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Research Article

Finite Difference Method for Hyperbolic Equations with the Nonlocal Integral Condition

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The stable difference schemes for the approximate solution of the nonlocal boundary value problem for multidimensional hyperbolic equations with dependent in space variable coefficients are presented. Stability of these difference schemes and of the first- and second-order difference derivatives is obtained. The theoretical statements for the solution of these difference schemes for one-dimensional hyperbolic equations are supported by numerical examples.

1. Introduction

Nonlocal problems have been a major research area in modern physics, biology, chemistry, and engineering when it is impossible to determine the boundary values of the unknown function. Numerical methods and theory of solutions of the nonlocal boundary value problems for partial differential equations of variable type were carried out in for example, [1–10] and the references therein. Hyperbolic equations with nonlocal integral conditions are widely used for chemical heterogeneity, plasma physics, thermoelasticity, and so forth. The solutions of hyperbolic equations with nonlocal integral conditions were investigated in [11–15]. The method of operators as a tool for investigation of the solution to hyperbolic equations in Hilbert and Banach spaces has been studied extensively (see, e.g., [16–28]).

In the present paper, the nonlocal boundary value problem for the multidimensional hyperbolic equation with nonlocal integral condition

$$\frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{m} (a_r(x) u_{x_r})_{x_r} = f(t,x),$$
$$x = (x_1, \dots, x_m) \in \Omega, \quad 0 < t < 1,$$

$$u(0,x) = \int_0^1 \alpha(\rho)u(\rho,x)d\rho + \varphi(x), \quad u_t(0,x) = \psi(x), \quad x \in \overline{\Omega},$$
$$u(t,x) = 0, \quad 0 < t < 1, \; x \in S \tag{1.1}$$

is considered. Here Ω is the unit open cube in the m-dimensional Euclidean space $\mathbb{R}^m = \{x = (x_1,\ldots,x_m) : 0 < x_j < 1, 1 \le j \le m\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$, $a_r(x)$ $(x \in \Omega)$, $\varphi(x)$, $\psi(x)$ $(x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (0,1), x \in \Omega)$ are given smooth functions, and $a_r(x) \ge a > 0$.

The first and second orders of approximation in t and the second order of approximation in space variables difference schemes for the approximate solution of nonlocal boundary value problem (1.1) are presented. Stability of these difference schemes and of the first- and second-order difference derivatives established. Error analysis is obtained by numerical solutions of one-dimensional hyperbolic equations with integral condition.

2. Difference Schemes and Stability Estimates

The discretization of problem (1.1) is carried out in two steps. In the first step, let us define the grid sets

$$\widetilde{\Omega}_h = \{x = x_r = (h_1 r_1,\ldots,h_m r_m),\; r = (r_1,\ldots,r_m),\; 0 \le r_j \le N_j,\; h_j N_j = 1,\; j = 1,\ldots,m\},$$
$$\Omega_h = \widetilde{\Omega}_h \cap \Omega, \quad S_h = \widetilde{\Omega}_h \cap S. \tag{2.1}$$

We introduce the Hilbert space $L_{2h} = L_2(\widetilde{\Omega}_h)$ of the grid functions

$$\varphi^h(x) = \{\varphi(h_1 r_1,\ldots,h_m r_m)\} \tag{2.2}$$

defined on $\widetilde{\Omega}_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_2(\widetilde{\Omega}_h)} = \left(\sum_{x\in\overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1,\cdots,h_m\right)^{1/2}. \tag{2.3}$$

