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Research Article

Qualitative Study of Solutions of Some Difference Equations

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We obtain in this paper the solutions of the following recursive sequences $x_{n+1} = x_n x_{n-3}/x_{n-2}(\pm 1 \pm x_n x_{n-3})$, $n = 0, 1, \ldots$, where the initial conditions are arbitrary real numbers and we study the behaviors of the solutions and we obtained the equilibrium points of the considered equations. Some qualitative behavior of the solutions such as the boundedness, the global stability, and the periodicity character of the solutions in each case have been studied. We presented some numerical examples by giving some numerical values for the initial values and the coefficients of each case. Some figures have been given to explain the behavior of the obtained solutions in the case of numerical examples by using the mathematical program Mathematica to confirm the obtained results.

1. Introduction

In this paper, we obtain the solutions of the following recursive sequences:

$$x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}(\pm 1 \pm x_n x_{n-3})}, \qquad n = 0, 1, \ldots, \tag{1.1}$$

where the initial conditions are arbitrary real numbers. Also, we study the behavior of the solutions.

Recently, there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and nonrational difference equations, one can refer to the papers [1–31] and references therein.

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Recently, Agarwal and Elsayed [4] investigated the global stability and periodicity character and gave the solution of some special cases of the difference equation:

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}. \tag{1.2}$$

Aloqeili [5] has obtained the solutions of the difference equation:

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}. \tag{1.3}$$

Çinar [8, 9] investigated the solutions of the following difference equations:

$$x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}}, \qquad x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}. \tag{1.4}$$

Elabbasy et al. [10, 12] investigated the global stability and periodicity character and gave the solution of special case of the following recursive sequences:

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \qquad x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}. \tag{1.5}$$

Ibrahim [19] got the solutions of the rational difference equation:

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})}. \tag{1.6}$$

Karatas et al. [20] got the form of the solution of the difference equation:

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}. \tag{1.7}$$

Simsek et al. [26] obtained the solutions of the following difference equations:

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}. \tag{1.8}$$

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I \tag{1.9}$$

be a continuously differentiable function. Then, for every set of initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \qquad n = 0, 1, \ldots, \tag{1.10}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [21].

Definition 1.1 (Equilibrium Point). A point $\overline{x} \in I$ is called an equilibrium point of (1.10) if

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}). \tag{1.11}$$

That is, $x_n = \overline{x}$, for $n \geq 0$, is a solution of (1.10), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.2 (Stability). (i) The equilibrium point $\overline{x}$ of (1.10) is locally stable if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta, \tag{1.12}$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \forall n \geq -k. \tag{1.13}$$

(ii) The equilibrium point $\overline{x}$ of (1.10) is locally asymptotically stable if $\overline{x}$ is locally stable solution of (1.10) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma, \tag{1.14}$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}. \tag{1.15}$$

(iii) The equilibrium point $\overline{x}$ of (1.10) is global attractor if, for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}. \tag{1.16}$$

(iv) The equilibrium point $\overline{x}$ of (1.10) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of (1.10).

(v) The equilibrium point $\overline{x}$ of (1.10) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of (1.10) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, \ldots, \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.17}$$

Theorem A (see [22]). *Assume that $p_i \in R$, $i = 1, 2, \ldots, k$, and $k \in \{0, 1, 2, \ldots\}$. Then,*

$$\sum_{i=1}^{k} |p_i| < 1 \tag{1.18}$$

is a sufficient condition for the asymptotic stability of the difference equation:

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \qquad n = 0, 1, \ldots. \tag{1.19}$$

Definition 1.3 (Periodicity). A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2. On the Equation $X_{n+1} = x_n x_{n-3}/(x_{n-2}(1 + x_n x_{n-3}))$

In this section, we give a specific form of the solution of the first equation in the form:

$$x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}(1 + x_n x_{n-3})}, \qquad n = 0, 1, \ldots, \tag{2.1}$$

where the initial values are arbitrary positive real numbers.

