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Research Article

Optimal Fuzzy Control for a Class of Nonlinear Systems

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The paper presents conditions suitable in design giving quadratic performances to stabilizing controllers for given class of continuous-time nonlinear systems, represented by Takagi-Sugeno models. Based on extended Lyapunov function and slack matrices, the design conditions are outlined in the terms of linear matrix inequalities to possess a stable structure closest to LQ performance, if premise variables are measurable. Simulation results illustrate the design procedure and demonstrate the performances of the proposed control design method.

1. Introduction

Since a generic method for design of a controller valid for all types of nonlinear systems has not been developed yet, an alternative seems to be fuzzy approach which benefits from the advantages of the approximation techniques approximating nonlinear system model equations. Using the Takagi-Sugeno (TS) fuzzy model [1], the nonlinear system is represented as a collection of fuzzy rules, where each rule utilizes the local dynamics by a linear system model. Since TS fuzzy models can well approximate a large class of nonlinear systems, and the TS-model-based approach can apprehend the nonlinear behavior of a system while keeping the simplicity of the linear models, by employing TS fuzzy models a control design methodology exploits fully advantage of the modern control theory, especially in the state space optimal and robust control.

The main idea of the TS-model-based controller design is to derive control rules so as to compensate each rule of a fuzzy system, determining the local feedback gains. It is known that the separate stabilization of these local modes does not ensure the stability of the overall system, and global design conditions have to be used to guarantee the global stability and control performance. Therefore, a range of stability conditions have been developed for TS fuzzy systems (see e.g., [2–5]), most of them relying on the feasibility of an associated system of linear matrix inequalities (LMIs) [6]. Therefore, the state control based on fuzzy TS system model gives control structures which can be designed using technique derived from equivalent LMIs (some principles and results are reported e.g., in [7–10]).

The problem of controlling a system in such way to optimize a performance index that represents the actual operating performance of a system has been an area of study for several decades [11–14]. In particular, if the attention is restricted to linear quadratic (LQ) control, several works following this approach over the years have been reported in the literature, where some new ones being for example, in [15]. Specifically, this approach has been often made in diverse practical problems for finite-time interval with time-varying feedback gains and full state measurable variables, to bring dynamical systems to a desired final states, as special interest in aircraft, spacecraft, and robots control and diagnosis [16, 17].

Following the given ideas in LQ control [18], the main contribution of the paper is to present new conditions for designing the stabilizing fuzzy state control with LQ performance for nonlinear MIMO systems approximated by a TS model and exploiting measurable premise variables. The proposed design method prefers methodology given in [7, 10] but is constructed on the extended form of quadratic Lyapunov function and enhanced evaluation of its time derivative [19, 20]. Because the Lyapunov synthesis approach is exploited to express global stability conditions in the form of a set of LMIs, resulting conservativeness of stability conditions is reduced, since while a common symmetric positive definite Lyapunov matrix verifying all inequalities is required, this approach eliminates products of this matrix with system model matrix parameters and extensive exploits affine properties of TS models.

The remainder of this paper is organized as follows. In Section 2 the general structure of TS models and LQ control is briefly described, and in Section 3 basic preliminaries are presented. The control design problem for systems with measurable premise variables is given in Sections 4 and 5, where especially new design conditions are derived and proven. Section 6 gives a numerical example to illustrate the effectiveness of the proposed approach and to confirm the validity of the control scheme. The last section draws conclusion remarks.

Throughout the paper, the following notations are used: $\mathbf{x}^T$, $\mathbf{X}^T$ denotes the transpose of the vector $\mathbf{x}$ and matrix $\mathbf{X}$, respectively, diag$[\cdot]$ denotes a block diagonal matrix, for a square matrix $\mathbf{X} > 0$ (resp. $\mathbf{X} < 0$) means that $\mathbf{X}$ is a symmetric positive definite matrix (resp., negative definite matrix), the symbol $\mathbf{I}_n$ represents the nth order unit matrix, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^{n\times r}$ the set of all $n \times r$ real matrices.

2. General Methodologies

2.1. Takagi-Sugeno Fuzzy Models

The systems under consideration is one class of multiinput and multioutput nonlinear (MIMO) dynamic systems, which in the state-space form is represented as

$$\dot{\mathbf{q}}(t) = \mathbf{a}(\mathbf{q}(t)) + \mathbf{B}\mathbf{u}(t), \tag{2.1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t), \tag{2.2}$$

where $\mathbf{q}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^r$, $\mathbf{y}(t) \in \mathbb{R}^m$ are vectors of the state, input, and output variables, respectively, and $\mathbf{B} \in \mathbb{R}^{n\times r}$ and $\mathbf{C} \in \mathbb{R}^{m\times n}$ are real finite values matrices. It is assumed that $\mathbf{a}(0) = \mathbf{0}$, and that $\mathbf{a}(\mathbf{q}(t))$ is bounded in associated sectors, that is, in the regions within the system will operate.

It is considered that the number of the nonlinear terms in the nonlinear part of model $\mathbf{a}(\mathbf{q}(t))$ is p, and that there exists a set of nonlinear sector functions of this properties

$$\begin{gathered} w_{lj}(\boldsymbol{\theta}(t)), \quad j = 1,2,\ldots,k,\ l = 1,2,\ldots,p, \\ w_{lj}(\boldsymbol{\theta}(t)) = w_{lj}(\theta_j(t)), \\ w_{l1}(\boldsymbol{\theta}(t)) = 1 - \sum_{j=2}^{k} w_{lj}(\boldsymbol{\theta}(t)), \end{gathered} \tag{2.3}$$

where k is the number of sector functions, and

$$\boldsymbol{\theta}(t) = \begin{bmatrix} \theta_1(t) & \theta_2(t) & \cdots & \theta_q(t) \end{bmatrix} \tag{2.4}$$

is the vector of premise variables. A premise variable represents any measurable variable and can be in a simple case a state variable.

Using a TS model, the conclusion part of a single rule consists no longer of a fuzzy set but determines a function with state variables as arguments, and the corresponding function is a local function for the fuzzy region that is described by the premise part of the rule [21]. Thus, using linear functions, a system state is described locally (in fuzzy regions) by linear models, and at the boundaries between regions a suitable interpolation is used between the corresponding local models. Thus, given a pair of $(\mathbf{q}(t), \mathbf{u}(t))$, the final state of the systems is inferred as follows

$$\begin{gathered} \dot{\mathbf{q}}(t) = \frac{\sum_{h=1}^{k} \cdots \sum_{j=1}^{k} w_{1h}(\theta_i(t)) \cdots w_{pj}(\theta_j(t)) \boldsymbol{\Omega}_{i\cdots j}}{\sum_{h=1}^{k} \cdots \sum_{j=1}^{k} w_{1h}(\theta_i(t)) \cdots w_{pj}(\theta_j(t))}, \\ \boldsymbol{\Omega}_{h\cdots j} = \mathbf{A}_{h\cdots j}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t), \end{gathered} \tag{2.5}$$

where $\boldsymbol{\Omega}_{h\cdots j}$ is the linear model associated with the $(h \cdots j)$ combination of sector function indexes. Constructing the aggregated function set $\{w_i(\boldsymbol{\theta}(t)),\ i = 1,2,\ldots,s,\ s = 2^k\}$ from all combinations of the sector functions, for example, ordered as follows,

$$\begin{gathered} w_1(\boldsymbol{\theta}(t)) = w_{11}(\theta_1(t)) \cdots w_{p1}(\theta_1(t)) \\ \vdots \\ w_s(\boldsymbol{\theta}(t)) = w_{1k}(\theta_k(t)) \cdots w_{pk}(\theta_k(t)) \end{gathered} \tag{2.6}$$

implies that

$$\dot{\mathbf{q}}(t) = \frac{\sum_{i=1}^{s} w_i(\boldsymbol{\theta}(t))\boldsymbol{\Omega}_i(t)}{\sum_{i=1}^{s} w_i(\boldsymbol{\theta}(t))} = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\boldsymbol{\Omega}_i(t), \tag{2.7}$$

$$\boldsymbol{\Omega}_i = \mathbf{A}_i\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t), \tag{2.8}$$

where

$$h_i(\boldsymbol{\theta}(t)) = \frac{w_i(\boldsymbol{\theta}(t))}{\sum_{i=1}^{s} w_i(\boldsymbol{\theta}(t))} \tag{2.9}$$

is the ith aggregated normalized membership function satisfying conditions:

$$0 \le h_i(\boldsymbol{\theta}(t)) \le 1, \quad \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) = 1, \quad \forall i \in \{1, \ldots, s\}, \tag{2.10}$$

and linear consequent equation represented by (2.8) is called a linear subsystem.

Therefore, the TS fuzzy approximation of (2.1) leads to (2.7), (2.8) where the matrix $\mathbf{A}_i \in \mathbb{R}^{n\times n}$ is Jacobian matrix of $\mathbf{a}(\mathbf{q}(t))$ with respect to $\mathbf{q}(t) = \mathbf{q}_i$, and $\mathbf{q}_i$ is the center of the ith sector (fuzzy region).

Now, the TS fuzzy model for (2.1), (2.2) takes the form

$$\dot{\mathbf{q}}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\mathbf{A}_i\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t)), \tag{2.11}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t). \tag{2.12}$$

Assumption 2.1. The matrices $\mathbf{B}$, $\mathbf{C}$ are the same for all local models.

Assumption 2.2. The pair $(\mathbf{a}(\mathbf{q}(t)), \mathbf{B})$ is locally controllable where

$$\mathbf{a}(\mathbf{q}(t)) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{A}_i, \tag{2.13}$$

and $\mathbf{B}$ is of full column rank.

2.2. Linear Quadratic Control Background

In order to build up the background of the proposed method, some basics on the continuous-time LQ control are recalled. Considering the linear model (2.1), (2.2), that is, $\mathbf{a}(\mathbf{q}(t)) = \mathbf{A}\mathbf{q}(t)$, the control design is possed as an optimal problem with certain combined quadratic performance on $\mathbf{q}(t)$ and $\mathbf{u}(t)$, and the control task is formulated as follows: find the nonzero control $\mathbf{u}(t)$ defined on $\langle 0, T\rangle$ such that the state $\mathbf{q}(t)$ is driven to the state coordinate origin at $t = T$, and the following performance index is minimized

$$J_T = \mathbf{q}^T(T)\mathbf{Q}^{\bullet}\mathbf{q}(T) + \int_0^T r(\mathbf{q}(t), \mathbf{u}(t)), \tag{2.14}$$

$$r(\mathbf{q}(t), \mathbf{u}(t)) = \mathbf{q}^T(t)\mathbf{Q}\mathbf{q}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t) = \begin{bmatrix}\mathbf{q}^T(t) & \mathbf{u}^T(t)\end{bmatrix}\mathbf{J}_J\begin{bmatrix}\mathbf{q}(t)\\ \mathbf{u}(t)\end{bmatrix}, \tag{2.15}$$

where $\mathbf{J}_J \in \mathbb{R}^{(n+r)\times(n+r)}$ takes the form:

$$\mathbf{J}_J = \operatorname{diag}\begin{bmatrix}\mathbf{Q} & \mathbf{R}\end{bmatrix} > 0, \tag{2.16}$$

$T > 0$ is finite, $\mathbf{Q} > 0$, $\mathbf{Q} \in \mathbb{R}^{n\times n}$, $\mathbf{R} > 0$, $\mathbf{R} \in \mathbb{R}^{m\times m}$, and $\mathbf{Q}^{\bullet} > 0$, $\mathbf{Q}^{\bullet} \in \mathbb{R}^{n\times n}$.

