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Research Article

Weak Solutions for Nonlinear Fractional Differential Equations in Banach Spaces

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We discuss the existence of weak solutions for a nonlinear boundary value problem of fractional differential equations in Banach space. Our analysis relies on the Mönch's fixed point theorem combined with the technique of measures of weak noncompactness.

1. Introduction

This paper is mainly concerned with the existence results for the following fractional differential equation:

$$\begin{gathered}{}^cD^{\alpha}_{0+}u(t) = f(t, u(t)), \quad t \in J := [0, T],\\ u(0) = \lambda_1 u(T) + \mu_1, \quad u'(0) = \lambda_2 u'(T) + \mu_2, \quad \lambda_1 \neq 1,\ \lambda_2 \neq 1,\end{gathered} \tag{1.1}$$

where $1 < \alpha \leq 2$ is a real number, ${}^cD^{\alpha}_{0+}$ is the Caputo's fractional derivative, $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$. $f : J \times E \to E$ is a given function satisfying some assumptions that will be specified later, and E is a Banach space with norm $\|u\|$.

Recently, fractional differential equations have found numerous applications in various fields of physics and engineering [1, 2]. It should be noted that most of the books and papers on fractional calculus are devoted to the solvability of initial value problems for differential equations of fractional order. In contrast, the theory of boundary value problems for nonlinear fractional differential equations has received attention quite recently and many aspects of this theory need to be explored. For more details and examples, see [3–19] and the references therein.

To investigate the existence of solutions of the problem above, we use Mönch's fixed point theorem combined with the technique of measures of weak noncompactness, which is an important method for seeking solutions of differential equations. This technique was mainly initiated in the monograph of Banaś and Goebel [20] and subsequently developed and used in many papers; see, for example, Banaś and Sadarangani [21], Guo et al. [22], Krzyśka and Kubiaczyk [23], Lakshmikantham and Leela [24], Mönch [25], O'Regan [26, 27], Szufla [28, 29], and the references therein. As far as we know, there are very few results devoted to weak solutions of nonlinear fractional differential equations [30–32]. Motivated by the above-mentioned papers [30–32], the purpose of this paper is to establish the existence results for the boundary value problem (1.1) by virtue of the Mönch's fixed point theorem combined with the technique of measures of weak noncompactness. Our results can be seen as a supplement of the results in [32] (see Remark 3.8).

The remainder of this is organized as follows. In Section 2, we provide some basic definitions, preliminaries facts, and various lemmas which are needed later. In Section 3, we give main results of problem (1.1). In the end, we also give an example for the illustration of the theories established in this paper.

2. Preliminaries and Lemmas

In this section, we present some basic notations, definitions, and preliminary results which will be used throughout this paper.

Let $J := [0, T]$ and $L^1(J, E)$ denote the Banach space of real-valued Lebesgue integrable functions on the interval J, $L^\infty(J, E)$ denote the Banach space of real-valued essentially bounded and measurable functions defined over J with the norm $\|\cdot\|_{L^\infty}$.

Let E be a real reflexive Banach space with norm $\|\cdot\|$ and dual E^*, and let $(E, \omega) = (E, \sigma(E, E^*))$ denote the space E with its weak topology. Here, $C(J, E)$ is the Banach space of continuous functions $x : J \to E$ with the usual supremum norm $\|x\|_\infty := \sup\{\|x(t)\| : t \in J\}$.

Moreover, for a given set V of functions $v : J \mapsto \mathbb{R}$, let us denote by $V(t) = \{v(t) : v \in V\}$, $t \in J$, and $V(J) = \{v(t) : v \in V,\ t \in J\}$.

Definition 2.1. A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any $(x_n)_n$ in E with $x_n(t) \to x(t)$ in (E, ω) then $h(x_n(t)) \to h(x(t))$ in (E, ω) for each $t \to J$).

Definition 2.2 (see [33]). The function $x : J \to E$ is said to be Pettis integrable on J if and only if there is an element $x_J \in E$ corresponding to each $I \subset J$ such that $\varphi(x_I) = \int_I \varphi(x(s))ds$ for all $\varphi \in E^*$, where the integral on the right is supposed to exist in the sense of Lebesgue. By definition, $x_I = \int_I x(s)ds$.

Let $P(J, E)$ be the space of all E-valued Pettis integrable functions in the interval J.