To the differential operator A^x generated by problem (1.1), we assign the difference operator A_h^x by the formula

$$A_h^x u_x^h = -\sum_{r=1}^m \left(a_r(x) u_{\overline{x_r}}^h\right)_{x_r r_j} \tag{2.4}$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\widetilde{\Omega}_h)$. With the help of A_h^x we arrive at the nonlocal boundary value problem

$$\begin{aligned}
&\frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x), \quad 0 < t < 1, \ x \in \widetilde{\Omega}_h, \\
&v^h(0,x) = \int_0^1 \alpha(\rho) v^h(\rho,x)\, d\rho + \varphi^h(x), \quad x \in \widetilde{\Omega}_h, \\
&\frac{dv^h(0,x)}{dt} = \psi^h(x), \quad x \in \widetilde{\Omega}_h
\end{aligned} \tag{2.5}$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (2.5) by the difference scheme

$$\begin{aligned}
&\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_{k+1}^h(x) = f_{k+1}^h(x), \\
&f_{k+1}^h(x) = f^h(t_{k+1}, x), \quad t_{k+1} = (k+1)\tau, \\
&\qquad 1 \le k \le N-1, \quad N\tau = 1, \quad x \in \widetilde{\Omega}_h, \\
&u_0^h(x) = \sum_{m=1}^{N} \alpha(t_m) u_m^h(x)\tau + \varphi^h(x), \quad x \in \widetilde{\Omega}_h, \\
&\left(I + \tau^2 A_h^x\right)\left(u_1^h(x) - u_0^h(x)\right)\tau^{-1} = \psi^h(x), \quad x \in \widetilde{\Omega}_h
\end{aligned} \tag{2.6}$$

of the first order accuracy in t.

Theorem 2.1. *Let τ and h be sufficiently small numbers. Then, the solutions of the difference scheme* (2.6) *satisfy the following stability estimates:*

$$\begin{aligned}
&\max_{0\le k\le N}\left\|u_k^h\right\|_{L_{2h}} + \max_{0\le k\le N}\sum_{r=1}^{m}\left\|\left(u_k^h\right)_{x_r r_j}\right\|_{L_{2h}} \\
&\quad \le M_1\left[\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{L_{2h}} + \sum_{r=1}^{m}\left\|\varphi^h_{\overline{x_r} r_j}\right\|_{L_{2h}}\right], \\
&\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right)\right\|_{L_{2h}} + \max_{0\le k\le N}\sum_{r=1}^{m}\left\|\left(u_k^h\right)_{\overline{x_r} x_r r_j}\right\|_{L_{2h}} \\
&\quad \le M_1\left[\left\|f_1^h\right\|_{L_{2h}} + \max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h - f_{k-1}^h\right)\right\|_{L_{2h}} + \sum_{r=1}^{m}\left\|\psi^h_{\overline{x_r} r_j}\right\|_{L_{2h}} + \sum_{r=1}^{m}\left\|\varphi^h_{\overline{x_r} x_r r_j}\right\|_{L_{2h}}\right],
\end{aligned} \tag{2.7}$$

where M_1 does not depend on τ, h, $\varphi^h(x)$, $\psi^h(x)$, and $f_k^h(x), 1 \le k < N$.

The proof of Theorem 2.1 is based on the symmetry property of difference operator A_h^x defined by the formula (2.4) and on the following theorem on coercivity inequality of the elliptic difference problem.

Theorem 2.2. *For the solutions of the elliptic difference problem*

$$A_h^x u^h(x) = \omega^h(x), \quad x \in \Omega_h, \qquad u^h(x) = 0, \quad x \in S_h, \tag{2.8}$$

the following coercivity inequality holds [29]:

$$\sum_{r=1}^{m} \left\| u^h_{\overline{x_r} x_r r_j} \right\|_{L_{2h}} \le M \left\| \omega^h \right\|_{L_{2h}}. \tag{2.9}$$

