_{(x,y)\in\partial W(S_k)} \frac{x^q}{y}\right]. \tag{2.40}$$

However,

$$\lim_{k\Rightarrow\infty}\left[\inf_{(x,y)\in\partial W(S_k)} \frac{x^q}{y}\right] = \lim_{k\Rightarrow\infty}\mu^2_{(F,q)}(T_k). \tag{2.41}$$

This implies that

$$\mu^2_{(F,q)}(T) = \lim_{k \Rightarrow \infty} \mu^2_{(F,q)}(T_k). \tag{2.42}$$

Since $\mu_{(F,q)}$ is positive a $\mu_{(F,q)}(T_k)$ is positive for each k, we have

$$\mu_{(F,q)}(T) = \lim_{k \Rightarrow \infty} \mu_{(F,q)}(T_k). \tag{2.43}$$

Theorem 2.7. *For any bounded accretive operator T there is a sequence of finitedimensional normal matrices $\{T_k\}$ such that*

$$\begin{gathered} \delta_1(T) \le \delta_1(T_k) \quad k = 1, 2, 3, \dots, \\ \delta_1(T) = \lim_{k \Rightarrow \infty} \delta_1(T_k). \end{gathered} \tag{2.44}$$

Proof. Recall that for any operator T we have

$$\delta_1^2(T) = \inf\left\{ f(x, y) = \frac{1}{xy} : x + iy \in W(S) \right\}, \tag{2.45}$$

where

$$S = T + iT^{-1}. \tag{2.46}$$

Using the notations in Theorem 2.1, for each k there is a normal operator S_k with $W(S_k) = G_k$. We can define S_k to be the diagonal matrix whose eigenvalues are $P_1, P_2, \dots, P_k$. By the spectral mapping theorem there exist a normal matrix T_k such that

$$S_k = T_k + iT_k^{-1}. \tag{2.47}$$

To see this, let the complex representation of the vertex P_j, $1 \le j \le k$, of G_k be

$$P_j = x_j + iy_j = z_j. \tag{2.48}$$

Take T_k to be any diagonal matrix whose eigenvalues λ_j satisfy

$$\frac{1}{\lambda_j + (1/\lambda_j)} = z_j. \tag{2.49}$$

To fund the eigenvalues of such a finite-dimensional diagonal matrix T_k explicitly, we solve the previous equation for λ_j. The solutions are

$$\lambda_j = -\frac{1}{2z_j}\left(\sqrt{-4z_j^2 + 1} - 1\right), \tag{2.50}$$

or

$$\lambda_j = \frac{1}{2z_j}\left(\sqrt{-4z_j^2+1}+1\right). \tag{2.51}$$

Since we have

$$\delta_1^2(T) = \inf_{(x,y)\in\partial W(S)} \frac{1}{xy} = \lim_{k\Rightarrow\infty}\left[\inf_{(x,y)\in\partial H_k} \frac{1}{xy},\right], \tag{2.52}$$

we have

$$\delta_1^2(T) = \lim_{k\Rightarrow\infty}\left[\inf_{(x,y)\in\partial W(S_k)} \frac{1}{xy}\right]. \tag{2.53}$$

However,

$$\lim_{k\Rightarrow\infty}\left[\inf_{(x,y)\in\partial W(S_k)} \frac{x^q}{y}\right] = \lim_{k\Rightarrow\infty}\delta_1^2(T_k). \tag{2.54}$$

This implies that

$$\delta_1^2(T) = \lim_{k\Rightarrow\infty}\delta_1^2(T_k). \tag{2.55}$$

Since $\mu_{(F,q)}$ is positive a $\mu_{(F,q)}(T_k)$ is positive for each k, we have

$$\delta_1(T) = \lim_{k\Rightarrow\infty}\delta_1(T_k). \tag{2.56}$$

□

In the proofs of Theorems 2.5 through 2.7 previous, we considered accretive normal matrices S_k whose spectrum are vertices of G_k, $k = 1,2,3,\ldots$. We could also consider matrices whose spectrums are H_k, $k = 1,2,3,\ldots$. However, notice that those matrices may not be accretive for small values of k.

The term antieigenvalue was initially defined by Gustafson for accretive operators. For an accretive operator T the quantity $\mu(T)$ is nonnegative. However, in some of our previous work we have computed $\mu(T)$ for normal operators or matrices which are not necessarily accretive (see [15, 16, 20, 21]). In Theorems 2.5 through 2.7 above we assumed T is bounded accretive to ensure that $W(S)$ in these theorems is a subset of the first quadrant. Thus, for each k, $W(S_k)$ in Theorems 2.5 through 2.7 is a finite polygon in the first quadrant, making it possible to compute $\mu(T_k)$ in terms of the first and the second critical eigenvalues (see (1.18)). If T is not accretive in Theorems 2.5 through 2.7, then we can only say that $W(S)$ is a subset of the upper-half plane which implies for each k, we can only say $W(S_k)$ is a bounded polygon in the upper-half plane. This is despite the fact that (2.22), (2.34), and (2.45) in the proofs of Theorems 2.5, 2.6, and 2.7, respectively, are still valid. Therefore, Theorems 2.5 through 2.7 are valid if the operator T in these theorems is not accretive. The only challenge in this case is computing $\mu(T_k)$, for each k, in these theorems. What we know from our previous work in [16] is that $\mu(T_k)$ can be expressed in terms of one or a pair of eigenvalues of T_k. However, we do not know which eigenvalue or which pair of eigenvalues of T_k expresses $\mu(T_k)$. Of course, one can use Theorem 2.2 of [16] to compute $\mu(T_k)$, however, this requires a lot of computations, particularly for large values of k.

Example 2.8. Consider a normal matrix T whose eigenvalues are $1+\sqrt{6}i$, $-2+\sqrt{7}i$, $3+4i$, $-4+\sqrt{34}i$, $1+\sqrt{15}i$, and $-5+2i$. For this matrix, we have $\mu(T)<0$. If $S=\operatorname{Re}T+iT^{*}T$, then $W(S)$ is the polygon whose vertices are $1+7i$, $-2+11i$, $3+25i$, $-4+50i$, $1+16i$, and $-5+29i$. This polygon is not a subset of the first quadrant. Therefore, $\mu(T)$, not readily found using the first and second critical eigenvalues. Using Theorem 2.2 of [16], we need to perform a set of $6C2+6=21$ computations and compare the values obtained to find $\mu(T)$.

In [20, 22] the concepts of slant antieigenvalues and symmetric antieigenvalues were introduced. These antieigenvalue-type quantities can also be approximated by their counterparts for normal matrices. However, since slant antieigenvalues or symmetric antieigenvalues are reduced to regular antieigenvalue of another operator (see [20]), we do not need to develop separate approximations for these two antieigenvalue-type quantities. If $\partial W(S)$ is simple enough, we can use more elementary methods to find the minimum of $f(x,y)$ on $\partial W(S)$ without approximating it with polygons. For example, in [21], we computed $\mu(T)$ by direct applications of the Lagrange multipliers when $\partial W(S)$ is just an ellipse. Also, in [23] we used Lagrange multipliers directly to compute $\mu(T)$, when T is a matrix of low dimension on the real field.

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