Theorem 2.1. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of* (2.1). *Then, for $n = 0, 1, \ldots$,*

$$\begin{aligned}
x_{6n-3} &= d\prod_{i=0}^{n-1}\left(\frac{1+6iad}{1+(6i+3)ad}\right), & x_{6n-2} &= c\prod_{i=0}^{n-1}\left(\frac{1+(6i+1)ad}{1+(6i+4)ad}\right),\\
x_{6n-1} &= b\prod_{i=0}^{n-1}\left(\frac{1+(6i+2)ad}{1+(6i+5)ad}\right), & x_{6n} &= a\prod_{i=0}^{n-1}\left(\frac{1+(6i+3)ad}{1+(6i+6)ad}\right),\\
x_{6n+1} &= \frac{ad}{c(1+ad)}\prod_{i=0}^{n-1}\left(\frac{1+(6i+4)ad}{1+(6i+7)ad}\right), & x_{6n+2} &= \frac{ad}{b(1+2ad)}\prod_{i=0}^{n-1}\left(\frac{1+(6i+5)ad}{1+(6i+8)ad}\right),
\end{aligned} \tag{2.2}$$

where $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_0 = a$.

Proof. For $n = 0$, the result holds. Now, suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
x_{6n-9} &= d\prod_{i=0}^{n-2}\left(\frac{1+6iad}{1+(6i+3)ad}\right), & x_{6n-8} &= c\prod_{i=0}^{n-2}\left(\frac{1+(6i+1)ad}{1+(6i+4)ad}\right),\\
x_{6n-7} &= b\prod_{i=0}^{n-2}\left(\frac{1+(6i+2)ad}{1+(6i+5)ad}\right), & x_{6n-6} &= a\prod_{i=0}^{n-2}\left(\frac{1+(6i+3)ad}{1+(6i+6)ad}\right),\\
x_{6n-5} &= \frac{ad}{c(1+ad)}\prod_{i=0}^{n-2}\left(\frac{1+(6i+4)ad}{1+(6i+7)ad}\right), & x_{6n-4} &= \frac{ad}{b(1+2ad)}\prod_{i=0}^{n-2}\left(\frac{1+(6i+5)ad}{1+(6i+8)ad}\right).
\end{aligned} \tag{2.3}$$

Now, it follows from (2.1) that

$$\begin{aligned}
&x_{6n-3}\\
&= \frac{x_{6n-4}x_{6n-7}}{x_{6n-6}(1+x_{6n-4}x_{6n-7})}\\
&= \frac{\dfrac{ad}{b(1+2ad)}\prod_{i=0}^{n-2}\left(\dfrac{1+(6i+5)ad}{1+(6i+8)ad}\right)b\prod_{i=0}^{n-2}\left(\dfrac{1+(6i+2)ad}{1+(6i+5)ad}\right)}{a\prod_{i=0}^{n-2}\left(\dfrac{1+(6i+3)ad}{1+(6i+6)ad}\right)\left(1+\dfrac{ad}{b(1+2ad)}\prod_{i=0}^{n-2}\left(\dfrac{1+(6i+5)ad}{1+(6i+8)ad}\right)b\prod_{i=0}^{n-2}\left(\dfrac{1+(6i+2)ad}{1+(6i+5)ad}\right)\right)}\\
&= \frac{\left(\dfrac{ad}{1+2ad}\right)\prod_{i=0}^{n-2}\left(\dfrac{1+(6i+5)ad}{1+(6i+8)ad}\right)\prod_{i=0}^{n-2}\left(\dfrac{1+(6i+2)ad}{1+(6i+5)ad}\right)}{a\prod_{i=0}^{n-2}\left(\dfrac{1+(6i+3)ad}{1+(6i+6)ad}\right)\left(1+\dfrac{ad}{(1+2ad)}\prod_{i=0}^{n-2}\left(\dfrac{1+(6i+2)ad}{1+(6i+8)ad}\right)\right)}\\
&= \frac{\left(\dfrac{ad}{(1+(6n+2)ad)}\right)}{\left(a\prod_{i=0}^{n-2}\left(\dfrac{1+(6i+3)ad}{1+(6i+6)ad}\right)\right)\left(1+\dfrac{ad}{(1+(6n+2)ad)}\right)}\\
&= \frac{d}{\left(\prod_{i=0}^{n-2}\left(\dfrac{1+(6i+3)ad}{1+(6i+6)ad}\right)\right)((1+(6n+2)ad)+ad)}\\
&= d\prod_{i=0}^{n-2}\left(\frac{1+(6i+6)ad}{1+(6i+3)\ ad}\right)\frac{1}{(1+(6n+3)ad)}.
\end{aligned} \tag{2.4}$$