Moreover, the second order of accuracy difference schemes

$$\begin{gathered}
\tau^{-2}\left[u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\right] + \frac{1}{2}A_h^x u_k^h(x) + \frac{1}{4}A_h^x\left[u_{k+1}^h(x) + u_{k-1}^h(x)\right] = f_k^h(x), \quad x \in \widetilde{\Omega}_h, \\
f_k^h = f_k^h(t_k, x), \quad t_k = k\tau, \ 1 \le k \le N-1, \ N\tau = 1, \\
u_0^h(x) = \sum_{j=1}^{N} \frac{\tau}{2}\left[\alpha(t_j)u^h(t_j, x) + \alpha(t_{j-1})u^h(t_{j-1}, x)\right] + \varphi^h(x), \quad x \in \widetilde{\Omega}_h, \\
\left(I + \frac{\tau^2 A_h^x}{2}\right)\tau^{-1}\left[u_1^h(x) - u_0^h(x)\right] - \frac{\tau}{2}\left[f_0^h(x) - A_h^x u_0^h(x)\right] = \psi^h(x), \\
f_0^h = f^h(0, x), \quad f_N^h = f^h(1, x), \quad x \in \widetilde{\Omega}_h,
\end{gathered} \tag{2.10}$$

and

$$\begin{gathered}
\tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right) + A_h^x u_k^h + \frac{\tau^2}{4}\left(A_h^x\right)^2 u_{k+1}^h = f_k^h(x), \\
f_k^h = f_k^h(t_k, x), \quad t_k = k\tau, \ 1 \le k \le N-1, \ N\tau = 1, \ x \in \widetilde{\Omega}_h, \\
u_0^h(x) = \sum_{j=1}^{N} \frac{\tau}{2}\left[\alpha(t_j)u^h(t_j, x) + \alpha(t_{j-1})u^h(t_{j-1}, x)\right] + \varphi^h(x), \quad x \in \widetilde{\Omega}_h, \\
\left(I + \frac{\tau^2 A_h^x}{4}\right)\left[\left(I + \frac{\tau^2 A_h^x}{4}\right)\tau^{-1}\left[u_1^h(x) - u_0^h(x)\right] - \frac{\tau}{2}\left[f_0^h(x) - A_h^x u_0^h(x)\right]\right] = \psi^h(x), \\
f_0^h = f^h(0, x), \quad f_N^h = f^h(1, x), \quad x \in \widetilde{\Omega}_h
\end{gathered} \tag{2.11}$$

for approximately solving the boundary value problem (1.1) is presented.

We have the following theorem.

Theorem 2.3. *Let τ and h be sufficiently small numbers. Then, for the solution of the difference schemes* (2.10) *and* (2.11) *the stability inequalities*

$$\begin{aligned}
&\max_{0\le k\le N}\left\|u_k^h\right\|_{L_{2h}}+\max_{0\le k\le N}\sum_{r=1}^{m}\left\|\left(u_k^h\right)_{x_r,j_r}\right\|_{L_{2h}}\\
&\quad\le M_2\left[\max_{0\le k\le N}\left\|f_k^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\sum_{r=1}^{m}\left\|\varphi^h_{\overline{x_r},j_r}\right\|_{L_{2h}}\right],\\
&\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}\\
&\quad\le M_2\left[\left\|f_0^h\right\|_{L_{2h}}+\max_{1\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}+\sum_{r=1}^{m}\left\|\psi^h_{\overline{x_r},j_r}\right\|_{L_{2h}}+\sum_{r=1}^{m}\left\|\varphi^h_{\overline{x_r},x_r,j_r}\right\|_{L_{2h}}\right]
\end{aligned}\tag{2.12}$$

hold, where M_2 is independent of τ, h, $\varphi^h(x)$, $\psi^h(x)$, and $f_k^h(x)$, $0\le k\le N$.

The proof of Theorem 2.3 is based on the symmetry property of difference operator A_h^x defined by formula (2.4) and on Theorem 2.2 on coercivity inequality of elliptic difference problem (2.8).

In Theorems 2.1 and 2.3, the constants M_1 and M_2 cannot be obtained sharply. Therefore, in the following section, we will study the accuracy of these difference schemes for solving the one-dimensional hyperbolic equations with the integral condition. Moreover, the method is supported by numerical experiments.