Proposition 2.3 (equivalent performance index). *If the linear system from* (2.1), (2.2) *is controllable, then the LQ control design task is optimized with respect to the equivalent quadratic cost function (performance index):*

$$J_T = \mathbf{q}^T(0)\mathbf{P}(0)\mathbf{q}(0) + \int_0^T p(\mathbf{q}(t), \mathbf{u}(t)), \tag{2.17}$$

$$p(\mathbf{q}(t), \mathbf{u}(t)) = \begin{bmatrix}\mathbf{q}^T(t) & \mathbf{u}^T(t)\end{bmatrix}\mathbf{J}(t)\begin{bmatrix}\mathbf{q}(t)\\ \mathbf{u}(t)\end{bmatrix}, \tag{2.18}$$

where

$$\mathbf{J}(t) = \begin{bmatrix}\mathbf{P}(t)\mathbf{A} + \mathbf{A}^T\mathbf{P}(t) + \dot{\mathbf{P}}(t) + \mathbf{Q} & \mathbf{P}(t)\mathbf{B}\\ \mathbf{B}^T\mathbf{P}(t) & \mathbf{R}\end{bmatrix}, \tag{2.19}$$

$\mathbf{P}(t) > 0$, $\mathbf{P}(t) \in \mathbb{R}^{n\times n}$, $\mathbf{J}(t) > 0$, $\mathbf{J}(t) \in \mathbb{R}^{(n+r)\times(n+r)}$, *respectively.*

Proof (compare e.g. [18]). Since now the system (2.1), (2.2) is linear in $\mathbf{q}(t)$, the quadratic Lyapunov function candidate can be chosen as

$$v(\mathbf{q}(t)) = \mathbf{q}^T(t)\mathbf{P}(t)\mathbf{q}(t), \tag{2.20}$$

and the derivative of the Lyapunov function candidate takes the form:

$$\begin{gathered}\dot{v}(\mathbf{q}(t), \mathbf{u}(t)) = \dot{\mathbf{q}}^T(t)\mathbf{P}(t)\mathbf{q}(t) + \mathbf{q}^T(t)\mathbf{P}(t)\dot{\mathbf{q}}(t) + \mathbf{q}^T(t)\dot{\mathbf{P}}(t)\mathbf{q}(t),\\ \dot{v}(\mathbf{q}(t), \mathbf{u}(t)) = \begin{bmatrix}\mathbf{q}^T(t) & \mathbf{u}^T(t)\end{bmatrix}\mathbf{J}_V(t)\begin{bmatrix}\mathbf{q}(t)\\ \mathbf{u}(t)\end{bmatrix},\end{gathered} \tag{2.21}$$

respectively, where

$$\mathbf{J}_V(t) = \begin{bmatrix} \mathbf{P}(t)\mathbf{A} + \mathbf{A}^T\mathbf{P}(t) + \dot{\mathbf{P}}(t) & \mathbf{P}(t)\mathbf{B} \\ \mathbf{B}^T\mathbf{P}(t) & \mathbf{0} \end{bmatrix}. \tag{2.22}$$

Defining, at the time instant T, the cumulative function V_T as

$$V_T = \int_0^T \dot{v}(\mathbf{q}(t), \mathbf{u}(t))dt, \tag{2.23}$$

which, in turn, is equivalent to

$$V_T = \mathbf{q}^T(T)\mathbf{P}(T)\mathbf{q}(T) - \mathbf{q}^T(0)\mathbf{P}(0)\mathbf{q}(0), \tag{2.24}$$

then adding (2.23) to (2.14), subtracting (2.24) from (2.14), and setting $\mathbf{P}(T) = \mathbf{Q}^{\bullet}$, the performance index (2.14) is brought to the form (2.17), where

$$p(\mathbf{q}(t), \mathbf{u}(i)) = r(\mathbf{q}(i), \mathbf{u}(i)) + \dot{v}(\mathbf{q}(t), \mathbf{u}(t)). \tag{2.25}$$

It is evident that with $\mathbf{J}(t) = \mathbf{J}_J + \mathbf{J}_V(t)$ then (2.16), (2.22) imply (2.19). □

Proposition 2.4 (infinite horizon LQ control). *LQ control that the control law gain has become constant value is given by*

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{q}(t), \tag{2.26}$$

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}, \tag{2.27}$$

where $\mathbf{P} > 0$ *is a solution of the algebraic Riccati equation (ARE)*

$$\mathbf{0} = \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}. \tag{2.28}$$

Proof (see e.g., [18]). Considering $\mathbf{P}(t) = \mathbf{P}$, $\mathbf{J}(t) = \mathbf{J}$, $\dot{\mathbf{P}}(t) = \mathbf{0}$, then (2.18), (2.19) imply

$$\frac{\partial p(\mathbf{q}(t), \mathbf{u}(t))}{\partial \mathbf{u}^T(t)} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_m \end{bmatrix} \mathbf{J} \begin{bmatrix} \mathbf{q}(t) \\ \mathbf{u}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{B}^T\mathbf{P} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{q}(t) \\ \mathbf{u}(t) \end{bmatrix} = \mathbf{0}, \tag{2.29}$$

$$\frac{\partial p(\mathbf{q}(t), \mathbf{u}(t))}{\partial \mathbf{q}^T(t)} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix} \mathbf{J} \begin{bmatrix} \mathbf{q}(t) \\ \mathbf{u}(t) \end{bmatrix} = \left(\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} + \mathbf{Q}\right)\mathbf{q}(t) + \mathbf{P}\mathbf{B}\mathbf{u}(t) = \mathbf{0}, \tag{2.30}$$

respectively. It is obvious that (2.29) implies (2.27), and by substituting (2.26), (2.27) into (2.30), (2.28) is obtained.

Note, it makes no practical sense to have a terminal cost term with terminal time being infinite in the performance index. □

Summarizing, (2.25) specifies the form of derivative of generalized Lyapunov function to formulate the infinite horizon LQ control design conditions using LMI.

3. Basic Preliminaries

The main concern of this section is to present basic concepts of nonlinear fuzzy control design for systems represented by TS model. Presented structure is partly motivated by minimizing the number of LMIs with respect to LMI solvers limitations.

Definition 3.1. Considering the general form of (2.11):

$$\dot{\mathbf{q}}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\mathbf{A}_i\mathbf{q}(t) + \mathbf{B}_i\mathbf{u}(t)) \tag{3.1}$$

and using the same set of membership function, the nonlinear fuzzy state controller is defined as

$$\mathbf{u}(t) = -\sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{K}_j\mathbf{q}(t). \tag{3.2}$$

Proposition 3.2. *If the set of aggregated normalized membership functions* (2.9) *satisfies* (2.10) *then*

$$\sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{K}_j^T\mathbf{R}\sum_{k=1}^{s} h_k(\boldsymbol{\theta}(t))\mathbf{K}_k < \sum_{l=1}^{s}\mathbf{K}_l^T\mathbf{R}\mathbf{K}_l. \tag{3.3}$$

Proof . Considering $s = 1$ the conditions (2.10) imply

$$h_1(\boldsymbol{\theta}(t))\mathbf{K}_1^T\mathbf{R}\ h_1(\boldsymbol{\theta}(t))\mathbf{K}_1 = h_1^2(\boldsymbol{\theta}(t))\mathbf{K}_1^T\mathbf{R}\mathbf{K}_1 < \mathbf{K}_1^T\mathbf{R}\mathbf{K}_1. \tag{3.4}$$

Providing the base of mathematical induction principle the number of functions is chosen as $s = 2$. Thus, left-hand side of (3.3) implies

$$\begin{gathered}\left(h_1(\boldsymbol{\theta}(t))\mathbf{K}_1^T + h_2(\boldsymbol{\theta}(t))\mathbf{K}_2^T\right)\mathbf{R}(h_1(\boldsymbol{\theta}(t))\,\mathbf{K}_1 + h_2(\boldsymbol{\theta}(t))\mathbf{K}_2) = \left[\mathbf{K}_1^T\mathbf{R}^{1/2}\ \ \mathbf{K}_2^T\mathbf{R}^{1/2}\right]\mathbf{H}_2(\boldsymbol{\theta}(t))\begin{bmatrix}\mathbf{R}^{1/2}\mathbf{K}_1\\ \mathbf{R}^{1/2}\mathbf{K}_2\end{bmatrix},\\ \mathbf{H}_2(\boldsymbol{\theta}(t)) = \begin{bmatrix} h_1^2(\boldsymbol{\theta}(t)) & h_1(\boldsymbol{\theta}(t))h_2(\boldsymbol{\theta}(t))\\ h_2(\boldsymbol{\theta}(t))h_1(\boldsymbol{\theta}(t)) & h_2^2(\boldsymbol{\theta}(t))\end{bmatrix},\end{gathered} \tag{3.5}$$

and right-hand side of (3.3) specifies

$$\mathbf{K}_1^T\mathbf{R}\mathbf{K}_1 + \mathbf{K}_2^T\mathbf{R}\mathbf{K}_2 = \left[\mathbf{K}_1^T\mathbf{R}^{1/2}\ \ \mathbf{K}_2^T\mathbf{R}^{1/2}\right]\mathbf{I}_2\begin{bmatrix}\mathbf{R}^{1/2}\mathbf{K}_1\\ \mathbf{R}^{1/2}\mathbf{K}_2\end{bmatrix}. \tag{3.6}$$

Applying the Schur complement property to $\mathbf{H}_2(\boldsymbol{\theta}(t))$ and $\mathbf{I}_2$ it is obvious that

$$\begin{gathered} 1 > h_2^2(\boldsymbol{\theta}(t)), \\ 1 > 0 = h_1^2(\boldsymbol{\theta}(t)) - h_2(\boldsymbol{\theta}(t))h_1(\boldsymbol{\theta}(t))h_2^{-2}(\boldsymbol{\theta}(t))h_1(\boldsymbol{\theta}(t))h_2(\boldsymbol{\theta}(t)), \end{gathered} \tag{3.7}$$

and the condition is satisfied in the sense of the proposition.