Hence, we have

$$x_{6n-3} = d\prod_{i=0}^{n-1}\left(\frac{1+6iad}{1+(6i+3)ad}\right). \tag{2.5}$$

Similarly,

$$\begin{aligned}
x_{6n+1} &= \frac{x_{6n}x_{6n-3}}{x_{6n-2}(1+x_{6n}x_{6n-3})} \\
&= \frac{ad\prod_{i=0}^{n-1}\left(\frac{1+(6i+3)ad}{1+(6i+6)ad}\right)d\prod_{i=0}^{n-1}\left(\frac{1+6iad}{1+(6i+3)ad}\right)}{c\prod_{i=0}^{n-1}\left(\frac{1+(6i+1)ad}{1+(6i+4)ad}\right)\left(1+a\prod_{i=0}^{n-1}\left(\frac{1+(6i+3)ad}{1+(6i+6)ad}\right)d\prod_{i=0}^{n-1}\left(\frac{1+6iad}{1+(6i+3)ad}\right)\right)} \\
&= \frac{ad\prod_{i=0}^{n-1}\left(\frac{1+6iad}{1+(6i+6)ad}\right)}{\left(c\prod_{i=0}^{n-1}\left(\frac{1+(6i+1)ad}{1+(6i+4)ad}\right)\right)\left(1+ad\prod_{i=0}^{n-1}\left(\frac{1+6iad}{1+(6i+6)ad}\right)\right)} \\
&= \frac{\left(\frac{ad}{(1+6nad)}\right)}{\left(c\prod_{i=0}^{n-1}\left(\frac{1+(6i+1)ad}{1+(6i+4)ad}\right)\right)\left(1+\frac{ad}{1+6nad}\right)} \\
&= \frac{ad}{\left(c\prod_{i=0}^{n-1}\left(\frac{1+(6i+1)ad}{1+(6i+4)ad}\right)\right)(1+6nad+ad)} \\
&= \prod_{i=0}^{n-1}\left(\frac{1+(6i+4)ad}{1+(6i+1)ad}\right)\left(\frac{ad}{c(1+7nad)}\right).
\end{aligned} \tag{2.6}$$

Hence, we have

$$x_{6n+1} = \frac{ad}{c(1+ad)}\prod_{i=0}^{n-1}\left(\frac{1+(6i+4)ad}{1+(6i+7)ad}\right). \tag{2.7}$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. □

Theorem 2.2. *Equation* (2.1) *has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Proof. For the equilibrium points of (2.1), we can write

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}\left(1+\overline{x}^3\right)}. \tag{2.8}$$

Then, we have

$$\begin{gathered}\overline{x}^2\left(1+\overline{x}^2\right) = \overline{x}^2, \\ \overline{x}^2\left(1+\overline{x}^2-1\right) = 0,\end{gathered} \tag{2.9}$$

or

$$\overline{x}^4 = 0. \tag{2.10}$$

Thus the equilibrium point of (2.1) is $\overline{x} = 0$.

Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w) = \frac{uw}{v(1+uw)}. \tag{2.11}$$

Therefore, it follows that

$$\begin{gathered}f_u(u,v,w) = \frac{w}{v(1+uw)^2}, \qquad f_v(u,v,w) = -\frac{uw}{v^2(1+uw)}, \\ f_w(u,v,w) = \frac{u}{v(1+uw)^2},\end{gathered} \tag{2.12}$$

we see that

$$f_u(\overline{x},\overline{x},\overline{x}) = 1, \qquad f_v(\overline{x},\overline{x},x) = 1, \qquad f_w(\overline{x},\overline{x},\overline{x}) = 1. \tag{2.13}$$

The proof follows by using Theorem A. □

Numerical Examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to (2.1).