3. Numerical Analysis

3.1. The First Order of Accuracy in Time Difference Scheme

In this section, the nonlocal boundary value problem

$$\begin{gathered}
\frac{\partial^2 u(t,x)}{\partial t^2}-(1+x)\frac{\partial^2 u(t,x)}{\partial x^2}-u_x(t,x)+u(t,x)=f(t,x),\\
f(t,x)=[(x+2)\sin x-\cos x]\left(e^t-1-t\right)+e^t\sin x,\quad 0<t<1,\ 0<x<\pi,\\
u(0,x)=\int_0^1 e^{-s}u(s,x)ds+\psi(x),\quad \psi(x)=\left(1-3e^{-1}\right)\sin x,\\
u_t(0,x)=0,\quad 0\le x\le\pi,\\
u(t,0)=u(t,\pi)=0,\quad 0\le t\le 1
\end{gathered}\tag{3.1}$$

for one dimensional hyperbolic equation is considered.

The exact solution of problem (3.1) is

$$u(t,x)=\left(e^t-1-t\right)\sin x.\tag{3.2}$$

Applying the formulas

$$\frac{u(x_{n+1})-u(x_{n-1})}{2h}-u'(x_n)=O\left(h^2\right),$$
$$\frac{u(\tau)-u(0)}{\tau}-u'(0)=O(\tau), \tag{3.3}$$
$$\frac{u(x_{n+1})-2u(x_n)+u(x_{n-1})}{h^2}-u''(x_n)=O\left(h^2\right)$$

and using the first order of accuracy in t implicit difference scheme (2.6), we obtain the difference scheme first order of accuracy in t and second order of accuracy in x

$$\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-(1+x_n)\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}-\frac{u_{n+1}^{k+1}-u_{n-1}^{k+1}}{2h}+u_n^{k+1}=f(t_{k+1},x_n),$$
$$f(t_{k+1},x_n)=[(x_n+2)\sin x_n-\cos x_n]\left(e^{t_{k+1}}-1-t_{k+1}\right)+e^{t_{k+1}}\sin x_n,$$
$$N\tau=1,\quad x_n=nh,\quad 1\le n\le M-1,\quad Mh=\pi,$$
$$u_n^0-\sum_{k=1}^{N}e^{-k\tau}\tau u_n^k=\varphi(x_n),\quad t_k=k\tau,\ 1\le k\le N-1, \tag{3.4}$$
$$\varphi(x_n)=\left(1-3e^{-1}\right)\sin x_n,\quad 0\le n\le M,$$
$$u_0^k=u_M^k=0,\quad u_n^1-u_n^0=0,\quad 0\le k\le N,\ 1\le n\le M-1$$

for approximate solutions of nonlocal boundary value problem (3.1). It can be written in the matrix form

$$A_nu_{n+1}+B_nu_n+C_nu_{n-1}=D\varphi_n,\quad 1\le n\le M-1,$$
$$u_0=\vec{0},\qquad u_M=\vec{0}. \tag{3.5}$$

Here

$$A_n=\begin{bmatrix}0&0&0&0&\cdots&0&0\\0&0&a_n&0&\cdots&0&0\\0&0&0&a_n&\cdots&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&\cdots&a_n&0\\0&0&0&0&\cdots&0&a_n\\0&0&0&0&\cdots&0&0\end{bmatrix}_{(N+1)\times(N+1)},$$

$$B_n = \begin{bmatrix} 1 & -\tau e^{-\tau} & -\tau e^{-2\tau} & -\tau e^{-3\tau} & \cdots & -\tau e^{-(N-1)\tau} & -\tau e^{-N\tau} \\ b & c & d_n & 0 & \cdots & 0 & 0 \\ 0 & b & c & d_n & \cdots & 0 & 0 \\ 0 & 0 & b & c & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & d_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & c & d_n \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$C_n = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & e_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & e_n & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & e_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & e_n \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$a_n = \frac{-(1+x_n)}{h^2} - \frac{1}{2h}, \qquad b = \frac{1}{\tau^2}, \qquad c = \frac{-2}{\tau^2},$$