Since the statement holds true for at least one value, it is assumed that it holds true for an arbitrary fixed value $s-1$

$$\mathbf{I}_{s-1} > \mathbf{H}_{s-1}(\boldsymbol{\theta}(t)) = \begin{bmatrix} h_1(\boldsymbol{\theta}(t)) \\ \vdots \\ h_{s-1}(\boldsymbol{\theta}(t)) \end{bmatrix} \begin{bmatrix} h_1(\boldsymbol{\theta}(t)) & \cdots & h_{s-1}(\boldsymbol{\theta}(t)) \end{bmatrix}. \tag{3.8}$$

To prove that the induction hypothesis holds true for all s let the sth membership function is included in prescribed way, that is,

$$\mathbf{H}_s(\boldsymbol{\theta}(t)) = \begin{bmatrix} \mathbf{H}_{s-1}(\boldsymbol{\theta}(t)) & \begin{bmatrix} h_1(\boldsymbol{\theta}(t)) \\ \vdots \\ h_{s-1}(\boldsymbol{\theta}(t)) \end{bmatrix} h_s(\boldsymbol{\theta}(t)) \\ * & h_s^2(\boldsymbol{\theta}(t)) \end{bmatrix}, \tag{3.9}$$

where here, and hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Now, comparing the Schur complements of $\mathbf{I}_s$ and $\mathbf{H}_s(\boldsymbol{\theta}(t))$, the first complement is satisfied since

$$1 > h_s^2(\boldsymbol{\theta}(t)), \tag{3.10}$$

and the second gives

$$\mathbf{I}_{s-1} > \mathbf{H}_{s-1}(\boldsymbol{\theta}(t)) - h_s^2(\boldsymbol{\theta}(t))\mathbf{H}_{s-1}(\boldsymbol{\theta}(t))h_s^{-2}(\boldsymbol{\theta}(t)) = \mathbf{H}_{s-1}(\boldsymbol{\theta}(t)) - \mathbf{H}_{s-1}(\boldsymbol{\theta}(t)) = \mathbf{0}. \tag{3.11}$$

Thus, (3.10), (3.11) imply (3.3). □

Proposition 3.3. *The equilibrium of the system* (3.1) *under control* (3.2) *is globally quadratic stable if there exists a positive definite symmetric matrix* $\mathbf{P} \in \mathbb{R}^{n \times n}$ *such that*

$$\mathbf{H}_{ii}^T\mathbf{P} + \mathbf{P}\mathbf{H}_{ii} < 0, \tag{3.12}$$

$$\frac{\mathbf{H}_{ij}^T + \mathbf{H}_{ji}^T}{2}\mathbf{P} + \mathbf{P}\frac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2} < 0, \tag{3.13}$$

for for all $i \in \langle 1, 2, \ldots s\rangle$, $i < j \le s$, $i, j \in \langle 1, 2, \ldots, s\rangle$ *and* $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$, *respectively, where*

$$\mathbf{H}_{ij} = \mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j. \tag{3.14}$$

Proof . Substituting (3.2) into (3.1) results in

$$\dot{\mathbf{q}}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) \left(\mathbf{A}_i \mathbf{q}(t) - \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t)) \mathbf{B}_i \mathbf{K}_j \mathbf{q}(t) \right). \tag{3.15}$$

Since $\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) = 1$ for all $i \in \{1, \ldots, s\}$, it yields

$$\dot{\mathbf{q}}(t) = \sum_{i=1}^{s} \sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t)) h_j(\boldsymbol{\theta}(t)) \left(\mathbf{A}_i - \mathbf{B}_i \mathbf{K}_j\right) \mathbf{q}(t) \tag{3.16}$$

and also, owing to the symmetry in summations:

$$\dot{\mathbf{q}}(t) = \sum_{i=1}^{s} \sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t)) h_j(\boldsymbol{\theta}(t)) \left(\mathbf{A}_j - \mathbf{B}_j \mathbf{K}_i\right) \mathbf{q}(t). \tag{3.17}$$

Thus, adding (3.16), (3.17) gives

$$2\dot{\mathbf{q}}(t) = \sum_{i=1}^{s} \sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t)) h_j(\boldsymbol{\theta}(t)) \left(\mathbf{H}_{ij} + \mathbf{H}_{ji}\right) \mathbf{q}(t). \tag{3.18}$$

Rearranging the computation, (3.18) can be written as

$$\begin{aligned} 2\dot{\mathbf{q}}(t) &= \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) h_i(\boldsymbol{\theta}(t)) (\mathbf{H}_{ii} + \mathbf{H}_{ii}) \mathbf{q}(t) + 2 \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} h_i(\boldsymbol{\theta}(t)) h_j(\boldsymbol{\theta}(t)) \left(\mathbf{H}_{ij} + \mathbf{H}_{ji}\right) \mathbf{q}(t), \\ \dot{\mathbf{q}}(t) &= \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) h_i(\boldsymbol{\theta}(t)) \mathbf{H}_{ii} \mathbf{q}(t) + 2 \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} h_i(\boldsymbol{\theta}(t)) h_j(\boldsymbol{\theta}(t)) \frac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2} \mathbf{q}(t), \end{aligned} \tag{3.19}$$

respectively. Defining Lyapunov function candidate of the form:

$$v(\mathbf{q}(t)) = \mathbf{q}^T(t) \mathbf{P} \mathbf{q}(t) > 0, \tag{3.20}$$

where $\mathbf{P} \in \mathbb{R}^{n\times n}$ is a positive definite symmetric matrix, then after evaluation the derivative of (3.20) with respect to t on a system trajectory it yields

$$\dot{v}(\mathbf{q}(t)) = \dot{\mathbf{q}}^T(t)\mathbf{P}\mathbf{q}(t) + \mathbf{q}^T(t)\mathbf{P}\dot{\mathbf{q}}(t) < 0 \tag{3.21}$$

$$\begin{aligned}\dot{v}(\mathbf{q}(t)) = {} & \mathbf{q}^T(t)\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))h_i(\boldsymbol{\theta}(t))\mathbf{H}_{ii}^T\mathbf{P}\mathbf{q}(t)\\ & + \mathbf{q}^T(t)\mathbf{P}\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))h_i(\boldsymbol{\theta}(t))\mathbf{H}_{ii}\mathbf{q}(t)\\ & + 2\mathbf{q}^T(t)\sum_{i=1}^{s-1}\sum_{j=i+1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\frac{\mathbf{H}_{ij}^T + \mathbf{H}_{ji}^T}{2}\mathbf{P}\mathbf{q}(t)\\ & + 2\mathbf{q}^T(t)\mathbf{P}\sum_{i=1}^{s-1}\sum_{j=i+1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\frac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2}\mathbf{q}(t) < 0,\end{aligned} \tag{3.22}$$

respectively. Then (3.21) can be compactly written as

$$\mathbf{q}^T(t)\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))h_i(\boldsymbol{\theta}(t))\mathbf{P}_{ii}^*\mathbf{q}(t) + 2\mathbf{q}^T(t)\sum_{i=1}^{s-1}\sum_{j=i+1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\mathbf{P}_{ij}^*\mathbf{q}(t) < 0, \tag{3.23}$$

where

$$\begin{aligned}\mathbf{P}_{ii}^* &= \mathbf{H}_{ii}^T\mathbf{P} + \mathbf{P}\mathbf{H}_{ii} < 0,\\ \mathbf{P}_{ij}^* &= \frac{\mathbf{H}_{ij}^T + \mathbf{H}_{ji}^T}{2}\mathbf{P} + \mathbf{P}\frac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2} < 0.\end{aligned} \tag{3.24}$$

Thus, (3.24) implies (3.12), (3.13). This concludes the proof. □

Remark 3.4. If $\mathbf{B}_i = \mathbf{B}$ for all $i \in \langle 1, 2, \ldots, s\rangle$ then

$$\mathbf{H}_{ij} + \mathbf{H}_{ji} = \mathbf{A}_i - \mathbf{B}\mathbf{K}_j + \mathbf{A}_j - \mathbf{B}\mathbf{K}_i = \mathbf{H}_{ii} + \mathbf{H}_{jj}, \tag{3.25}$$

and (3.13) for $i < j \le s$, $i, j \in \langle 1, 2, \ldots, s\rangle$ takes the form

$$\frac{\mathbf{H}_{ii}^T + \mathbf{H}_{jj}^T}{2}\mathbf{P} + \mathbf{P}\frac{\mathbf{H}_{ii} + \mathbf{H}_{jj}}{2} < 0, \tag{3.26}$$

which implies

$$\mathbf{H}_{ii}^T\mathbf{P} + \mathbf{P}\mathbf{H}_{ii} < 0, \qquad \mathbf{H}_{jj}^T\mathbf{P} + \mathbf{P}\mathbf{H}_{jj} < 0. \tag{3.27}$$

It is evident that with $\mathbf{H}_{ii}, \mathbf{H}_{jj}$ satisfying (3.12) also (3.27) is satisfied.

Proposition 3.5. *The equilibrium of the system* (3.1) *under control* (3.2) *is globally asymptotically stable if there exist a positive definite symmetric matrix* $\mathbf{X} \in \mathbb{R}^{n\times n}$ *and matrices* $\mathbf{Y}_j \in \mathbb{R}^{r\times n}$ *such that*

$$\mathbf{X} = \mathbf{X}^T > 0, \qquad \mathbf{T}_{ii} = \mathbf{T}_{ii}^T < 0, \qquad \mathbf{T}_{ij} = \mathbf{T}_{ij}^T < 0, \tag{3.28}$$

for all $i \in \langle 1,2,\dots,s\rangle$ *and* $i < j \le s$, $i,j \in \langle 1,2,\dots,s\rangle$ *and* $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$, *respectively, where*

$$\begin{gathered} \mathbf{T}_{ii} = \mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} - \mathbf{Y}_i^T\mathbf{B}_i^T - \mathbf{B}_i\mathbf{Y}_i, \\ \mathbf{T}_{ij} = \frac{\left(\mathbf{X}\mathbf{A}_i^T - \mathbf{Y}_j^T\mathbf{B}_i^T\right) + \left(\mathbf{X}\mathbf{A}_j^T - \mathbf{Y}_i^T\mathbf{B}_j^T\right)}{2} + \frac{\left(\mathbf{A}_i\mathbf{X} - \mathbf{B}_i\mathbf{Y}_j\right) + \left(\mathbf{A}_j\mathbf{X} - \mathbf{B}_j\mathbf{Y}_i\right)}{2}. \end{gathered} \tag{3.29}$$

Then, the set of control law gain matrices are given as follows:

$$\mathbf{K}_j = \mathbf{Y}_j\mathbf{X}^{-1}, \quad j = 1,2,\dots,s. \tag{3.30}$$

Proof. Since $\mathbf{P}$ is considered to be a positive definite matrix, it is obvious that $\mathbf{P}^{-1}$ is also positive definite, and premultiplying left-hand and right-hand side of (3.12), as well as (3.13) by $\mathbf{P}^{-1}$ leads to the inequalities:

$$\begin{gathered} \mathbf{P}^{-1}(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i)^T + (\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i)\mathbf{P}^{-1} < 0, \\ \mathbf{P}^{-1}\frac{\left(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j\right)^T + \left(\mathbf{A}_j - \mathbf{B}_j\mathbf{K}_i\right)^T}{2} + \frac{\left(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j\right) + \left(\mathbf{A}_j - \mathbf{B}_j\mathbf{K}_i\right)}{2}\mathbf{P}^{-1} < 0. \end{gathered} \tag{3.31}$$

Thus, with the notation

$$\mathbf{X} = \mathbf{P}^{-1}, \qquad \mathbf{Y}_j = \mathbf{K}_j\mathbf{X}, \tag{3.32}$$

(3.31) implies (3.29), respectively. This concludes the proof. □

Proposition 3.6. *The equilibrium of the fuzzy system* (2.11) *controlled by the fuzzy controller* (3.2) *is globally asymptotically stable if there exists a positive definite matrix* $\mathbf{X} \in \mathbb{R}^{n\times n}$ *and matrices* $\mathbf{Y}_j \in \mathbb{R}^{r\times n}$ *such that*

$$\mathbf{X} = \mathbf{X}^T > 0, \qquad \mathbf{T}_{ij}^{\diamond} = \mathbf{T}_{ij}^{\diamond T} < 0, \tag{3.33}$$

for $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$, $i,j = 1,2,\dots,s$, *where*

$$\mathbf{T}_{ij}^{\diamond} = \mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} - \mathbf{Y}_j^T\mathbf{B}^T - \mathbf{B}\mathbf{Y}_j. \tag{3.34}$$

The set of control law gain matrices is given by (3.30).

Proof. It Implies directly from Remark 3.4 and Proposition 3.5. □

4. Fuzzy Controller with Quadratic Performances

The controller design is accomplished using the concept of asymptotic stability by analyzing the existence of an extended Lyapunov function. The fuzzy static output controller is designed using the concept of parallel distributed compensation, in which the fuzzy controller shares the same sets of normalized membership functions like the TS fuzzy system model. The goal is to achieve a certain level of performance using a guaranteed-cost approach results known from LQ control theory.