Lemma 2.3 (see [33]). *If $x(\cdot)$ is Pettis integrable and $h(\cdot)$ is a measurable and essentially bounded real-valued function, then $x(\cdot)h(\cdot)$ is Pettis integrable.*

Definition 2.4 (see [34]). Let E be a Banach space, Ω_E the set of all bounded subsets of E, and B_1 the unit ball in E. The *De Blasi* measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by $\beta(X) = \inf\{\epsilon > 0 :$ there exists a weakly compact subset Ω of E such that $X \subset \epsilon B_1 + \Omega\}$.

Lemma 2.5 (see [34]). *The De sBlasi measure of noncompactness satisfies the following properties:*

(a) $S \subset T \Rightarrow \beta(S) \leq \beta(T)$;

(b) $\beta(S) = 0 \Leftrightarrow S$ *is relatively weakly compact;*

(c) $\beta(S \cup T) = \max\{\beta(S), \beta(T)\}$;

(d) $\beta(\overline{S}^{\omega}) = \beta(S)$, *where* $\overline{S}^{\omega}$ *denotes the weak closure of* S;

(e) $\beta(S + T) \leq \beta(S) + \beta(T)$;

(f) $\beta(aS) = |a|\alpha(S)$;

(g) $\beta(\text{ conv } (S)) = \beta(S)$;

(h) $\beta(\cup_{|\lambda| \leq h} \lambda S) = h\beta(S)$.

The following result follows directly from the Hahn-Banach theorem.

Lemma 2.6. *Let* E *be a normed space with* $x_0 \neq 0$. *Then there exists* $\varphi \in E^*$ *with* $\|\varphi\| = 1$ *and* $\varphi(x_0) = \|x_0\|$.

For completeness, we recall the definitions of the Pettis-integral and the Caputo derivative of fractional order.

Definition 2.7 (see [26]). Let $h : J \to E$ be a function. The fractional Pettis integral of the function h of order $\alpha \in \mathbb{R}^+$ is defined by

$$I^{\alpha} h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds, \tag{2.1}$$

where the sign "$\int$" denotes the Pettis integral and Γ is the Gamma function.

Definition 2.8 (see [3]). For a function $f : J \to E$, the Caputo fractional-order derivative of f is defined by

$$\left({}^{c}D_{a+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad n-1 < \alpha < n, \tag{2.2}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.9 (see [28]). *Let* D *be a closed convex and equicontinuous subset of a metrizable locally convex vector space* $C(J, E)$ *such that* $0 \in D$. *Assume that* $A : D \to D$ *is weakly sequentially continuous. If the implication*

$$\overline{V} = \overline{\text{conv}}\,(\{0\} \cup A(V)) \Longrightarrow V \text{ is relatively weakly compact}, \tag{2.3}$$

holds for every subset V *of* D, *then* A *has a fixed point.*

3. Main Results

Let us start by defining what we mean by a solution of the problem (1.1).

Definition 3.1. A function $x \in C(J, E_\omega)$ is said to be a solution of the problem (1.1) if x satisfies the equation ${}^cD_{0+}^{\alpha}u(t) = f(t, u(t))$ on J and satisfies the conditions $u(0) = \lambda_1 u(T) + \mu_1,\ u'(0) = \lambda_2 u'(T) + \mu_2$.

For the existence results on the problem (1.1), we need the following auxiliary lemmas.

Lemma 3.2 (see [3, 7]). *For $\alpha > 0$, the general solution of the fractional differential equation ${}^cD_{0+}^{\alpha}u(t) = 0$ is given by*

$$h(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1}t^{n-1}, \quad C_i \in \mathbb{R},\ i = 0, 1, 2, \ldots,\ n-1,\ n = [\alpha] + 1. \tag{3.1}$$

Lemma 3.3 (see [3, 7]). *Assume that $h \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha}\, {}^cD_{0+}^{\alpha}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1} \tag{3.2}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where $n = [\alpha] + 1$.