$$d_n = \frac{1}{\tau^2} + \frac{2(1+x_n)}{h^2} + 1, \qquad e_n = \frac{-(1+x_n)}{h^2} + \frac{1}{2h},$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$\varphi_n^{k+1} = f(t_{k+1}, x_n) = [(x_n+2)\sin x_n - \cos x_n]\left(e^{t_k} - 1 - t_k\right) + e^{t_k}\sin x_n,$$
$$x_n = nh, \quad t_k = k\tau,\ 1 \le k \le N-1, \tag{3.6}$$

and $D = I_{N+1}$ is the identity matrix.

$$U_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, \quad s = n-1, n, n+1. \tag{3.7}$$

This type system was used by Samarskii and Nikolaev [30] for difference equations. For the solution of the matrix equation (3.5), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, \ldots, 2, 1, \tag{3.8}$$

where $u_M = \vec{0}$, α_j $(j = 1, \ldots, M-1)$ are $(N+1) \times (N+1)$ square matrices, β_j $(j = 1, \ldots, M-1)$ are $(N+1) \times 1$ column matrices, α_1, β_1 are zero matrices, and

$$\begin{gathered}\alpha_{n+1} = -\left(B_n + C_n \alpha_n\right)^{-1} A_n, \\ \beta_{n+1} = \left(B_n + C_n \alpha_n\right)^{-1} \left(D_n \varphi_n - C_n \beta_n\right), \quad n = 1, 2, 3, \ldots, M-1.\end{gathered} \tag{3.9}$$

3.2. The Second Order of Accuracy in Time Difference Scheme

Applying (3.3) and using the second order of accuracy in t implicit difference scheme (2.10), we obtain the second order of accuracy difference scheme in t and in x

$$\begin{gathered}
\frac{u_n^{k+1} - 2u_n^k + u_{n-1}^{k-1}}{\tau^2} - (1 + x_n)\left[\frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{4h^2} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2h^2} - \frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{4h^2}\right] \\
-\left[\frac{u_{n+1}^{k-1} - u_{n-1}^{k-1}}{8h} + \frac{u_{n+1}^k - u_{n-1}^k}{4h} + \frac{u_{n+1}^{k+1} - u_{n-1}^{k+1}}{8h}\right] + \frac{1}{2} u_n^k + \frac{u_n^{k+1} + u_n^{k-1}}{4} = \varphi_n^k, \\
\varphi_n^k = \left[(x_n + 2)\sin x_n - \cos x_n\right]\left(e^{t_k} - 1 - t_k\right) + e^{t_k} \sin x_n, \\
Mh = \pi, \quad x_n = nh, \quad 1 \le n \le M-1, \\
N\tau = 1, \quad t_k = k\tau, \quad 1 \le k \le N-1, \\
u_n^0 = \sum_{k=1}^{N} \frac{\tau}{2}\left[e^{-k\tau} u_n^k + e^{-(k-1)\tau} u_n^{k-1}\right] + \psi(x_n), \quad 0 \le n \le M, \\
\psi(x_n) = \left(1 - 3e^{-1}\right) \sin x_n, \quad 0 \le n \le M, \\
u_n^1 - u_n^0 = \frac{\tau^2}{2}\left[\frac{u_{n+1}^0 - u_{n-1}^0}{2h} + (1 + x_n)\frac{u_{n+1}^0 - 2u_n^0 + u_{n-1}^0}{h^2} - u_n^0 + \sin x_n\right], \quad 1 \le n \le M-1, \\
u_0^k = u_M^k = 0, \quad 0 \le k \le N
\end{gathered} \tag{3.10}$$

for approximate solutions of the nonlocal boundary value problem (3.1). We have again $(N+1) \times (M+1)$ system of linear equations. We can write the system as a matrix equation (3.5).