Theorem 4.1. *The equilibrium of the system* (3.1)*, controlled by the fuzzy controller* (3.2)*, is globally asymptotically stable if there exist positive definite symmetric matrices* $\mathbf{X} \in \mathbb{R}^{n\times n}$*,* $\mathbf{Q}^{\diamond} \in \mathbb{R}^{n\times n}$*,* $\mathbf{R}^{\diamond} \in \mathbb{R}^{r\times r}$*, and matrices* $\mathbf{Y}_j \in \mathbb{R}^{r\times n}$*, such that with* (3.29)

$$\mathbf{X} = \mathbf{X}^T > 0, \qquad \mathbf{Q}^{\diamond} = \mathbf{Q}^{\diamond T} > 0, \qquad \mathbf{R}^{\diamond} = \mathbf{R}^{\diamond T} > 0,$$

$$\begin{bmatrix} \mathbf{T}_{ii} & \mathbf{X} & \mathbf{Y}_1^T & \cdots & \mathbf{Y}_s^T \\ * & -\mathbf{Q}^{\diamond} & \mathbf{0} & \cdots & \mathbf{0} \\ * & * & -\mathbf{R}^{\diamond} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & -\mathbf{R}^{\diamond} \end{bmatrix} < 0, \qquad \begin{bmatrix} \mathbf{T}_{ij} & \mathbf{X} & \mathbf{Y}_1^T & \cdots & \mathbf{Y}_s^T \\ * & -\mathbf{Q}^{\diamond} & \mathbf{0} & \cdots & \mathbf{0} \\ * & * & -\mathbf{R}^{\diamond} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & -\mathbf{R}^{\diamond} \end{bmatrix} < 0, \tag{4.1}$$

for all $i \in \langle 1,2,\ldots,s\rangle$, $i < j \leq s$, $i,j \in \langle 1,2,\ldots,s\rangle$*, and* $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$*, respectively. The set of control law gain matrices can be found directly as*

$$\mathbf{K}_j = \mathbf{Y}_j\mathbf{X}^{-1}, \quad j = 1,2,\ldots,s. \tag{4.2}$$

Proof. Considering (3.16) and defining with respect to (2.15), (2.25), then the quadratic positive Lyapunov function is as follows

$$v(\mathbf{q}(t)) = \mathbf{q}^T(t)\mathbf{P}\mathbf{q}(t) + \int_0^t r(\mathbf{q}(r),\mathbf{u}(r))dr > 0, \tag{4.3}$$

where $\mathbf{P}$ is a positive definite symmetric matrix, then it yields

$$\dot{v}(\mathbf{q}(t)) = \dot{\mathbf{q}}^T(t)\mathbf{P}\mathbf{q}(t) + \mathbf{q}^T(t)\mathbf{P}\dot{\mathbf{q}}(t) + r(\mathbf{q}(r),\mathbf{u}(r)) < 0. \tag{4.4}$$

Substituting (2.15), (3.1) and (3.2) into (4.4) gives

$$\dot{v}(\mathbf{q}(t)) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{q}_j^{\bullet T}(t)\boldsymbol{\Pi}_i\mathbf{q}_k^{\bullet}(t) < 0, \tag{4.5}$$

where

$$\mathbf{\Pi}_i = \begin{bmatrix} \mathbf{Q} + \mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i & \mathbf{P}\mathbf{B}_i \\ \mathbf{B}_i^T\mathbf{P} & \mathbf{R} \end{bmatrix} < 0,$$
$$\mathbf{q}_j^{\bullet T}(t) = \left[\mathbf{q}^T(t) \quad -\mathbf{q}^T(t)\sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{K}_j^T \right], \tag{4.6}$$

and after straightforward computation it can be obtained

$$\dot{v}(\mathbf{q}(t)) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{q}^T(t)\mathbf{\Upsilon}_{ijk}(\boldsymbol{\theta}(t))\mathbf{q}(t) < 0, \tag{4.7}$$

with

$$\mathbf{\Upsilon}_{ijk}(\boldsymbol{\theta}(t)) = \mathbf{Q} + \mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{K}_j^T\mathbf{R}\sum_{k=1}^{s} h_k(\boldsymbol{\theta}(t))\mathbf{K}_k$$
$$- \mathbf{P}\mathbf{B}_i\sum_{k=1}^{s} h_k(\boldsymbol{\theta}(t))\mathbf{K}_k - \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{K}_j^T\mathbf{B}_i^T\mathbf{P}. \tag{4.8}$$

Now, exploiting (3.3), then (4.7), (4.8) can be rewritten as

$$\dot{v}(\mathbf{q}(t)) < \mathbf{q}^T(t)\sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\mathbf{\Phi}_{ij}\mathbf{q}(t) < 0, \tag{4.9}$$

where

$$\mathbf{\Phi}_{ij} = \left(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j\right)^T\mathbf{P} + \mathbf{P}\left(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j\right) + \mathbf{Q} + \sum_{l=1}^{s}\mathbf{K}_l^T\mathbf{R}\mathbf{K}_l < 0. \tag{4.10}$$

Analogously to (3.23) then (4.10) can be written as

$$\dot{v}(\mathbf{q}(t)) < \mathbf{q}^T(t)\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))h_i(\boldsymbol{\theta}(t))\mathbf{\Phi}_{ii}^{\diamond}\mathbf{q}(t) + 2\mathbf{q}^T(t)\sum_{i=1}^{s-1}\sum_{j=i+1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\mathbf{\Phi}_{ij}^{\diamond}\mathbf{q}(t) < 0, \tag{4.11}$$

where, with $\mathbf{H}_{ij}$ defined in (3.14), it is

$$\Phi_{ii}^{\diamond} = \mathbf{H}_{ii}^{T}\mathbf{P} + \mathbf{P}\mathbf{H}_{ii} + \mathbf{Q} + \sum_{l=1}^{s}\mathbf{K}_{l}^{T}\mathbf{R}\mathbf{K}_{l} < 0,$$
$$\Phi_{ij}^{\diamond} = \frac{\mathbf{H}_{ij}^{T} + \mathbf{H}_{ji}^{T}}{2}\mathbf{P} + \mathbf{P}\frac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2} + \mathbf{Q} + \sum_{l=1}^{s}\mathbf{K}_{l}^{T}\mathbf{R}\mathbf{K}_{l} < 0, \tag{4.12}$$

for all $i \in \langle 1,2,\ldots,s\rangle$, $i < j \leq s$, $i,j \in \langle 1,2,\ldots,s\rangle$ and $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$, respectively.

Since $\mathbf{P}$ is a regular positive definite square matrix, then premultiplying left-hand side and right-hand side of (4.12) by $\mathbf{P}^{-1}$ give

$$\mathbf{P}^{-1}\mathbf{H}_{ii}^{T} + \mathbf{H}_{ii}\mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{Q}\mathbf{P}^{-1} + \sum_{l=1}^{s}\mathbf{P}^{-1}\mathbf{K}_{l}^{T}\mathbf{R}\mathbf{K}_{l}\mathbf{P}^{-1} < 0,$$
$$\mathbf{P}^{-1}\frac{\mathbf{H}_{ij}^{T} + \mathbf{H}_{ji}^{T}}{2} + \frac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2}\mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{Q}\mathbf{P}^{-1} + \sum_{l=1}^{s}\mathbf{P}^{-1}\mathbf{K}_{l}^{T}\mathbf{R}\mathbf{K}_{l}\mathbf{P}^{-1} < 0. \tag{4.13}$$

Thus, using (3.29) and the notations

$$\mathbf{X} = \mathbf{P}^{-1}, \qquad \mathbf{Y}_j = \mathbf{K}_j\mathbf{X}, \qquad \mathbf{Q}^{\diamond} = \mathbf{Q}^{-1}, \qquad \mathbf{R}^{\diamond} = \mathbf{R}^{-1}, \tag{4.14}$$

it yields

$$\mathbf{T}_{ii} + \mathbf{X}(\mathbf{Q}^{\diamond})^{-1}\mathbf{X} + \sum_{l=1}^{s}\mathbf{Y}_{l}^{T}(\mathbf{R}^{\diamond})^{-1}\mathbf{Y}_{l} < 0, \tag{4.15}$$

$$\mathbf{T}_{ij} + \mathbf{X}(\mathbf{Q}^{\diamond})^{-1}\mathbf{X} + \sum_{l=1}^{s}\mathbf{Y}_{l}^{T}(\mathbf{R}^{\diamond})^{-1}\mathbf{Y}_{l} < 0, \tag{4.16}$$

for all $i \in \langle 1,2,\ldots,s\rangle$, $i < j \leq s$, $i,j \in \langle 1,2,\ldots,s\rangle$, $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$, respectively. It is evident that (4.15) implies (4.2), and (4.16) implies (4.2). □

Theorem 4.2. *The equilibrium of the fuzzy system* (2.11) *controlled by the fuzzy controller* (3.2) *is globally asymptotically stable if there exist positive definite matrices* $\mathbf{X} \in \mathbb{R}^{n\times n}$, $\mathbf{Q}^{\diamond} \in \mathbb{R}^{n\times n}$, $\mathbf{R}^{\diamond} \in \mathbb{R}^{r\times r}$, *and matrices* $\mathbf{Y}_j \in \mathbb{R}^{r\times n}$ *such that*

$$\mathbf{X} = \mathbf{X}^{T} > 0, \qquad \mathbf{Q}^{\diamond} = \mathbf{Q}^{\diamond T} > 0, \qquad \mathbf{R}^{\diamond} = \mathbf{R}^{\diamond T} > 0, \tag{4.17}$$

$$\begin{bmatrix} \mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} - \mathbf{Y}_j^T\mathbf{B}^T - \mathbf{B}\mathbf{Y}_j & \mathbf{X} & \mathbf{Y}_1^T & \cdots & \mathbf{Y}_s^T \\ * & -\mathbf{Q}^{\diamond} & \mathbf{0} & \cdots & \mathbf{0} \\ * & * & -\mathbf{R}^{\diamond} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & -\mathbf{R}^{\diamond} \end{bmatrix} < 0, \tag{4.18}$$

for all $i, j \in \langle 1, 2, \ldots, s\rangle$, $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$. *The set of control law gain matrices is given by* (4.7).

Proof. Since $\mathbf{B}_i = \mathbf{B}$ for all $i \in \langle 1, 2, \ldots, s\rangle$, then (4.10) implies

$$\boldsymbol{\Phi}_{ij} = \left(\mathbf{A}_i - \mathbf{B}\mathbf{K}_j\right)^T\mathbf{P} + \mathbf{P}\left(\mathbf{A}_i - \mathbf{B}\mathbf{K}_j\right) + \mathbf{Q} + \sum_{l=1}^{s}\mathbf{K}_l^T\mathbf{R}\mathbf{K}_l < 0. \tag{4.19}$$

Thus, premultiplying the both side of (4.19) by $\mathbf{P}^{-1}$ gives

$$\mathbf{P}^{-1}\left(\mathbf{A}_i - \mathbf{B}\mathbf{K}_j\right)^T + \left(\mathbf{A}_i - \mathbf{B}\mathbf{K}_j\right)\mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{Q}\mathbf{P}^{-1} + \sum_{l=1}^{s}\mathbf{P}^{-1}\mathbf{K}_l^T\mathbf{R}\mathbf{K}_l\mathbf{P}^{-1} < 0. \tag{4.20}$$

and with (3.34) and the notations (4.14) then (4.20) implies (4.18). □

5. Enhanced Controller with Quadratic Performance

The previous section was detailed how to find the fuzzy controller with quadratic performance ensuring the global asymptotic stability of the system. To extend the affine TS model principle by introducing the slack matrix variables into the LMIs, the system matrices are now decoupled from the equivalent Lyapunov matrix.