We derive the corresponding Green's function for boundary value problem (1.1) *which will play major role in our next analysis.*

Lemma 3.4. *Let $\rho \in C(J, E)$ be a given function, then the boundary-value problem*

$$\begin{gathered} {}^cD_{0+}^{\alpha}u(t) = \rho(t), \quad t \in (0, T),\ 1 < \alpha \leq 2, \\ u(0) = \lambda_1 u(T) + \mu_1, \qquad u'(0) = \lambda_2 u'(T) + \mu_2, \quad \lambda_1 \neq 1,\ \lambda_2 \neq 1 \end{gathered} \tag{3.3}$$

has a unique solution

$$u(t) = \int_0^T G(t,s)\rho(s)\,ds + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_1 - 1)(\lambda_2 - 1)} - \frac{\mu_1}{(\lambda_1 - 1)}, \tag{3.4}$$

where $G(t,s)$ is defined by the formula

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{\lambda_1 (T-s)^{\alpha-1}}{(\lambda_1 - 1)\Gamma(\alpha)} + \dfrac{\lambda_2[\lambda_1 T + (1-\lambda_1)t](T-s)^{\alpha-2}}{(\lambda_1 - 1)(\lambda_2 - 1)\Gamma(\alpha - 1)}, & \text{if } \ 0 \leq s \leq t \leq T, \\ -\dfrac{\lambda_1 (T-s)^{\alpha-1}}{(\lambda_1 - 1)\Gamma(\alpha)} + \dfrac{\lambda_2[\lambda_1 T + (1-\lambda_1)t](T-s)^{\alpha-2}}{(\lambda_1 - 1)(\lambda_2 - 1)\Gamma(\alpha - 1)}, & \text{if } \ 0 \leq t \leq s \leq T. \end{cases} \tag{3.5}$$

Here $G(t,s)$ is called the Green's function of boundary value problem (3.3).

Proof. By the Lemma 3.3, we can reduce the equation of problem (3.3) to an equivalent integral equation

$$u(t) = I_{0^+}^{\alpha}\rho(t) - c_1 - c_2 t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\rho(s)\,ds - c_1 - c_2 t \tag{3.6}$$

for some constants $c_1, c_2 \in \mathbb{R}$. On the other hand, by relations $D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t)$ and $I_{0^+}^{m} I_{0^+}^{n} u(t) = I_{0^+}^{m+n} u(t)$, for $m, n > 0,\ u \in L(0,1)$, we have

$$u'(t) = -c_2 + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \rho(s) ds = -c_2 + I_{0^+}^{\alpha-1} \rho(t). \tag{3.7}$$

Applying the boundary conditions (3.3), we have

$$\begin{gathered} c_1 = \frac{\lambda_1}{\lambda_1 - 1} \left[\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \rho(s) ds - \frac{T\lambda_2}{\lambda_2 - 1} \left(\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \rho(s) ds + \frac{\mu_2}{\lambda_2} \right) + \frac{\mu_1}{\lambda_1} \right], \\ c_2 = \frac{\lambda_2}{(\lambda_2 - 1)} \frac{1}{\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} \rho(s) ds + \frac{\mu_2}{(\lambda_2 - 1)}. \end{gathered} \tag{3.8}$$

Therefore, the unique solution of problem (3.3) is

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(s) ds - c_1 - c_2 t \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(s) ds - \frac{\lambda_1}{\lambda_1 - 1} \left[\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \rho(s) ds \right. \\ &\qquad \left. - \frac{T\lambda_2}{\lambda_2 - 1} \left(\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \rho(s) ds + \frac{\mu_2}{\lambda_2} \right) + \frac{\mu_1}{\lambda_1} \right] \\ &\quad - \left[\frac{\lambda_2}{(\lambda_2 - 1)} \frac{1}{\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} \rho(s) ds + \frac{\mu_2}{(\lambda_2 - 1)} \right] t \\ &= \int_0^T G(t,s) \rho(s) ds + \frac{\mu_2 [\lambda_1 T + (1-\lambda_1) t]}{(\lambda_1 - 1)(\lambda_2 - 1)} - \frac{\mu_1}{(\lambda_1 - 1)}, \end{aligned} \tag{3.9}$$

which completes the proof. □

Remark 3.5. From the expression of $G(t,s)$, it is obvious that $G(t,s)$ is continuous on $J \times J$. Denote

$$G^* = \sup \left\{ \int_0^T |G(t,s)| ds : t \in J \right\}. \tag{3.10}$$

Remark 3.6. Letting $\xi_1 = 1/(\lambda_1 - 1)$, $\xi_2 = 1/(\lambda_1 - 1)(\lambda_2 - 1)$, $g(t) = \mu_2[\lambda_1 T + (1-\lambda_1)t]/(\lambda_1 - 1)(\lambda_2 - 1) - \mu_1/(\lambda_1 - 1) = \mu_2[\lambda_1 T + (1-\lambda_1)t]\xi_2 - \mu_1 \xi_1$, it is obvious that $g(t)$ is continuous in J, denoting $g^* = \sup\{|g(t)|,\ t \in J\}$.