Example 2.3. We assume $x_{-3} = 11, x_{-2} = 7,\ x_{-1} = 13,\ x_0 = 3$, (see Figure 1).

Example 2.4. See Figure 2, since $x_{-3} = 2,\ x_{-2} = 9,\ x_{-1} = 3,\ x_0 = 5$.

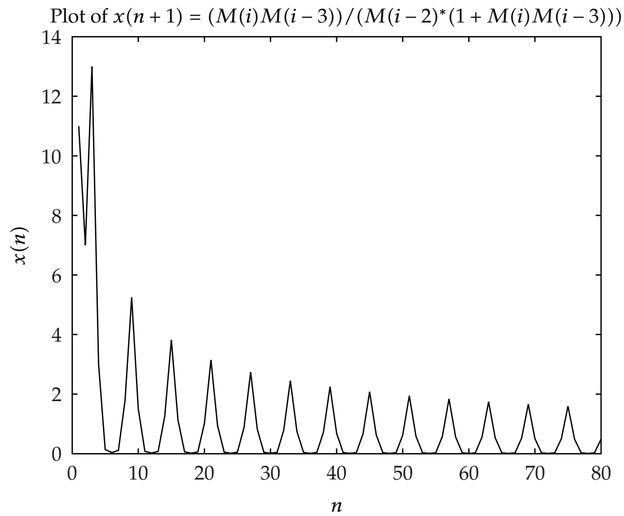

Figure 1

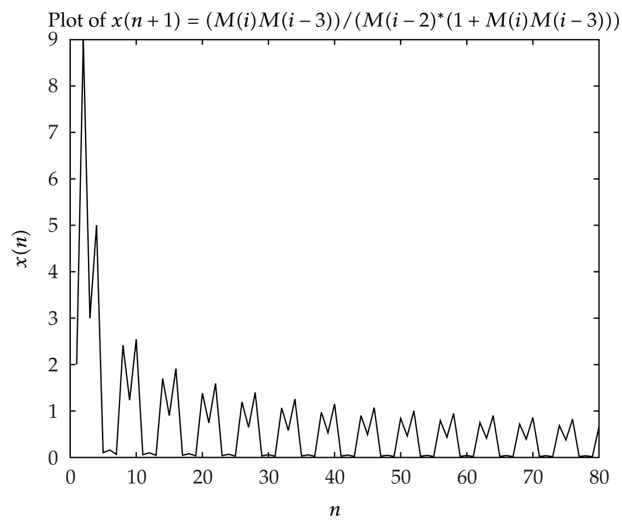

Figure 2

3. On the Equation $X_{n+1} = x_n x_{n-3}/(x_{n-2}(-1 + x_n x_{n-3}))$

In this section, we obtain the solution of the second equation in the form:

$$x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}(-1 + x_n x_{n-3})}, \qquad n = 0, 1, \ldots, \tag{3.1}$$

where the initial values are arbitrary nonzero real numbers with $x_0 x_{-3} \neq 1$.

Theorem 3.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (3.1). *Then,* (3.1) *has unboundedness solution and, for* $n = 0, 1, \ldots,$

$$\begin{aligned} x_{6n-3} &= \frac{d}{(-1+ad)^n}, & x_{6n-2} &= c(-1+ad)^n, & x_{6n-1} &= \frac{b}{(-1+ad)^n}, \\ x_{6n} &= a(-1+ad)^n, & x_{6n+1} &= \frac{ad}{c(-1+ad)^{n+1}}, & x_{6n+2} &= \frac{ad}{b}(-1+ad)^n, \end{aligned} \tag{3.2}$$

where $x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a$.