5.1. Stability Conditions

Theorem 5.1. *The equilibrium of the system* (3.1) *under control* (3.2) *is globally asymptotically stable if there exist positive definite symmetric matrices* $\mathbf{R} \in \mathbb{R}^{r\times r}$, $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{S}_1$, $\mathbf{S}_2 \in \mathbb{R}^{n\times n}$, *such that*

$$\mathbf{P} = \mathbf{P}^T > 0, \qquad \mathbf{Q} = \mathbf{Q}^T > 0, \qquad \mathbf{R} = \mathbf{R}^T > 0, \qquad \mathbf{S}_1 = \mathbf{S}_1^T > 0, \qquad \mathbf{S}_2 = \mathbf{S}_2^T > 0, \tag{5.1}$$

$$\begin{bmatrix} \mathbf{W}_{ii} & \mathbf{P} + \mathbf{S}_1 - \mathbf{H}_{ii}^T\mathbf{S}_2 \\ * & 2\mathbf{S}_2 \end{bmatrix} < 0, \qquad \begin{bmatrix} \mathbf{W}_{ij} & \mathbf{P} + \mathbf{S}_1 - \dfrac{\mathbf{H}_{ij}^T + \mathbf{H}_{ji}^T}{2}\mathbf{S}_2 \\ * & 2\mathbf{S}_2 \end{bmatrix} < 0, \tag{5.2}$$

for all $i \in \langle 1,2,\ldots,s\rangle$, $i < j \leq s$, $i,j \in \langle 1,2,\ldots,s\rangle$ *and* $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$, *respectively, where with* $\mathbf{H}_{ij}$ *defined in* (3.14) *it is*

$$\mathbf{W}_{ii} = \mathbf{Q} - \mathbf{H}_{ii}^T\mathbf{S}_1 - \mathbf{S}_1\mathbf{H}_{ii} + \sum_{l=1}^{s}\mathbf{K}_l^T\mathbf{R}\mathbf{K}_l,$$
$$\mathbf{W}_{ij} = \mathbf{Q} - \frac{\mathbf{H}_{ij}^T + \mathbf{H}_{ji}^T}{2}\mathbf{S}_1 - \mathbf{S}_1\frac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2} + \sum_{l=1}^{s}\mathbf{K}_l^T\mathbf{R}\mathbf{K}_l. \tag{5.3}$$

Proof. Since (3.1) implies

$$\dot{\mathbf{q}}(t) - \sum_{i=1}^{s}h_i(\boldsymbol{\theta}(t))(\mathbf{A}_i\mathbf{q}(t) + \mathbf{B}_i\mathbf{u}(t)) = \mathbf{0}, \tag{5.4}$$

using arbitrary regular symmetric square matrices $\mathbf{S}_1, \mathbf{S}_2 \in \mathbb{R}^{n\times n}$ it yields

$$\left(\mathbf{q}^T(t)\mathbf{S}_1 + \dot{\mathbf{q}}^T(t)\mathbf{S}_2\right)\left(\dot{\mathbf{q}}(t) - \sum_{i=1}^{s}h_i(\boldsymbol{\theta}(t))(\mathbf{A}_i\mathbf{q}(t) + \mathbf{B}_i\mathbf{u}(t))\right) = 0. \tag{5.5}$$

Adding (5.5), and transposition of (5.5) to (4.4), and then inserting (3.2) give

$$\begin{aligned}
\dot{v}(\mathbf{q}(t)) = {} & \mathbf{q}^T(t)\mathbf{Q}\mathbf{q}(t) + \mathbf{q}^T(t)\sum_{j=1}^{s}h_j(\boldsymbol{\theta}(t))\mathbf{K}_j^T\mathbf{R}\sum_{k=1}^{s}h_k(\boldsymbol{\theta}(t))\mathbf{K}_k\mathbf{q}(t) \\
& + \mathbf{q}^T(t)\mathbf{S}_1\dot{\mathbf{q}}(t) + \dot{\mathbf{q}}^T(t)\mathbf{S}_1\mathbf{q}(t) + 2\dot{\mathbf{q}}^T(t)\mathbf{S}_2\dot{\mathbf{q}}(t) + \mathbf{q}^T(t)\mathbf{P}\dot{\mathbf{q}}(t) + \dot{\mathbf{q}}^T(t)\mathbf{P}\mathbf{q}(t) \\
& - \mathbf{q}^T(t)\mathbf{S}_1\sum_{i=1}^{s}\sum_{j=1}^{s}h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\left(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j\right)\mathbf{q}(t) \\
& - \dot{\mathbf{q}}^T(t)\mathbf{S}_2\sum_{i=1}^{s}\sum_{j=1}^{s}h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\left(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j\right)\mathbf{q}(t) \\
& - \mathbf{q}^T(t)\sum_{i=1}^{s}\sum_{j=1}^{s}h_i(\boldsymbol{\theta}(t))h_i(\boldsymbol{\theta}(t))\left(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j\right)^T\mathbf{S}_1\mathbf{q}(t) \\
& - \mathbf{q}^T(t)\sum_{i=1}^{s}\sum_{j=1}^{s}h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\left(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j\right)^T\mathbf{S}_2\dot{\mathbf{q}}(t) < 0.
\end{aligned} \tag{5.6}$$

Then, using the notation:

$$\mathbf{q}^{\circ T}(t) = \left[\mathbf{q}^T(t) \quad \dot{\mathbf{q}}^T(t)\right] \tag{5.7}$$

after straightforward computation it can be obtained

$$\dot{v}(\mathbf{q}(t)) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{q}^{\circ T}(t)\boldsymbol{\Phi}_i^{\circ}(\boldsymbol{\theta}(t))\mathbf{q}^{\circ}(t) < 0, \tag{5.8}$$

where

$$\boldsymbol{\Phi}_i^{\circ}(\boldsymbol{\theta}(t)) = \begin{bmatrix} \boldsymbol{\Lambda}_i^{\circ}(\boldsymbol{\theta}(t)) & \boldsymbol{\Gamma}_i(\boldsymbol{\theta}(t)) \\ * & 2\mathbf{S}_2 \end{bmatrix}, \tag{5.9}$$

$$\begin{aligned} \boldsymbol{\Lambda}_i^{\circ}(\boldsymbol{\theta}(t)) = \mathbf{Q} - \mathbf{A}_i^T\mathbf{S}_1 - \mathbf{S}_1\mathbf{A}_i + \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{K}_j^T\mathbf{R}\sum_{k=1}^{s} h_k(\boldsymbol{\theta}(t))\mathbf{K}_k \\ + \mathbf{S}_1\mathbf{B}_i\sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{K}_j + \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{K}_j^T\mathbf{B}_i^T\mathbf{S}_1, \end{aligned} \tag{5.10}$$

$$\boldsymbol{\Gamma}_i^{\circ}(\boldsymbol{\theta}(t)) = \mathbf{P} + \mathbf{S}_1 - \mathbf{A}_i^T\mathbf{S}_2 + \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{K}_j^T\mathbf{B}_i^T\mathbf{S}_2. \tag{5.11}$$

Exploiting (3.3), then (5.10) can be rewritten as

$$\boldsymbol{\Lambda}_i^{\circ}(\boldsymbol{\theta}(t)) < \boldsymbol{\Lambda}_i(\boldsymbol{\theta}(t)) = \mathbf{Q} - \mathbf{A}_i^T\mathbf{S}_1 - \mathbf{S}_1\mathbf{A}_i + \sum_{l=1}^{s}\mathbf{K}_l^T\mathbf{R}\mathbf{K}_l + \mathbf{S}_1\mathbf{B}_i\sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{K}_j + \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{K}_j^T\mathbf{B}_i^T\mathbf{S}_1, \tag{5.12}$$

and using (5.12) it yields

$$\dot{v}(\mathbf{q}(t)) < \mathbf{q}^{\circ T}(t)\sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\mathbf{P}_{ij}^{\circ}\mathbf{q}^{\circ}(t) < 0, \tag{5.13}$$

where

$$\mathbf{P}_{ij}^{\circ} = \begin{bmatrix} \mathbf{W}_{ij} & \mathbf{P} + \mathbf{S}_1 - \left(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j\right)^T\mathbf{S}_2 \\ * & 2\mathbf{S}_2 \end{bmatrix} < 0. \tag{5.14}$$

Analogously to (3.23) and (4.11) now (5.13) can be written as

$$\dot{v}(\mathbf{q}(t)) < \mathbf{q}^{\circ T}(t)\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))h_i(\boldsymbol{\theta}(t))\mathbf{P}_{ii}^{\circ}\mathbf{q}^{\circ}(t) + 2\mathbf{q}^{\circ T}(t)\sum_{i=1}^{s-1}\sum_{j=i+1}^{s} h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t))\mathbf{P}_{ij}^{\circ}\mathbf{q}^{\circ}(t) < 0, \tag{5.15}$$

with $\mathbf{W}_{ii}$, $\mathbf{W}_{ij}$ defined in (5.3), respectively, and

$$\mathbf{P}_{ii}^{\circ} = \begin{bmatrix} \mathbf{W}_{ii} & \mathbf{P} + \mathbf{S}_1 - \mathbf{H}_{ii}^T \mathbf{S}_2 \\ * & 2\mathbf{S}_2 \end{bmatrix} < 0, \qquad \mathbf{P}_{ij}^{\circ} = \begin{bmatrix} \mathbf{W}_{ij} & \mathbf{P} + \mathbf{S}_1 - \dfrac{\mathbf{H}_{ij}^T + \mathbf{H}_{ji}^T}{2} \mathbf{S}_2 \\ * & 2\mathbf{S}_2 \end{bmatrix} < 0. \tag{5.16}$$

Since (5.16) implies (5.1), this concludes the proof. □

Corollary 5.2. *If* $\mathbf{B}_i = \mathbf{B}$ *for all* $i \in \langle 1, 2, \ldots, s \rangle$ *then* (5.14) *implies that the equilibrium of the system* (2.11) *under control* (3.2) *is globally asymptotically stable if there exist positive definite symmetric matrices* $\mathbf{R} \in \mathbb{R}^{r \times r}$, $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{S}_1$, $\mathbf{S}_2 \in \mathbb{R}^{n \times n}$, *such that*

$$\mathbf{P} = \mathbf{P}^T > 0, \qquad \mathbf{Q} = \mathbf{Q}^T > 0, \qquad \mathbf{R} = \mathbf{R}^T > 0, \qquad \mathbf{S}_1 = \mathbf{S}_1^T > 0, \qquad \mathbf{S}_2 = \mathbf{S}_2^T > 0, \tag{5.17}$$

$$\begin{bmatrix} \mathbf{Q} - \mathbf{H}_{ij}^{\circ T} \mathbf{S}_1 - \mathbf{S}_1 \mathbf{H}_{ij}^{\circ} + \sum_{l=1}^{s} \mathbf{K}_l^T \mathbf{R} \mathbf{K}_l & \mathbf{P} + \mathbf{S}_1 - \mathbf{H}_{ij}^{\circ T} \mathbf{S}_2 \\ * & 2\mathbf{S}_2 \end{bmatrix} < 0, \tag{5.18}$$

for all $i, j \in \langle 1, 2, \ldots, s \rangle$, $h_i(\boldsymbol{\theta}(t)) h_j(\boldsymbol{\theta}(t)) \neq 0$, *where*

$$\mathbf{H}_{ij}^{\circ} = \mathbf{A}_i - \mathbf{B}\mathbf{K}_j, \quad \forall i, j \in \langle 1, 2, \ldots, s \rangle. \tag{5.19}$$

The importance of Theorem 5.1 is that it separates $\mathbf{P}$ from system matrices $\mathbf{A}_i$, $\mathbf{B}_i$, that is, there are no terms containing product of $\mathbf{P}$ and any of them. This enables to derive design conditions with respect to natural affine properties of TS models.