To prove the main results, we need the following assumptions:

(H1) for each $t \in J$, the function $f(t, \cdot)$ is weakly sequentially continuous;

(H2) for each $x \in C(J, E)$, the function $f(\cdot, x(\cdot))$ is Pettis integrable on J;

(H3) there exists $p_f \in L^\infty(J,\mathbb{R}^+)$ such that $\|f(t,u)\| \le p_f(t)\|u\|$, for a.e. $t \in J$ and each $u \in E$;

(H3)′ there exists $p_f \in L^\infty(J,E)$ and a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that $\|f(t,u)\| \le p_f(t)\psi(\|u\|)$, for a.e. $t \in J$ and each $u \in E$;

(H4) for each bounded set $D \subset E$, and each $t \in J$, the following inequality holds

$$\beta(f(t,D)) \le p_f(t) \cdot \beta(D); \tag{3.11}$$

(H5) there exists a constant $R > 0$ such that

$$\frac{R}{g^* + \|p_f\|_{L^\infty}\psi(R)G^*} > 1, \tag{3.12}$$

where $\|p_f\|_{L^\infty} = \sup\{p_f(t) : t \in J\}$.

Theorem 3.7. *Let E be a reflexive Banach space and assume that (H1)–(H3) are satisfied. If*

$$\|p_f\|_{L^\infty}G^* < 1, \tag{3.13}$$

then the problem (1.1) *has at least one solution on J.*

Proof. Let the operator $\mathcal{A} : C(J,E) \to C(J,E)$ defined by the formula

$$(\mathcal{A}u)(t) := \int_0^T G(t,s)f(s,u(s))ds + g(t), \tag{3.14}$$

where $G(\cdot,\cdot)$ is the Green's function defined by (3.5). It is well known the fixed points of the operator $\mathcal{A}$ are solutions of the problem (1.1).

First notice that, for $x \in C(J,E)$, we have $f(\cdot,x(\cdot)) \in P(J,E)$ (assumption (H2)). Since, $s \mapsto G(t,s) \in L^\infty(J)$, then $G(t,\cdot)f(\cdot,x(\cdot))$ is Pettis integrable for all $t \in J$ by Lemma 2.3, and so the operator $\mathcal{A}$ is well defined.

Let

$$R \ge \frac{g^*}{1 - \|p_f\|_{L^\infty}G^*}, \tag{3.15}$$

and consider the set

$$\begin{aligned} D = \Big\{ x \in C(J,E) : \|x\|_\infty \le R,\ \|x(t_1) - x(t_2)\| \le \left|\mu_2(1-\lambda_1)\xi_2\right| \cdot |t_2 - t_1| \\ + R\|p_f\|_{L^\infty} \int_0^T |G(t_2,s) - G(t_1,s)|ds \text{ for } t_1,t_2 \in J \Big\}. \end{aligned} \tag{3.16}$$

Clearly, the subset D is closed, convex, and equicontinuous. We shall show that $\mathcal{A}$ satisfies the assumptions of Lemma 2.9. The proof will be given in three steps.

Step 1. We will show that the operator $\mathcal{A}$ maps D into itself.

Take $x \in D$, $t \in J$ and assume that $\mathcal{A}x(t) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|\mathcal{A}x(t)\| = \varphi(\mathcal{A}x(t))$. Thus

$$\begin{aligned}
\|(\mathcal{A}x)(t)\| = \varphi((\mathcal{A}x)(t)) &= \varphi\left(g(t) + \int_0^T G(t,s) f(s, y(s)) ds \right) \\
&\leq \varphi(g(t)) + \int_0^T |G(t,s)| \cdot \varphi(f(s, x(s))) ds \\
&\leq \|g(t)\| + \int_0^T |G(t,s)| \cdot p_f(s) \cdot \|x(s)\| ds \\
&\leq g^* + \|p_f\|_{L^\infty} R G^* \\
&\leq R.
\end{aligned} \tag{3.17}$$

Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and $\forall x \in D$, so $\mathcal{A}x(\tau_2) - \mathcal{A}x(\tau_1) \neq 0$. Then there exists $\varphi \in E^*$, such that $\|\mathcal{A}x(\tau_2) - \mathcal{A}x(\tau_1)\| = \varphi(\mathcal{A}x(\tau_2) - \mathcal{A}x(\tau_1))$. Hence,

$$\begin{aligned}
\|\mathcal{A}x(\tau_2) - \mathcal{A}x(\tau_1)\| &= \varphi\left(g(\tau_2) - g(\tau_1) + \int_0^T [G(\tau_2, s) - G(\tau_1, s)] \cdot f(s, x(s)) ds \right) \\
&\leq \varphi(g(\tau_2) - g(\tau_1)) + \int_0^T |G(\tau_2, s) - G(\tau_1, s)| \cdot \|f(s, x(s))\| ds \\
&\leq \|g(\tau_2) - g(\tau_1)\| + R\|p_f\|_{L^\infty} \int_0^T |G(\tau_2, s) - G(\tau_1, s)| ds \\
&\leq |\mu_2(1 - \lambda_1)\xi_2| \cdot |t_2 - t_1| \\
&\quad + R\|p_f\|_{L^\infty} \int_0^T |G(\tau_2, s) - G(\tau_1, s)| ds;
\end{aligned} \tag{3.18}$$

this means that $\mathcal{A}(D) \subset D$.

Step 2. We will show that the operator $\mathcal{A}$ is weakly sequentially continuous.

Let (x_n) be a sequence in D and let $(x_n(t)) \to x(t)$ in (E, w) for each $t \in J$. Fix $t \in J$. Since f satisfies assumptions (H1), we have $f(t, x_n(t))$ converge weakly uniformly to $f(t, x(t))$. Hence, the Lebesgue Dominated Convergence Theorem for Pettis integrals implies that $\mathcal{A}x_n(t)$ converges weakly uniformly to $\mathcal{A}x(t)$ in E_ω. Repeating this for each $t \in J$ shows $\mathcal{A}x_n \to \mathcal{A}x$. Then $\mathcal{A} : D \to D$ is weakly sequentially continuous.

Step 3. The implication (2.3) holds. Now let V be a subset of D such that $V \subset \overline{\text{conv}}(\mathcal{A}(V) \cup \{0\})$. Clearly, $V(t) \subset \overline{\text{conv}}(\mathcal{A}(V) \cup \{0\})$ for all $t \in J$. Hence, $\mathcal{A}V(t) \subset \mathcal{A}D(t), t \in J$, is bounded in E. Thus, $\mathcal{A}V(t)$ is weakly relatively compact since a subset of a reflexive Banach space is weakly relatively compact if and only if it is bounded in the norm topology. Therefore,

$$\begin{aligned} v(t) &\le \beta(\mathcal{A}(V)(t) \cup \{0\}) \\ &\le \beta(\mathcal{A}(V)(t)) \\ &= 0, \end{aligned} \tag{3.19}$$

thus, V is relatively weakly compact in E. In view of Lemma 2.9, we deduce that $\mathcal{A}$ has a fixed point which is obviously a solution of the problem (1.1). This completes the proof. □

Remark 3.8. In the Theorem 3.7, we presented an existence result for weak solutions of the problem (1.1) in the case where the Banach space E is reflexive. However, in the nonreflexive case, conditions (H1)–(H3) are not sufficient for the application of Lemma 2.9; the difficulty is with condition (2.3). Our results can be seen as a supplement of the results in [32] (see Remark 3.8).

Theorem 3.9. *Let E be a Banach space, and assume assumptions (H1), (H2), (H3), (H4) are satisfied. If* (3.13) *holds, then the problem* (1.1) *has at least one solution on J.*

Theorem 3.10. *Let E be a Banach space, and assume assumptions (H1), (H2), (H3)′, (H4), (H5) are satisfied. If* (3.13) *holds, then the problem* (1.1) *has at least one solution on J.*

Proof. Assume that the operator $\mathcal{A} : C(J,E) \to C(J,E)$ is defined by the formula (3.14). It is well known the fixed points of the operator $\mathcal{A}$ are solutions of the problem (1.1).

First notice that, for $x \in C(J,E)$, we have $f(\cdot, x(\cdot)) \in P(J,E)$ (assumption (H2)). Since, $s \mapsto G(t,s) \in L^\infty(J)$, then $G(t,\cdot)f(\cdot,x(\cdot))$ for all $t \in J$ is Pettis integrable (Lemma 2.3) and thus the operator $\mathcal{A}$ makes sense.