Here

$$A_n = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_n & 2a_n & a_n & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_n & 2a_n & a_n & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_n & 2a_n & a_n \\ w_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B_n = \begin{bmatrix} 1-\dfrac{\tau}{2} & -\tau e^{-\tau} & -\tau e^{-2\tau} & \cdots & -\tau e^{-(N-1)\tau} & -\dfrac{\tau}{2}e^{-N\tau} \\ b_n & d_n & b_n & \cdots & 0 & 0 \\ 0 & b_n & d_n & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_n & 0 \\ 0 & 0 & 0 & \cdots & d_n & b_n \\ y_n & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$C_n = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_n & 2c_n & c_n & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_n & 2c_n & c_n & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & c_n & 2c_n & c_n \\ z_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.11}$$

$$a_n = \frac{-(1+x_n)}{4h^2} - \frac{1}{8h}, \qquad b_n = \frac{1}{\tau^2} + \frac{1+x_n}{2h^2} + \frac{1}{4},$$

$$c_n = \frac{-(1+x_n)}{4h^2} + \frac{1}{8h}, \qquad d_n = \frac{-2}{\tau^2} + \frac{1+x_n}{h^2} + \frac{1}{2},$$

$$y_n = -1 + \frac{(1+x_n)\tau^2}{h^2} + \frac{\tau^2}{2}, \qquad w_n = -\frac{\tau^2}{4h} - \frac{(1+x_n)\tau^2}{2h^2}, \qquad z_n = \frac{\tau^2}{4h} - \frac{(1+x_n)\tau^2}{2h^2},$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$\varphi_n^k = f(t_k, x_n) = [(x_n+2)\sin x_n - \cos x_n]\left(e^{t_k} - 1 - t_k\right) + e^{t_k}\sin x_n, \quad 1 \le k \le N-1,$$

$$D = I_{N+1}, \qquad U_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, \quad s = n-1, n, n+1.$$

For the solution of the matrix equation (3.5), we used the same algorithm as in the first order of accuracy difference scheme.

3.3. The Second Order of Accuracy in Time Difference Scheme Generated by A^2

Applying (3.3) and formulas

$$\frac{u(x_{n+2}) - 4u(x_{n+1}) + 6u(x_n) - 4u(x_{n-1}) + u(x_{n-2})}{h^4} - u^{(iv)}(x_n) = O\left(h^2\right),$$
$$\frac{2u(0) - 5u(h) + 4u(2h) - u(3h)}{h^2} - u''(0) = O\left(h^2\right), \tag{3.12}$$
$$\frac{2u(\pi) - 5u(\pi - h) + 4u(\pi - 2h) - u(\pi - 3h)}{h^2} - u''(\pi) = O\left(h^2\right)$$

and using difference scheme (2.11), we obtain the second order of accuracy difference scheme

