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Research Article

Extended Duality in Fuzzy Optimization Problems

Tingting Zou

Information Science and Technology College, Dalian Maritime University, Dalian 116026, China

Correspondence should be addressed to Tingting Zou; zoutingt@163.com

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Duality theorem is an attractive approach for solving fuzzy optimization problems. However, the duality gap is generally nonzero for nonconvex problems. So far, most of the studies focus on continuous variables in fuzzy optimization problems. And, in real problems and models, fuzzy optimization problems also involve discrete and mixed variables. To address the above problems, we improve the extended duality theory by adding fuzzy objective functions. In this paper, we first define continuous fuzzy nonlinear programming problems, discrete fuzzy nonlinear programming problems, and mixed fuzzy nonlinear programming problems and then provide the extended dual problems, respectively. Finally we prove the weak and strong extended duality theorems, and the results show no duality gap between the original problem and extended dual problem.

1. Introduction

Nonlinear programming problems (NLPs) play an important role in both manufacturing systems and industrial processes and have been widely used in the fields of operations research, planning and scheduling, optimal control, engineering designs, and production management [1–4]. Due to its significance in both academic and engineering applications, different kinds of approaches have been proposed to solve NLPs and obtained some achievements [5–9]. In [10], we present three algorithms using reverse bridge theorem (RBTH) for solving discrete nonlinear programming problems (DNLPs), continuous nonlinear programming problems (CNLPs), and mixed constrained nonlinear programming problems (MINLPs), respectively, and finally prove the soundness and completeness of these algorithms.

In fact, many practical problems encountered by designers and decision makers would take place in an environment in which the statements might be vague or imprecise. Therefore, in 1970, Bellman and Zadeh first introduced fuzzy optimization problem, which combined the fuzzy decision and fuzzy goals [11]. Since then, there are many articles with regard to the fuzzy optimization problems [12–14]. In 2008, Wu proposed continuous and differentiable fuzzy-valued objective function with real constraints and presented the sufficient optimality conditions for obtaining the nondominated solution of fuzzy optimization problem [15]. Later, he adopted the Karush-Kuhn-Tucker optimality conditions to solve the fuzzy optimization problems [16]. Furthermore, Pathak and Pirzada presented the necessary and sufficient Kuhn-Tucker like optimality conditions for nonlinear fuzzy optimization problems with fuzzy-valued objective function and fuzzy-valued constraints [17]. Jameel and Sadeghi showed that the results solution of fuzzy optimization is a generalization of the solution of the crisp optimization problem [18]. Moreover, Baykasoğlu and Göçken gave the review of fuzzy mathematical programming models according to fuzzy components [19]. So far, most of the studies focus on treating continuous variables in fuzzy optimization problems. However, in real life problems and models, fuzzy optimization problems also involve discrete and mixed variables. Therefore, in this paper, we define continuous fuzzy nonlinear programming problems (CFNPs) with continuous variables, discrete fuzzy nonlinear programming problems (DFNPs) with discrete variables, and mixed fuzzy nonlinear programming problems (MFNPs) with continuous and discrete variables. Compared to previous formula of fuzzy optimization problems, the above three problems increase equality constraints of the variables.

On the other hand, duality theorem has been proved to be an attractive approach for solving fuzzy optimization problems recently [20–27]. The most important aspect of duality is the existence of the duality gap, which is the difference between the optimal solution by solving the original problem and the lower bound of the dual problem. However, for nonconvex problems, the duality gap is generally nonzero and may be large value for some problems. Thus, the duality approach cannot be directly used for solving fuzzy optimization problems with nonconvex functions [28, 29]. Recently, Y. Chen and M. Chen proposed an extended duality theory for nonlinear optimization and proved that there was zero duality gap for general nonconvex optimization problems [30]. To deal with nonconvex fuzzy optimization problems with continuous, discrete, and mixed variable, we improve the extended duality theory by adding fuzzy objective functions. In this paper, we define extended duality theory of fuzzy nonlinear optimization with continuous, discrete, and mixed spaces and prove the weak and strong extended duality theorems, and the results show no duality gap between the original problem and extended dual fuzzy optimization problems.