Proof. For $n = 0$ the result holds. Now, suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned} x_{6n-9} &= \frac{d}{(-1+ad)^{n-1}}, & x_{6n-8} &= c(-1+ad)^{n-1}, & x_{6n-7} &= \frac{b}{(-1+ad)^{n-1}}, \\ x_{6n-6} &= a(-1+ad)^{n-1}, & x_{6n-5} &= \frac{ad}{c(-1+ad)^n}, & x_{6n-4} &= \frac{ad}{b}(-1+ad)^{n-1}. \end{aligned} \tag{3.3}$$

Now, it follows from (3.1) that

$$\begin{aligned} x_{6n-3} &= \frac{x_{6n-4}x_{6n-7}}{x_{6n-6}(-1+x_{6n-4}x_{6n-7})} = \frac{(ad/b)(-1+ad)^{n-1}b/(-1+ad)^{n-1}}{a(-1+ad)^{n-1}\left(-1+(ad/b)(-1+ad)^{n-1}b/(-1+ad)^{n-1}\right)} \\ &= \frac{d}{(-1+ad)^{n-1}(-1+ad)} = \frac{d}{(-1+ad)^n}, \\ x_{6n-2} &= \frac{x_{6n-3}x_{6n-6}}{x_{6n-5}(-1+x_{6n-3}x_{6n-6})} = \frac{\left(d/(-1+ad)^n\right)a(-1+ad)^{n-1}}{ad/c(-1+ad)^n\left(-1+\left(d/(-1+ad)^n\right)a(-1+ad)^{n-1}\right)} \\ &= \frac{c(-1+ad)^{n-1}}{(-1+(ad/(-1+ad)))}\frac{(-1+ad)}{(-1+ad)} = c(-1+ad)^n, \\ x_{6n-1} &= \frac{x_{6n-2}x_{6n-5}}{x_{6n-4}(-1+x_{6n-2}x_{6n-5})} = \frac{c(-1+ad)^n ad/c(-1+ad)^n}{(ad/b)(-1+ad)^{n-1}(-1+c(-1+ad)^n(ad/c(-1+ad)^n))} \\ &= \frac{b}{(-1+ad)^{n-1}(-1+ad)} = \frac{b}{(-1+ad)^n}. \end{aligned} \tag{3.4}$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. □

Theorem 3.2. *Equation* (3.1) *has a periodic solution of period six iff* $ad = 2$ *and will take the form:* $\{d, c, b, a, ad/c, ad/b, d, c, b, a, ad/c, ad/b, \ldots\}$.

Proof. First suppose that there exists a prime period six solution:

$$d, c, b, a, \frac{ad}{c}, \frac{ad}{b}, d, c, b, a, \frac{ad}{c}, \frac{ad}{b}, \ldots, \tag{3.5}$$

of (3.1), we see from the form of the solution of (3.1) that

$$\begin{aligned} d &= \frac{d}{(-1+ad)^n}, \qquad c = c(-1+ad)^n, \qquad b = \frac{b}{(-1+ad)^n}, \\ a &= a(-1+ad)^n, \qquad \frac{ad}{c} = \frac{ad}{c(-1+ad)^{n+1}}, \qquad \frac{ad}{b} = \frac{ad}{b}(-1+ad)^n, \end{aligned} \tag{3.6}$$

or

$$(-1+ad)^n = 1. \tag{3.7}$$

Then,

$$ad = 2. \tag{3.8}$$

Second, assume that $ad = 2$. Then, we see from the form of the solution of (3.1) that

$$x_{6n-3} = d, \qquad x_{6n-2} = c, \qquad x_{6n-1} = b, \qquad x_{6n} = a, \qquad x_{6n+1} = \frac{ad}{c}, \qquad x_{6n+2} = \frac{ad}{b}. \tag{3.9}$$

Thus, we have a periodic solution of period six and the proof is complete. □

Theorem 3.3. *Equation* (3.1) *has two equilibrium points which are* $0, \sqrt[3]{2}$ *and these equilibrium points are not locally asymptotically stable.*

Proof. For the equilibrium points of (3.1), we can write

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}\left(-1+\overline{x}^2\right)}. \tag{3.10}$$

Then, we have

$$\overline{x}^2\left(-1+\overline{x}^2\right) = \overline{x}^2, \tag{3.11}$$

or

$$\overline{x}^2\left(\overline{x}^2 - 2\right) = 0, \tag{3.12}$$

Thus, the equilibrium points of (3.1) are $0, \pm\sqrt{2}$.