5.2. Control Parameter Design

In the next theorems, a scalar $\delta > 0$, $\delta \in \mathbb{R}$ is involved in the set of LMIs. The tuning parameter δ was added in the LMIs in an attempt to obtain less conservative stability conditions than Theorems 4.1 and 4.2, respectively. This procedure of adding scalar in LMIs has been widely explored in literature (see e.g., [19]).

Theorem 5.3. *The equilibrium of the system* (3.1) *controlled by the fuzzy controller* (3.2) *is globally asymptotically stable if for given* $\delta > 0$, $\delta \in \mathbb{R}$ *there exist positive definite symmetric matrices* $\mathbf{X}$, $\mathbf{Z}$, $\mathbf{Q}^{\bullet} \in \mathbb{R}^{n \times n}$, $\mathbf{R}^{\bullet} \in \mathbb{R}^{r \times r}$, *and matrices* $\mathbf{Y}_j \in \mathbb{R}^{r \times n}$, *such that*

$$\mathbf{X} = \mathbf{X}^T > 0, \quad \mathbf{Z} = \mathbf{Z}^T > 0, \quad \mathbf{Q}^{\bullet} = \mathbf{Q}^{\bullet T} > 0, \quad \mathbf{R}^{\bullet} = \mathbf{R}^{\bullet T} > 0, \tag{5.20}$$

$$\begin{bmatrix} \mathbf{V}_{ii} & \mathbf{U}_{ii} & \mathbf{Y}_1^T & \cdots & \mathbf{Y}_s^T \\ * & -2\delta\mathbf{X} & \mathbf{0} & \cdots & \mathbf{0} \\ * & * & -\mathbf{R}^{\bullet} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & -\mathbf{R}^{\bullet} \end{bmatrix} < 0, \qquad \begin{bmatrix} \mathbf{V}_{ij} & \mathbf{U}_{ij} & \mathbf{Y}_1^T & \cdots & \mathbf{Y}_s^T \\ * & -2\delta\mathbf{X} & \mathbf{0} & \cdots & \mathbf{0} \\ * & * & -\mathbf{R}^{\bullet} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & -\mathbf{R}^{\bullet} \end{bmatrix} < 0, \tag{5.21}$$

where

$$\mathbf{V}_{ii} = \mathbf{T}_{ii} + \mathbf{Q}^{\bullet}, \qquad \mathbf{V}_{ij} = \mathbf{T}_{ij} + \mathbf{Q}^{\bullet}, \tag{5.22}$$

$$\mathbf{U}_{ii} = \mathbf{Z} - \delta\mathbf{X} + \mathbf{X}\mathbf{A}_i^T - \mathbf{Y}_i^T\mathbf{B}_i^T, \tag{5.23}$$

$$\mathbf{U}_{ij} = \mathbf{Z} - \delta\mathbf{X} + \frac{1}{2}\mathbf{X}\left(\mathbf{A}_i^T + \mathbf{A}_j^T\right) - \frac{1}{2}\left(\mathbf{Y}_j^T\mathbf{B}_i^T + \mathbf{Y}_i^T\mathbf{B}_j^T\right), \tag{5.24}$$

for all $i \in \langle 1,2,\ldots,s\rangle$, $i < j \le s$, $i,j \in \langle 1,2,\ldots,s\rangle$, $h_i(\boldsymbol{\theta}(t))h_j(\boldsymbol{\theta}(t)) \neq 0$, *respectively. The set of control law gain matrices is given as in* (3.30).

Proof. Since $\mathbf{S}_1, \mathbf{S}_2$ are considered to be symmetric positive definite, introducing the congruence transform matrix:

$$\mathbf{T} = \operatorname{diag}\left[\mathbf{S}_1^{-1} \;\; \mathbf{S}_2^{-1}\right] \tag{5.25}$$

and premultiplying left-hand as well as right-hand sides of (5.16) by (5.25) gives

$$\begin{gathered}
\mathbf{T}\mathbf{P}_{ii}^{\circ}\mathbf{T} = \begin{bmatrix} \mathbf{W}_{ii}^{\circ} & \mathbf{S}_1^{-1}\mathbf{P}\mathbf{S}_2^{-1} + \mathbf{S}_2^{-1} - \mathbf{S}_1^{-1}\mathbf{H}_{ii}^T \\ * & 2\mathbf{S}_2^{-1} \end{bmatrix} < 0, \\
\mathbf{T}\mathbf{P}_{ij}^{\circ}\mathbf{T} = \begin{bmatrix} \mathbf{W}_{ij}^{\circ} & \mathbf{S}_1^{-1}\mathbf{P}\mathbf{S}_2^{-1} + \mathbf{S}_2^{-1} - \mathbf{S}_1^{-1}\dfrac{\mathbf{H}_{ij}^T + \mathbf{H}_{ji}^T}{2} \\ * & 2\mathbf{S}_2^{-1} \end{bmatrix} < 0, \\
\mathbf{W}_{ii}^{\circ} = \mathbf{S}_1^{-1}\mathbf{W}_{ii}\mathbf{S}_1^{-1} = -\mathbf{S}_1^{-1}\mathbf{H}_{ii}^T - \mathbf{H}_{ii}\mathbf{S}_1^{-1} + \mathbf{S}_1^{-1}\mathbf{Q}\mathbf{S}_1^{-1} + \sum_{l=1}^{s}\mathbf{S}_1^{-1}\mathbf{K}_l^T\mathbf{R}\mathbf{K}_l\mathbf{S}_1^{-1}, \\
\mathbf{W}_{ij}^{\circ} = \mathbf{S}_1^{-1}\mathbf{W}_{ij}\mathbf{S}_1^{-1} = \sum_{l=1}^{s}\mathbf{S}_1^{-1}\mathbf{K}_l^T\mathbf{R}\mathbf{K}_l\mathbf{S}_1^{-1} + \mathbf{S}_1^{-1}\mathbf{Q}\mathbf{S}_1^{-1} - \mathbf{S}_1^{-1}\frac{\mathbf{H}_{ij}^T + \mathbf{H}_{ji}^T}{2} - \frac{\mathbf{H}_{ij} + \mathbf{H}_{ji}}{2}\mathbf{S}_1^{-1}.
\end{gathered} \tag{5.26}$$

Thus, with $\delta > 0$, $\delta \in \mathbb{R}$ and with the notations

$$\begin{gathered}
\mathbf{X} = -\mathbf{S}_1^{-1}, \qquad \delta\mathbf{X} = -\mathbf{S}_2^{-1}, \qquad \mathbf{Y}_j = \mathbf{K}_j\mathbf{X}, \\
\mathbf{Z} = \mathbf{S}_1^{-1}\mathbf{P}\mathbf{S}_2^{-1}, \qquad \mathbf{Q}^{\bullet} = \mathbf{S}_1^{-1}\mathbf{Q}\mathbf{S}_1^{-1}, \qquad \mathbf{R}^{\bullet} = \mathbf{R}^{-1},
\end{gathered} \tag{5.27}$$

it yields

$$\mathbf{W}_{ii}^{\circ} = \mathbf{T}_{ii} + \mathbf{Q}^{\bullet} + \sum_{l=1}^{s}\mathbf{Y}_l^T(\mathbf{R}^{\bullet})^{-1}\mathbf{Y}_l, \qquad \mathbf{W}_{ij}^{\circ} = \mathbf{T}_{ij} + \mathbf{Q}^{\bullet} + \sum_{l=1}^{s}\mathbf{Y}_l^T(\mathbf{R}^{\bullet})^{-1}\mathbf{Y}_l, \tag{5.28}$$

and considering (5.22), we have

$$\begin{bmatrix} \mathbf{V}_{ii} + \sum_{l=1}^{s} \mathbf{Y}_l^T (\mathbf{R}^\bullet)^{-1} \mathbf{Y}_l & \mathbf{Z} - \delta\mathbf{X} + \mathbf{X}\mathbf{A}_i^T - \mathbf{Y}_i^T \mathbf{B}_i^T \\ * & -2\delta\mathbf{X} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathbf{V}_{ij} + \sum_{l=1}^{s} \mathbf{Y}_l^T (\mathbf{R}^\bullet)^{-1} \mathbf{Y}_l & \mathbf{Z} - \delta\mathbf{X} + \frac{1}{2}\mathbf{X}\left(\mathbf{A}_i^T + \mathbf{A}_j^T\right) - \frac{1}{2}\left(\mathbf{Y}_j^T \mathbf{B}_i^T + \mathbf{Y}_i^T \mathbf{B}_j^T\right) \\ * & -2\delta\mathbf{X} \end{bmatrix} < 0. \tag{5.29}$$

Using Schur complement property, then (5.29) implies (5.21)–(5.24). □

Theorem 5.4. *The equilibrium of the system* (2.11) *controlled by the fuzzy controller* (3.2) *is globally asymptotically stable if for given* $\delta > 0$, $\delta \in \mathbb{R}$ *there exist positive definite symmetric matrices* $\mathbf{X}, \mathbf{Z}, \mathbf{Q}^\bullet \in \mathbb{R}^{n\times n}$, $\mathbf{R}^\bullet \in \mathbb{R}^{r\times r}$, *and matrices* $\mathbf{Y}_j \in \mathbb{R}^{r\times n}$, *such that*

$$\mathbf{X} = \mathbf{X}^T > 0, \qquad \mathbf{Z} = \mathbf{Z}^T > 0, \qquad \mathbf{Q}^\bullet = \mathbf{Q}^{\bullet T} > 0, \qquad \mathbf{R}^\bullet = \mathbf{R}^{\bullet T} > 0, \tag{5.30}$$

$$\begin{bmatrix} \mathbf{T}_{ij}^\diamond + \mathbf{Q}^\bullet & \mathbf{Z} - \delta\mathbf{X} + \mathbf{X}\mathbf{A}_i^T - \mathbf{Y}_j^T\mathbf{B}^T & \mathbf{Y}_1^T & \cdots & \mathbf{Y}_s^T \\ * & -2\delta\mathbf{X} & \mathbf{0} & \cdots & \mathbf{0} \\ * & * & -\mathbf{R}^\bullet & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & -\mathbf{R}^\bullet \end{bmatrix} < 0, \tag{5.31}$$

for all $i, j \in \langle 1, 2, \ldots, s\rangle$. *Then, the set of control law gain matrices is given as in* (3.30).

Proof. If $\mathbf{B}_i = \mathbf{B}$ for all $i \in \langle 1, 2, \ldots, s\rangle$ then (5.18) implies

$$\begin{bmatrix} \mathbf{W}_{ij}^\diamond & \mathbf{P} + \mathbf{S}_1 - \left(\mathbf{A}_i - \mathbf{B}\mathbf{K}_j\right)^T \mathbf{S}_2 \\ * & 2\mathbf{S}_2 \end{bmatrix} < 0, \tag{5.32}$$

$$\mathbf{W}_{ij}^\diamond = \mathbf{Q} - \left(\mathbf{A}_i - \mathbf{B}\mathbf{K}_j\right)^T \mathbf{S}_1 - \mathbf{S}_1\left(\mathbf{A}_i - \mathbf{B}\mathbf{K}_j\right) + \sum_{l=1}^{s} \mathbf{K}_l^T \mathbf{R}\mathbf{K}_l. \tag{5.33}$$

Premultiplying left-hand side and right-hand side of (5.32) by (5.25) gives

$$\begin{bmatrix} \mathbf{W}_{ij}^\bullet & \mathbf{S}_1^{-1}\mathbf{P}\mathbf{S}_2^{-1} + \mathbf{S}_2^{-1} - \mathbf{S}_1^{-1}\mathbf{A}_i^T + \mathbf{S}_1^{-1}\mathbf{K}_j^T\mathbf{B}^T \\ * & 2\mathbf{S}_2^{-1} \end{bmatrix} < 0,$$

$$\mathbf{W}_{ij}^\bullet = -\mathbf{S}_1^{-1}\left(\mathbf{A}_i - \mathbf{B}\mathbf{K}_j\right)^T - \left(\mathbf{A}_i - \mathbf{B}\mathbf{K}_j\right)\mathbf{S}_1^{-1} + \mathbf{S}_1^{-1}\mathbf{Q}\mathbf{S}_1^{-1} + \sum_{l=1}^{s} \mathbf{S}_1^{-1}\mathbf{K}_l^T\mathbf{R}\mathbf{K}_l\mathbf{S}_1^{-1}, \tag{5.34}$$

and with the notations (5.27), (5.32) then (5.34) implies (5.31). □

Note, the forms (5.2), (5.18) are suitable to optimize a solution with respect to LMI variables in an LMI structure. Conversely, the forms (5.21), (5.31) behave LMI structure only if δ is a prescribed constant design parameter. In the opposite case, the design task has to be formulated as BMI problem.