$$\begin{gathered}
\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - (1 + x_n)\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} - \frac{u_{n+1}^k - u_{n-1}^k}{2h} + u_n^k \\
+\frac{\tau^2}{4}\Bigg[(1 + x_n)^2\frac{u_{n+2}^{k+1} - 4u_{n+1}^{k+1} + 6u_n^{k+1} - 4u_{n-1}^{k+1} + u_{n-2}^{k+1}}{h^4} \\
+4(1 + x_n)\frac{u_{n+2}^{k+1} - 2u_{n+1}^{k+1} + 2u_{n-1}^{k+1} - u_{n-2}^{k+1}}{2h^3} - 2x_n\frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} \\
-\frac{u_{n+1}^{k+1} - u_{n-1}^{k+1}}{h} + u_n^{k+1}\Bigg] = \varphi_n^k, \\
\varphi_n^k = [(x_n + 2)\sin x_n - \cos x_n]\left(e^{t_k} - 1 - t_k\right) + e^{t_k}\sin x_n, \\
Mh = \pi, \quad x_n = nh, \quad 2 \le n \le M - 2, \\
N\tau = 1, \quad t_k = k\tau, \quad 1 \le k \le N - 1, \\
u_n^0 = \sum_{k=1}^{N}\frac{\tau}{2}\left[e^{-k\tau}u_n^k + e^{-(k-1)\tau}u_n^{k-1}\right] = \psi(x_n), \quad 0 \le n \le M, \\
\psi(x_n) = \left(1 - 3e^{-1}\right)\sin x_n, \quad 0 \le n \le M, \\
u_n^1 - u_n^0 = \frac{\tau^2}{2}\left[\frac{u_{n+1}^0 - u_{n-1}^0}{2h} + (1 + x_n)\frac{u_{n+1}^0 - 2u_n^0 + u_{n-1}^0}{h^2} - u_n^0 + \sin x_n\right], \quad 2 \le n \le M - 2, \\
u_0^k = u_M^k = 0, \quad 0 \le k \le N, \\
u_1^k = \frac{4}{5}u_2^k - \frac{1}{5}u_3^k, \quad 0 \le k \le N, \\
u_{M-1}^k = \frac{4}{5}u_{M-2}^k - \frac{1}{5}u_{M-3}^k, \quad 0 \le k \le N
\end{gathered} \tag{3.13}$$

for approximate solutions of problem (3.1). One can write the $(N+1)\times(M+1)$ system of linear equations (3.13) as the matrix equation

$$\begin{gathered}
A_n u_{n+2} + B_n u_{n+1} + C_n u_n + D_n u_{n-1} + E_n u_{n-2} = R\varphi_n, \\
u_0 = \vec{0}, \qquad u_M = \vec{0}, \quad 2 \le n \le M-2, \\
u_1 = \frac{4}{5}u_2 - \frac{1}{5}u_3, \\
u_{M-1} = \frac{4}{5}u_{M-2} - \frac{1}{5}u_{M-3}.
\end{gathered} \tag{3.14}$$

Here, A_n, B_n, C_n, D_n, E_n, and R are $(N+1)\times(N+1)$ square matrices:

$$A_n = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_n & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_n & c_n & \cdots & 0 & 0 \\ 0 & 0 & b_n & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & c_n & 0 \\ 0 & 0 & 0 & \cdots & b_n & c_n \\ y_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$C_n = \begin{bmatrix} 1-\dfrac{\tau}{2} & -\tau e^{-\tau} & -\tau e^{-2\tau} & \cdots & -\tau e^{-(N-2)\tau} & -\tau e^{-(N-1)\tau} & -\dfrac{\tau}{2}e^{-N\tau} \\ d & e_n & f_n & \cdots & 0 & 0 & 0 \\ 0 & d & e_n & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & e_n & f_n & 0 \\ 0 & 0 & 0 & \cdots & d & e_n & f_n \\ w_n & 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$D_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & g_n & l_n & \cdots & 0 & 0 & 0 \\ 0 & 0 & g_n & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & g_n & l_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & g_n & l_n \\ z_n & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$E_n = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & m_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & m_n & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & m_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & m_n \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{3.15}$$