Let $f : (0,\infty)^3 \to (0,\infty)$ be a function defined by

$$f(u,v,w) = \frac{uw}{v(-1+uw)}. \tag{3.13}$$

Therefore, it follows that

$$f_u(u,v,w) = -\frac{w}{v(-1+uw)^2}, \qquad f_v(u,v,w) = -\frac{uw}{v^2(-1+uw)}, \\ f_w(u,v,w) = -\frac{u}{v(-1+uw)^2}, \tag{3.14}$$

we see that

$$f_u(\overline{x},\overline{x},\overline{x}) = -1, \qquad f_v(\overline{x},\overline{x},x) = \pm 1, \qquad f_w(\overline{x},\overline{x},\overline{x}) = -1. \tag{3.15}$$

The proof follows by using Theorem A. □

Numerical Examples

Here, we will represent different types of solutions of (3.1).

Example 3.4. We consider $x_{-3} = 2, x_{-2} = 9,\ \ x_{-1} = 3,\ \ x_0 = 5$, (see Figure 3).

Example 3.5. See Figure 4 since $x_{-3} = 7,\ \ x_{-2} = 2,\ \ x_{-1} = 8,\ \ x_0 = 2/7$.

The following cases can be proved similarly.

4. On the Equation $X_{n+1} = x_n x_{n-3}/(x_{n-2}(1 - x_n x_{n-3}))$

In this section, we get the solution of the third following equation:

$$x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}(1 - x_n x_{n-3})}, \qquad n = 0,1,\ldots, \tag{4.1}$$

where the initial values are arbitrary positive real numbers.

Theorem 4.1. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of* (4.1). *Then, for $n = 0,1,\ldots$,*

$$\begin{gathered} x_{6n-3} = d\prod_{i=0}^{n-1}\left(\frac{1-6iad}{1-(6i+3)ad}\right), \qquad x_{6n-2} = c\prod_{i=0}^{n-1}\left(\frac{1-(6i+1)ad}{1-(6i+4)ad}\right), \\ x_{6n-1} = b\prod_{i=0}^{n-1}\left(\frac{1-(6i+2)ad}{1-(6i+5)ad}\right), \qquad x_{6n} = a\prod_{i=0}^{n-1}\left(\frac{1-(6i+3)ad}{1-(6i+6)ad}\right), \\ x_{6n+1} = \frac{ad}{c(1-ad)}\prod_{i=0}^{n-1}\left(\frac{1-(6i+4)ad}{1-(6i+7)ad}\right), \qquad x_{6n+2} = \frac{ad}{b(1-2ad)}\prod_{i=0}^{n-1}\left(\frac{1-(6i+5)ad}{1-(6i+8)ad}\right). \end{gathered} \tag{4.2}$$