Let $R > 0$, and consider the set

$$\begin{aligned} \mathfrak{D} = \Bigg\{ & x \in C(J,E) : \|x\|_\infty \le R,\ \|x(t_1) - x(t_2)\| \le \left|\mu_2(1-\lambda_1)\xi_2\right| \cdot |t_2 - t_1| \\ & + \|p_f\|_{L^\infty}\psi(R)\int_0^T |G(t_2,s) - G(t_1,s)|ds \text{ for } t_1,t_2 \in J \Bigg\}. \end{aligned} \tag{3.20}$$

Clearly the subset $\mathfrak{D}$ is closed, convex and equicontinuous. We shall show that $\mathcal{A}$ satisfies the assumptions of Lemma 2.9. The proof will be given in three steps.

Step 1. We will show that the operator $\mathcal{A}$ maps $\mathfrak{D}$ into itself.

Take $x \in \mathfrak{D}$, $t \in J$ and assume that $\mathcal{A}x(t) \ne 0$. Then there exists $\psi \in E^*$ such that $\|\mathcal{A}x(t)\| = \psi(\mathcal{A}x(t))$. Thus

$$\|(\mathcal{A}x)(t)\| = \psi((\mathcal{A}x)(t)) = \psi\left(g(t) + \int_0^T G(t,s)f(s,y(s))ds \right)$$

$$
\begin{aligned}
&\le \psi(g(t)) + \int_0^T |G(t,s)| \cdot \psi(f(s,x(s)))ds \\
&\le \psi(g(t)) + \int_0^T |G(t,s)| \cdot p_f(s) \cdot \psi(\|x(s)\|)ds \\
&\le g^* + \|p_f\|_{L^\infty} \psi(R) G^* \\
&\le R.
\end{aligned}
\tag{3.21}
$$

Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and $\forall x \in \mathfrak{D}$, so $\mathcal{A}x(\tau_2) - \mathcal{A}x(\tau_1) \neq 0$. Then there exist $\psi \in E^*$ such that

$$
\|\mathcal{A}x(\tau_2) - \mathcal{A}x(\tau_1)\| = \psi(\mathcal{A}x(\tau_2) - \mathcal{A}x(\tau_1)). \tag{3.22}
$$

Thus

$$
\begin{aligned}
\|\mathcal{A}x(\tau_2) - \mathcal{A}x(\tau_1)\| &= \psi\left(g(\tau_2) - g(\tau_1) + \int_0^T [G(\tau_2,s) - G(\tau_1,s)] \cdot f(s,x(s))ds \right) \\
&\le \psi(g(\tau_2) - g(\tau_1)) + \int_0^T |G(\tau_2,s) - G(\tau_1,s)| \cdot \|f(s,x(s))\|ds \\
&\le \|g(\tau_2) - g(\tau_1)\| + \psi(R)\|p_f\|_{L^\infty} \int_0^T |G(\tau_2,s) - G(\tau_1,s)|ds \\
&\le |\mu_2(1-\lambda_1)\xi_2| \cdot |t_2 - t_1| + \psi(R)\|p_f\|_{L^\infty} \int_0^T |G(\tau_2,s) - G(\tau_1,s)|ds;
\end{aligned}
\tag{3.23}
$$

this means that $\mathcal{A}(\mathfrak{D}) \subset \mathfrak{D}$.

Step 2. We will show that the operator $\mathcal{A}$ is weakly sequentially continuous.

Let (x_n) be a sequence in $\mathfrak{D}$ and let $(x_n(t)) \to x(t)$ in (E, w) for each $t \in J$. Fix $t \in J$. Since f satisfies assumptions (H1), we have $f(t, x_n(t))$, converging weakly uniformly to $f(t, x(t))$. Hence the Lebesgue Dominated Convergence theorem for Pettis integral implies $\mathcal{A}x_n(t)$ converging weakly uniformly to $\mathcal{A}x(t)$ in E_ω. We do it for each $t \in J$ so $\mathcal{A}x_n \to \mathcal{A}x$. Then $\mathcal{A} : \mathfrak{D} \to \mathfrak{D}$ is weakly sequentially continuous.