6. Illustrative Examples

The nonlinear dynamics of the hydrostatic transmission were taken from [22], and this MIMO model was used at first in control design and simulation.

The hydrostatic transmission dynamics is represented by a nonlinear fourth-order state-space model:

$$\begin{aligned} \dot{q}_1(t) &= -a_{11}q_1(t) + b_{11}u_1(t), \\ \dot{q}_2(t) &= -a_{22}q_2(t) + b_{22}u_2(t), \\ \dot{q}_3(t) &= a_{31}q_1(t)p(t) - a_{33}q_3(t) - a_{34}q_2(t)q_4(t), \\ \dot{q}_4(t) &= a_{43}q_2(t)q_3(t) - a_{44}q_4(t), \end{aligned} \tag{6.1}$$

where $q_1(t)$ is the normalized hydraulic pump angle, $q_2(t)$ is the normalized hydraulic motor angle, $q_3(t)$ is the pressure difference [bar], $q_4(t)$ is the hydraulic motor speed [rad/s], $p(t)$ is the speed of hydraulic pump [rad/s], $u_1(t)$ is the normalized control signal of the hydraulic pump, and $u_2(t)$ is the normalized control signal of the hydraulic motor. It is supposed that the external variable $p(t)$ as well as the second state variable $q_2(t)$ are measurable. In given working points the parameters are

$$\begin{aligned} &a_{11} = 7.6923 \quad && a_{22} = 4.5455 \quad && a_{33} = 7.6054.10^{-4}, \\ &a_{31} = 0.7877 \quad && a_{34} = 0.9235 \quad && b_{11} = 1.8590.10^{3}, \\ &a_{43} = 12.1967 \quad && a_{44} = 0.4143 \quad && b_{22} = 1.2879.10^{3}. \end{aligned} \tag{6.2}$$

Since the variables $p(t) \in \langle 105, 300\rangle$ and $q_2(t) \in \langle 0.001, 1\rangle$ are bounded on the prescribed sectors then vector of the premise variables can be chosen as follows:

$$\boldsymbol{\theta}(t) = \left[\theta_1(t) \;\; \theta_2(t)\right] = \left[q_2(t) \;\; p(t)\right]. \tag{6.3}$$

Thus, the set of nonlinear sector functions:

$$\begin{aligned} w_{11}(q_2(t)) &= \frac{b_1 - q_2(t)}{b_1 - b_2}, \quad w_{12}(q_2(t)) = \frac{q_2(t) - b_2}{b_1 - b_2} = 1 - w_{11}(q_2(t)), \quad b_1 = 0.001, \; b_2 = 1, \\ w_{21}(p(t)) &= \frac{c_1 - p(t)}{c_1 - c_2}, \quad w_{22}(p(t)) = \frac{p(t) - c_2}{c_1 - c_2} = 1 - w_{21}(p(t)), \quad c_1 = 105, \; c_2 = 300, \end{aligned} \tag{6.4}$$

implies the next set of normalized membership functions:

$$h_1(q_2(t),p(t)) = w_{11}(q_2(t))w_{21}(p(t)), \qquad h_2(q_2(t),p(t)) = w_{12}(q_2(t))w_{21}(p(t)),$$
$$h_3(q_2(t),p(t)) = w_{11}(q_2(t))w_{22}(p(t)), \qquad h_4(q_2(t),p(t)) = w_{12}(q_2(t))w_{22}(p(t)). \tag{6.5}$$

The transformation of nonlinear differential equations of the system into a TS fuzzy system in standard form gives

$$\mathbf{A}_i = \begin{bmatrix} -a_{11} & 0 & 0 & 0 \\ 0 & -a_{22} & 0 & 0 \\ a_{31}c_k & 0 & -a_{33} & -a_{34}b_l \\ 0 & 0 & a_{43}b_l & -a_{44} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{C}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{6.6}$$

with the associations

$$i = 1 \longleftarrow (l = 1,\ k = 1) \qquad i = 2 \longleftarrow (l = 2,\ k = 1),$$
$$i = 3 \longleftarrow (l = 1,\ k = 2) \qquad i = 4 \longleftarrow (l = 2,\ k = 2). \tag{6.7}$$

Thus, solving (5.30)-(5.31) for given $\delta = 20$ with respect to the LMI matrix variables $\mathbf{X}$, $\mathbf{Z}$, $\mathbf{Q}^{\bullet}$, $\mathbf{R}^{\bullet}$, and $\mathbf{Y}_j$, $j = 1,2,3,4$ using Self-Dual-Minimization (SeDuMi) package for Matlab [23], the feedback gain matrix design problem was feasible with the results:

$$\mathbf{X} = \begin{bmatrix} 0.0042 & 0.0000 & -0.0042 & 0.0002 \\ 0.0000 & 0.0259 & 0.0000 & 0.0000 \\ -0.0042 & 0.0000 & 0.0269 & -0.0104 \\ -0.0002 & 0.0000 & -0.0104 & 0.0537 \end{bmatrix},$$

$$\mathbf{Z} = \begin{bmatrix} 1.6360 & 0.0000 & -0.3994 & -0.0349 \\ 0.0000 & 1.2977 & 0.0000 & 0.0000 \\ -0.3994 & 0.0000 & 0.9726 & -0.1850 \\ -0.0349 & 0.0000 & -0.1850 & 1.0982 \end{bmatrix},$$

$$\mathbf{Q}^{\bullet} = \begin{bmatrix} 1.8612 & 0.0000 & -0.5123 & -0.0126 \\ 0.0000 & 0.6409 & 0.0000 & 0.0000 \\ -0.5123 & 0.0000 & 0.2197 & 0.0121 \\ -0.0126 & 0.0000 & 0.0121 & 0.0314 \end{bmatrix}, \qquad \mathbf{R}^{\bullet} = \begin{bmatrix} 0.9920 & 0.0000 \\ 0.0000 & 0.9920 \end{bmatrix},$$

$$\mathbf{K}_j = \begin{bmatrix} 0.2386 & 0.0000 & 0.0350 & 0.0075 \\ 0.0000 & 0.0207 & 0.0000 & 0.0000 \end{bmatrix}, \quad j = 1,2,3,4, \tag{6.8}$$

which rise up a stable set of closed-loop subsystems.

Comparing with the standard approach, presented method tends to produce the same control gain matrices if $\mathbf{B}_i = \mathbf{B}$ for all i, which radically reduce the control structure, since the result is stabilizing linear control law with quadratic performance for the nonlinear system. Moreover, such control is robust with respect to a premise variable sensor fault.

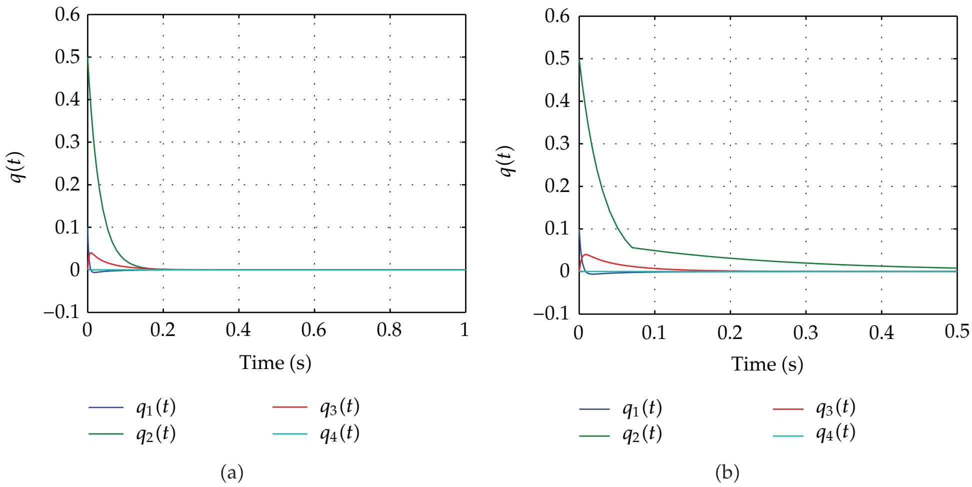

Figure 1: (a) Fault-free system state response, (b) faulty system state response.

Specifying simulation conditions for unforced (autonomous) regime and $t \geq 0$ as follows:

$$\mathbf{p}(t) = c_1, \qquad \mathbf{q}^T(0) = \begin{bmatrix} 0.1 & 0.5 & 0 & 0 \end{bmatrix}, \tag{6.9}$$

then Figure 1(a) shows the solution for constant external system signal $p(t)$ and nonzero initial condition.

If the second variable (premise) sensor fault was modeled as the step function, and the system in unforced regime was controlled by the nominal state control before, and after the sensor fault occurrence time instant $t_f = 0.07\,\mathrm{s}$, then the state responses of the system are shown in Figure 1(b). Evidently, closed-loop system stayed stable.

Using the decoupling control principe [24], also the forced regime was simulated with the control policy:

$$\mathbf{u}(t) = \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\left(-\mathbf{K}_j\mathbf{q}(t) + \mathbf{W}_{wj}\mathbf{w}(t)\right), \tag{6.10}$$

where $\mathbf{w}(t) \in \mathbb{R}^r$ was the desired output vector, and $\mathbf{W}_{wj} \in \mathbb{R}^{r\times r}$, $j = 1,\ldots,s$ was the set of signal gain matrices. Using method given in [25], the signal gain matrices were computed as

$$\mathbf{W}_{wj} \doteq \frac{1}{4}\sum_{i=1}^{4}\left(\mathbf{C}\left(-\left(\mathbf{A}_j - \mathbf{B}\mathbf{K}_i\right)\right)^{-1}\mathbf{B}\right)^{-1},$$

$$\mathbf{W}_{w1} = \begin{bmatrix} 0.0000 & 0.1401 \\ 0.0970 & 0.0000 \end{bmatrix}, \qquad \mathbf{W}_{w2} = \begin{bmatrix} 0.0000 & 0.1401 \\ 0.0970 & 0.0000 \end{bmatrix},$$

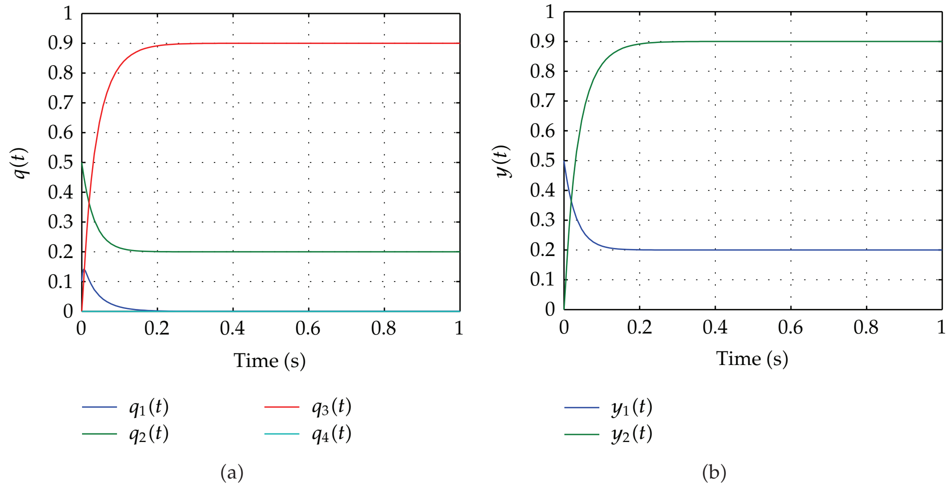

Figure 2: (a) System state response, (b) system output response.

$$\mathbf{W}_{w3} = \begin{bmatrix} 0.0000 & 1.3434 \\ 0.0970 & 0.0000 \end{bmatrix}, \qquad \mathbf{W}_{w4} = \begin{bmatrix} 0.0000 & 1.1360 \\ 0.0970 & 0.0000 \end{bmatrix},$$
$$\mathbf{w}^{T}(t) = \begin{bmatrix} 0.2000 & 0.9000 \end{bmatrix}. \tag{6.11}$$

The simulation results for forced regime are shown in Figures 2(a) and 2(b), reflecting the closed-loop system state, as well as system output responses.