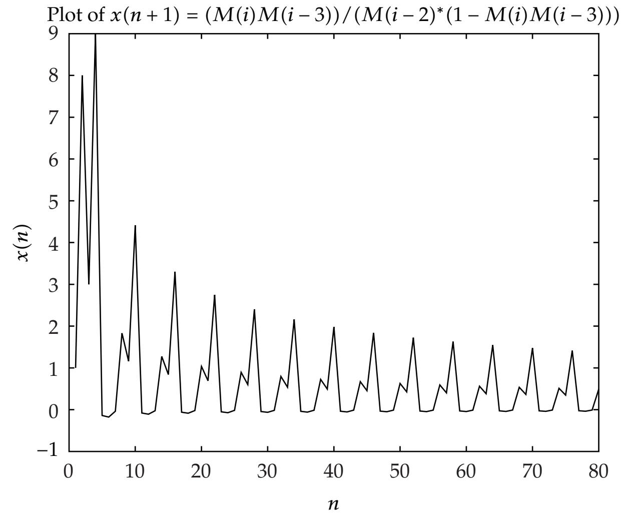

Figure 3

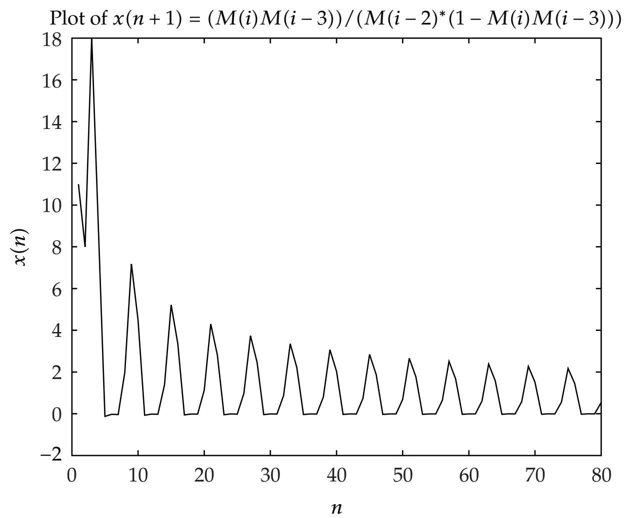

Figure 4

Theorem 4.2. *Equation* (4.1) *has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Example 4.3. Assume that $x_{-3} = 1, x_{-2} = 8,\ x_{-1} = 3,\ x_0 = 9,$ (see Figure 5).

Example 4.4. See Figure 6 since $x_{-3} = 11,\ x_{-2} = 8,\ x_{-1} = 18,\ x_0 = 9.$

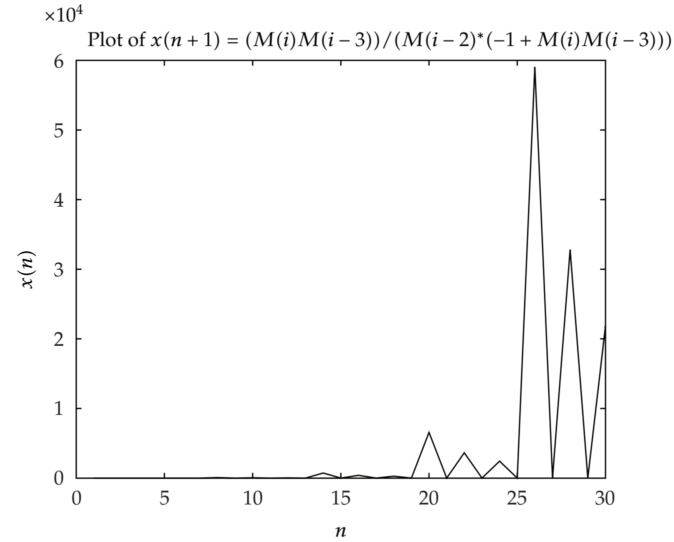

Figure 5

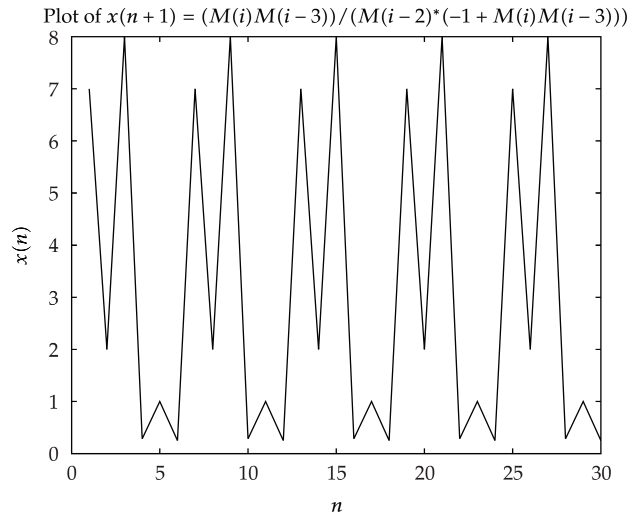

Figure 6

5. On the Equation $X_{n+1} = x_n x_{n-3}/(x_{n-2}(-1 - x_n x_{n-3}))$

Here, we obtain a form of the solutions of the equation

$$x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}(-1 - x_n x_{n-3})}, \qquad n = 0, 1, \ldots, \tag{5.1}$$

where the initial values are arbitrary nonzero real numbers with $x_{-3}x_0 \neq -1$.