We denote

$$a_n = \frac{\tau^2}{4}\left(\frac{(1+x_n)^2}{h^4} + \frac{2(1+x_n)}{h^3}\right), \qquad b_n = -\frac{(1+x_n)}{h^2} - \frac{1}{2h},$$
$$d = \frac{1}{\tau^2}, \qquad e_n = -\frac{2}{\tau^2} + \frac{2(1+x_n)}{h^2} + 1, \qquad g_n = \frac{1}{2h} - \frac{1+x_n}{h^2},$$
$$c_n = \frac{\tau^2}{4}\left(\frac{-4(1+x_n)^2}{h^4} - \frac{4(1+x_n)}{h^3} - \frac{2x_n}{h^2} - \frac{1}{h}\right),$$
$$l_n = \frac{\tau^2}{4}\left(\frac{-4(1+x_n)^2}{h^4} - \frac{4(1+x_n)}{h^3} - \frac{2x_n}{h^2} + \frac{1}{h}\right),$$
$$f_n = \frac{1}{\tau^2} + \frac{\tau^2}{4}\left(\frac{6(1+x_n)^2}{h^4} + \frac{4x_n}{h^2} + 1\right),$$
$$m_n = \frac{\tau^2}{4}\left(\frac{(1+x_n)^2}{h^4} - \frac{2(1+x_n)}{h^3}\right),$$
$$z_n = \frac{\tau^2}{4h} - \frac{(1+x_n)\tau^2}{2h^2},$$
$$w_n = -1 + \frac{(1+x_n)\tau^2}{h^2} + \frac{\tau^2}{2}, \qquad y_n = -\frac{\tau^2}{4h} - \frac{(1+x_n)\tau^2}{2h^2},$$
$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$
$$\varphi_n^k = f(t_k, x_n) = [(x_n+2)\sin x_n - \cos x_n](e^{t_k} - 1 - t_k) + e^{t_k}\sin x_n, \quad 1 \le k \le N-1,$$
$$R = I_{N+1} \text{ is the identity matrix}, \qquad U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ \vdots \\ U_s^N \end{bmatrix}_{(N+1)\times 1}, \tag{3.16}$$

where $s = n \pm 2, n \pm 1, n$.

Table 1

Difference schemes	$N = M = 20$	$N = M = 40$	$N = M = 80$
Difference schemes (3.4)	0.0743	0.0357	0.0175
Difference schemes (3.10)	0.0012	0.0003009	0.0000756
Difference schemes (3.13)	0.0005453	0.0001231	0.00002923

For the solution of matrix equation (3.14), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following formula:

$$U_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}u_{n+2} + \gamma_{n+1}, \quad n = M-2, \dots, 2, 1, \tag{3.17}$$

where α_j, β_j $(j = 1, \dots, M-1)$ are $(N+1) \times (N+1)$ square matrices, γ_j $(j = 1, \dots, M-1)$ are $(N+1) \times 1$ column matrices, and α_1, β_1, γ_1, γ_2 are zero matrices. α_2, β_2 are

$$\begin{gathered}
\alpha_2 = \frac{4}{5}I_{N+1}, \qquad \beta_2 = -\frac{1}{5}I_{N+1}, \\
F_n = C_n + D_n\alpha_n + E_n\alpha_{n-1}\alpha_n + E_n\beta_{n-1} \\
\alpha_{n+1} = F_n^{-1}\left[-B_n - D_n\beta_n - E_n\alpha_{n-1}\beta_n\right], \qquad \beta_{n+1} = -F_n^{-1}A_n, \\
\gamma_{n+1} = F_n^{-1}\left[R\varphi_n - D_n\gamma_n + E_n\alpha_{n-1}\gamma_n + E_n\gamma_{n-1}\right].
\end{gathered} \tag{3.18}$$

For solution of the last difference equation, we need to find u_M, u_{M-1}

$$\begin{gathered}
u_M = \vec{0}, \\
u_{M-1} = \left((\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}\right)^{-1}((4I - \alpha_{M-2}))\gamma_{M-1} - \gamma_{M-2}.
\end{gathered} \tag{3.19}$$

3.4. Error Analysis

The errors are computed by

$$E_M^N = \max_{1 \le k \le N-1, 1 \le n \le M-1} \left| u(t_k, x_n) - u_n^k \right| \tag{3.20}$$

of the numerical solutions, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n) and the results are given in Table 1.

Thus, the results show that the second order of accuracy difference schemes (3.10) and (3.13) are more accurate comparing with the first order of accuracy difference scheme (3.4).

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