Example 6.1. Consider and give later the problem of balancing an inverted pendulum on a cart adopted from [10, 26], where the objective is to control its state trajectories to the state origin, the state equation of motion of this nonlinear SISO system is,

$$\begin{aligned} \dot{q}_1(t) &= q_2(t), \\ \dot{q}_2(t) &= \frac{g\sin(q_1(t)) - cml\, q_2^2(t)\sin(2q_1(t)) - c\cos(q_1(t))\; u(t)}{(4/3)l - cml\cos^2(q_1(t))}, \end{aligned} \tag{6.12}$$

where $q_1(t)$ denotes the angle of the pendulum from the vertical axis [rad], $q_2(t)$ is the angular velocity [rad/s], $u(t)$ is the force applied to the cart [N], m is the mass of the pendulum [kg], M is the mass of the cart [kg], l is the length from the center of mass of the pendulum to the shaft axis [m], and g is the gravity constant [$9.81\,\mathrm{m/s^2}$]. In the design and simulation, the pendulum parameters were set as

$$l = 0.5, \qquad m = 2, \qquad M = 8, \qquad c = (m+M)^{-1} = 0.1. \tag{6.13}$$

Since the premise variable $q_1(t) \in \langle -\pi/3, \pi/3\rangle$ is bounded on this interval, the next sectors

$$\left\langle -\frac{\pi}{3}, -\frac{\pi}{9}\right\rangle, \quad \left\langle -\frac{2\pi}{9}, 0\right\rangle, \quad \left\langle -\frac{\pi}{9}, \frac{\pi}{9}\right\rangle, \quad \left\langle 0, \frac{2\pi}{9}\right\rangle, \quad \left\langle \frac{\pi}{9}, \frac{\pi}{3}\right\rangle \tag{6.14}$$

were chosen to approximate the system dynamics. The transformation of nonlinear differential equations of the system into a TS fuzzy model gives

$$\begin{gathered}
\mathbf{A}_i = \begin{bmatrix} 0 & 1 \\ a_{21i} & 0 \end{bmatrix}, \qquad \mathbf{b}_i = \begin{bmatrix} 0 \\ b_{2i} \end{bmatrix}, \qquad \mathbf{c}^T = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad i = 1,2,3,4, \\
a_{211} = \frac{g}{(4/3)l - cml}, \qquad a_{212} = \frac{(9/\pi)\sin(\pi/9)g}{(4/3)l - cml\cos^2(\pi/9)}, \\
a_{213} = \frac{(9/2\pi)\sin(2\pi/9)g}{(4/3)l - cml\cos^2(2\pi/9)}, \qquad a_{214} = \frac{(3/\pi)\sin(\pi/3)g}{(4/3)l - cml\cos^2(\pi/3)}, \\
b_{21} = \frac{-c}{(4/3)l - cml}, \qquad b_{22} = \frac{-\cos(\pi/9)c}{(4/3)l - cml\cos^2(\pi/9)}, \\
b_{23} = \frac{-\cos(2\pi/9)c}{(4/3)l - cml\cos^2(2\pi/9)}, \qquad b_{24} = \frac{-\cos(\pi/3)c}{(4/3)l - cml\cos^2(\pi/3)}.
\end{gathered} \tag{6.15}$$

Now, (3.1), (2.12) take the forms:

$$\begin{gathered}
\dot{\mathbf{q}}(t) = \sum_{i=1}^{4} h_i(\boldsymbol{\theta}(t))(\mathbf{A}_i\mathbf{q}(t) + \mathbf{b}_i u(t)), \\
y(t) = \mathbf{c}^T\mathbf{q}(t),
\end{gathered} \tag{6.16}$$

where

$$\begin{gathered}
\mathbf{q}^T(t) = \begin{bmatrix} q_1(t) & q_2(t) \end{bmatrix}^T, \qquad \boldsymbol{\theta}(t) = \begin{bmatrix} \theta_1(t) & \theta_2(t) & \theta_3(t) & \theta_4(t) \end{bmatrix}^T, \\
\theta_i(t) = \begin{cases} \theta_1(t) & \text{if } q_1(t) \text{ is about } 0, \\ \theta_2(t) & \text{if } q_1(t) \text{ is about } \pm\dfrac{\pi}{9}, \\ \theta_3(t) & \text{if } q_1(t) \text{ is about } \pm\dfrac{2\pi}{9}, \\ \theta_4(t) & \text{if } q_1(t) \text{ is about } \pm\dfrac{\pi}{3}, \end{cases} \\
h_1(\theta_1(t)) = 1 - \frac{9}{\pi}|\theta_1(t)|, \qquad h_2(\theta_2(t)) = 1 - \frac{9}{\pi}\left|\theta_2(t) - \frac{\pi}{9}\operatorname{sign}(\theta_2(t))\right|, \\
h_3(\theta_3(t)) = 1 - \frac{9}{2\pi}\left|\theta_3(t) - \frac{2\pi}{9}\operatorname{sign}(\theta_3(t))\right|, \qquad h_4(\theta_4(t)) = \frac{3}{\pi}\left|\theta_4(t) - \frac{2\pi}{9}\operatorname{sign}(\theta_4(t))\right|.
\end{gathered} \tag{6.17}$$

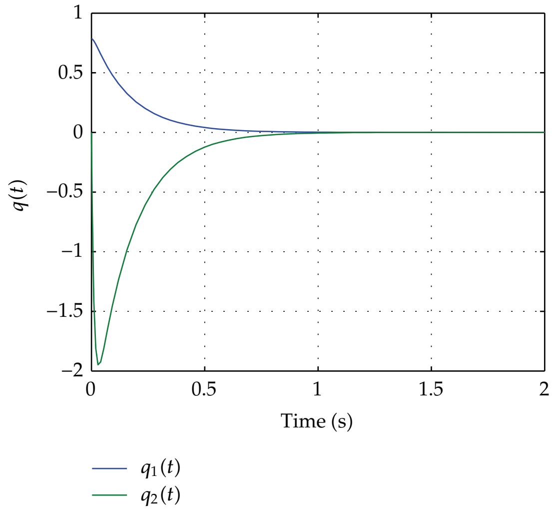

Figure 3: Responses of the pendulum angle $q_1(t)$ and angular velocity $q_2(t)$ of the inverted pendulum on a cart with fuzzy control.

Thus, solving (5.20)–(5.24) for given $\delta = 20$ with respect to the LMI matrix variables $\mathbf{X}$, $\mathbf{Z}$, $\mathbf{Q}^{\bullet}$, $\mathbf{R}^{\bullet}$, and $\mathbf{Y}_j$, $j = 1,2,3,4$ using SeDuMi, the feedback gain matrix design problem was feasible with the results:

$$\mathbf{X} = \begin{bmatrix} 0.0018 & -0.0066 \\ -0.0066 & 0.0244 \end{bmatrix}, \qquad \mathbf{Z} = \begin{bmatrix} 0.0446 & -0.1647 \\ -0.1647 & 0.6136 \end{bmatrix},$$
$$\mathbf{Q}^{\bullet} = \begin{bmatrix} 0.0008 & -0.0028 \\ -0.0028 & 0.0101 \end{bmatrix}, \qquad R^{\bullet} = 1.1301, \tag{6.18}$$
$$\mathbf{K}_j = \begin{bmatrix} -725.0702 & -195.4523 \end{bmatrix}, \quad j = 1,2,3,4,$$

which rise up a stable set of closed-loop subsystems.

Comparing with the standard approach, the method tends in this case also to produce the same control law gain matrices although $\mathbf{B}_i \neq \mathbf{B}$ for all i. Note that the nonlinear controller (3.2) does not apply for $\pi/3 < |x_1(t)| < \pi$.

Define the simulation conditions for unforced regime as follows:

$$\mathbf{q}^T(0) = \begin{bmatrix} \frac{\pi}{4} & 0 \end{bmatrix}. \tag{6.19}$$

Figure 3 shows the solution for control of the system with nonzero initial condition.

The second example was included into the paper to demonstrate more complexity of design. Moreover, functionality properties of proposed method can be verified for example, on the flexible-joint robot arm model [27], Lorenz chaotic system model [28], converter model [29], and so forth.

7. Concluding Remarks

New approach to design of the state control with quadratic performance for a class of continuous-time TS fuzzy systems is presented in this paper. This is achieved by application of TS fuzzy model relating to multimodel approximation structure, the extended Lyapunov function, and its enhanced derivative. Presented version is derived in terms of optimization over LMI constraints using standard LMI numerical optimization procedures to manipulate the global stability of the system. The limitation of this approach is that some state variables must be measurable to construct the fuzzy controller. This is a common limitation for control system design on TS fuzzy approach.

The global quadratic stability of the closed-loop system, solved in the sense of enhanced Lyapunov function derivative, was formulated considering measurable premise variables. Since the stability conditions based on the standard form of the quadratic Lyapunov function are very conservative as a common symmetric positive definite matrix verifying all Lyapunov inequalities is required, the presented principle, naturally exploiting the affine properties of TS fuzzy models and incorporating linear quadratic performance, strictly decouples Lyapunov matrix and the system parameter matrices in the resulting LMIs and significantly reduces the conservativeness in the fuzzy control design. Such LMI-based fuzzy controller is to minimize an upper bound of the performance index.

Used technique denotes that the controller shares the same fuzzy sets with the fuzzy system in the premise parts. As the presented sufficient stability condition did not consider the membership functions of both the TS fuzzy model and the fuzzy controller, the design conditions are valid for any arbitrary membership functions set. Since there is generally no restriction on the design of the membership functions, only the structural complexity of the fuzzy controller may be increased when the membership functions of the TS fuzzy model are more complex. Thus, if it is possible to minimize the design effort and complexity with respect to given system nonlinear sectors, a compromise between complexity and error approximation criterion can be used. In opposite cases, for example, if affine fuzzy system is highly nonlinear, shift operations or diffeomorphic transformations have to be used to transform it to linear types, or staircase membership functions can be employed to approximate the continuous membership functions of the TS fuzzy model and fuzzy controller and to include the membership functions into the stability conditions [30]. However, since this kind of approach yields another structure of matrix inequality conditions, it was not employed in the paper.

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