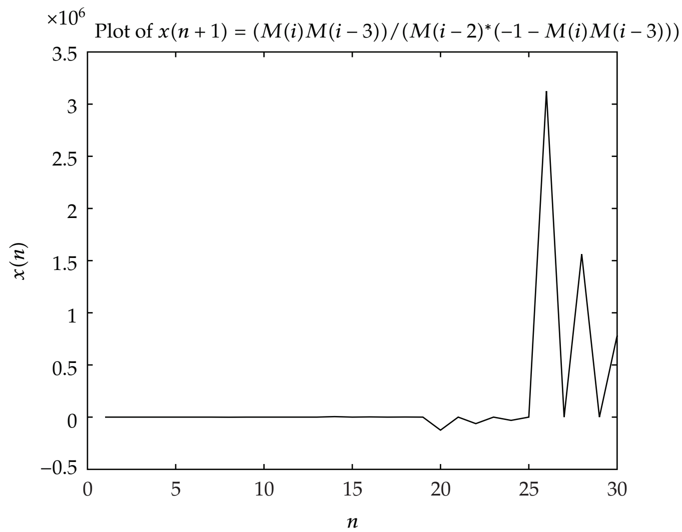

Figure 7

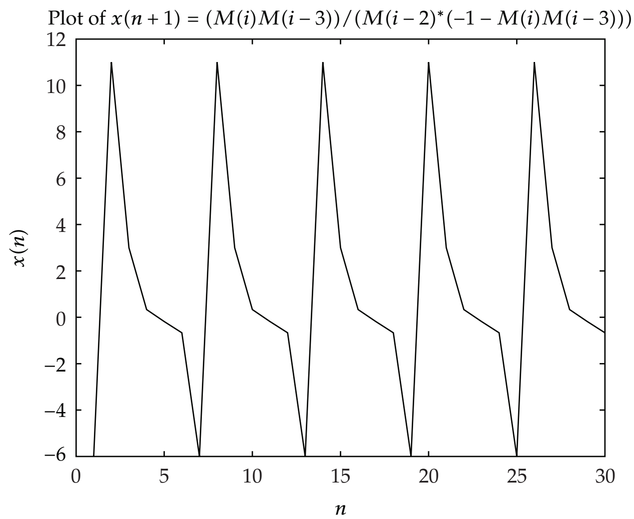

Figure 8

Theorem 5.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (5.1). *Then,* (5.1) *has unboundedness solution and, for* $n = 0, 1, \ldots,$

$$x_{6n-3} = \frac{d}{(-1-ad)^n}, \qquad x_{6n-2} = c(-1-ad)^n, \qquad x_{6n-1} = \frac{b}{(-1-ad)^n},$$
$$x_{6n} = a(-1-ad)^n, \qquad x_{6n+1} = \frac{ad}{c(-1-ad)^{n+1}}, \qquad x_{6n+2} = \frac{ad}{b}(-1-ad)^n. \tag{5.2}$$

Theorem 5.2. *Equation* (5.1) *has a periodic solution of period six if and only if* $ad = -2$ *and will take the form:*$\{d, c, b, a, ad/c, ad/b, d, c, b, a, ad/c\ , ad/b\ , \ldots\}$.

Theorem 5.3. *Equation* (5.1) *has a unique equilibrium point which is* 0, *and this equilibrium point is not locally asymptotically stable.*

Example 5.4. Consider $x_{-3} = 6,\ x_{-2} = 8,\ x_{-1} = 12,\ x_0 = 4,$ (see Figure 7).

Example 5.5. Figure 8 shows the solutions when $x_{-3} = -6,\ x_{-2} = 11,\ x_{-1} = 3,\ x_0 = 2/6$.

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