Step 3. The implication (2.3) holds. Now let V be a subset of $\mathfrak{D}$ such that $V \subset \overline{\text{conv}}(\mathcal{A}(V) \cup \{0\})$. Clearly, $V(t) \subset \overline{\text{conv}}(\mathcal{A}(V) \cup \{0\})$ for all $t \in J$. Hence, $\mathcal{A}V(t) \subset \mathcal{A}\mathfrak{D}(t)$, $t \in J$, is bounded in E. Since function g is continuous on J, the set $\overline{\{g(t), t \in J\}} \subset E$ is compact, so $\beta(g(t)) = 0$. Using this fact, assumption (H4), Lemma 2.5 and the properties of the measure β, we have for each $t \in J$

$$
\begin{aligned}
v(t) &\le \beta(\mathcal{A}(V)(t) \cup \{0\}) \\
&\le \beta(\mathcal{A}(V)(t))
\end{aligned}
$$

$$
\begin{aligned}
&= \beta\left\{\int_0^T G(t,s) f(s,V(s))ds\right\} \\
&\leq \int_0^T |G(t,s)| \cdot p_f(s) \cdot \beta(V(s))ds \\
&\leq \left\|p_f\right\|_{L^\infty} \cdot \int_0^T |G(t,s)| \cdot v(s)ds \\
&\leq \left\|p_f\right\|_{L^\infty} \cdot \|v\|_\infty \cdot G^*,
\end{aligned}
\tag{3.24}
$$

which gives

$$\|v\|_\infty \leq \left\|p_f\right\|_{L^\infty} \cdot \|v\|_\infty \cdot G^*. \tag{3.25}$$

This means that

$$\|v\|_\infty \cdot \left[1 - \left\|p_f\right\|_{L^\infty} \cdot G^*\right] \leq 0. \tag{3.26}$$

By (3.13) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively weakly compact in E. In view of Lemma 2.9, we deduce that $\mathcal{A}$ has a fixed point which is obviously a solution of the problem (1.1). This completes the proof. □

4. An Example

In this section we give an example to illustrate the usefulness of our main result.

Example 4.1. Let us consider the following fractional boundary value problem:

$$
\begin{gathered}
{}^cD^\alpha u = \frac{2}{19+e^t}\frac{\|u\|}{1+\|u\|}, \quad t \in J := [0,T],\ 1 < \alpha \leq 2, \\
u(0) = \lambda_1 u(T) + \mu_1, \qquad u'(0) = \lambda_2 u'(T) + \mu_2.
\end{gathered}
\tag{4.1}
$$

Set $T = 1$, $f(t,u) = (2/(19+e^t))(\|u\|/(1+\|u\|))$, $(t,u) \in J \times E$, $\lambda_1 = \lambda_2 = -1$, $\mu_1 = \mu_2 = 0$. Clearly conditions (H1), (H2), and (H3) hold with $p_f(t) = 2/(19+e^t)$. From (3.5), we have

$$
G(t,s) = \begin{cases}
\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \dfrac{(1-2t)(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}, & \text{if } 0 \leq s \leq t \leq 1, \\[2ex]
-\dfrac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \dfrac{(1-2t)(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}, & \text{if } 0 \leq t \leq s \leq 1.
\end{cases}
\tag{4.2}
$$

We have

$$
\begin{aligned}
\int_0^1 G(t,s)ds &= \int_0^t G(t,s)ds + \int_t^1 G(t,s)ds \\
&= \int_0^t \left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(1-2t)(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}\right]ds \\
&\quad + \int_t^1 \left[-\frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(1-2t)(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}\right]ds \\
&= \frac{4t^\alpha - 2}{4\Gamma(\alpha+1)} + \frac{1-2t}{4\Gamma(\alpha)}, \\
&\qquad \|p_f\|_{L^\infty} = \frac{1}{10}.
\end{aligned}
\tag{4.3}
$$

A simple computation gives

$$
G^* < \frac{1}{4\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha+1)}. \tag{4.4}
$$

We shall check that condition (3.13) is satisfied. Indeed

$$
\|p\|_{L^\infty} G^* < \frac{1}{10}\left[\frac{1}{4\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha+1)}\right] < 1, \tag{4.5}
$$

which is satisfied for some $\alpha \in (1,2]$. Then by Theorem 3.7, the problem (4.1) has at least one solution on J for values of α satisfying (4.5).

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