The remainder of this paper is organized as follows. After an introduction, we recall some basic notions and work related to fuzzy optimization problems in Section 2. Then in Section 3, we define fuzzy nonlinear programming problems in continuous, discrete, and mixed spaces and the extended dual problem, respectively. In Section 4, we prove the weak and strong extended duality theorems. Last section is the conclusion of the paper.

2. Related Previous Works

In this section, we recall some basic definitions and work related to fuzzy optimization problems. Let U be a universal set. A fuzzy subset $\tilde{c}$ of U is a mapping $\mu_{\tilde{c}} : U \rightarrow [0,1]$. The α-level of $\tilde{c}$ denoted by $\tilde{c}_\alpha$ is defined by $\tilde{c}_\alpha = \{x \in U : \mu_{\tilde{c}}(x) \geq \alpha\}$ for all $\alpha \in (0,1]$. The 0-level set $\tilde{c}_0$ is defined as the closure of the set $\{x \in U : \mu_{\tilde{c}}(x) > 0\}$.

Definition 1. We denote by $F(U)$ the set of all fuzzy subset $\tilde{c}$ of U with membership function $\mu_{\tilde{c}}$ satisfying the following conditions:

(1) $\tilde{c}$ is normal; that is, there exists an $x \in U$ such that $\mu_{\tilde{c}}(x) = 1$;

(2) $\mu_{\tilde{c}}$ is quasi concave; that is, $\mu_{\tilde{c}}(\lambda x + (1-\lambda) y) \geq \min\{\mu_{\tilde{c}}(x), \mu_{\tilde{c}}(y)\}$ for all $\lambda \in [0,1]$;

(3) $\mu_{\tilde{c}}$ is upper semicontinuous; that is, $\{x : \mu_{\tilde{c}}(x) \geq \alpha\}$ is a closed subset of U for all $\alpha \in [0,1]$;

(4) the 0-level set $\tilde{c}_0$ is a compact subset of U.

Throughout this paper, the universal set U is the set of all real number $\mathfrak{R}$. The member $\tilde{c}$ in $F(\mathfrak{R})$ is called a fuzzy number. For all $\alpha \in [0,1]$, we can denote the α-level of $\tilde{c}$ by $\tilde{c}_\alpha = [\tilde{c}^L_\alpha, \tilde{c}^U_\alpha]$.

Let $\tilde{x} \in F^n(\mathfrak{R}) \equiv F(\mathfrak{R}) \times \cdots \times F(\mathfrak{R})$; that is, $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$, where $\tilde{x}_i \in F(\mathfrak{R})$ for $i = 1, \ldots, n$. We also write $\tilde{x}^L_\alpha = (\tilde{x}^L_{1\alpha}, \ldots, \tilde{x}^L_{n\alpha})$ and $\tilde{x}^U_\alpha = (\tilde{x}^U_{1\alpha}, \ldots, \tilde{x}^U_{n\alpha})$, where $\tilde{x}^L_{i\alpha} \equiv (\tilde{x}_i)^L_\alpha$ and $\tilde{x}^U_{i\alpha} \equiv (\tilde{x}_i)^U_\alpha$ for all $i = 1, \ldots, n$.

Definition 2. The fuzzy scalar product of the fuzzy vectors $\tilde{x}$ and $\tilde{y}$ in $F^n(\mathfrak{R})$ is defined by

$$\langle\langle \tilde{x}, \tilde{y} \rangle\rangle = (\tilde{x}_1 \otimes \tilde{y}_1) \oplus \cdots \oplus (\tilde{x}_n \otimes \tilde{y}_n). \tag{1}$$

Let $\tilde{c}, \tilde{d} \in F(\mathfrak{R})$. We write $\tilde{d} \succeq \tilde{c}$ if and only if $\tilde{d}^L_\alpha \geq \tilde{c}^L_\alpha$ and $\tilde{d}^U_\alpha \geq \tilde{c}^U_\alpha$ for all $\alpha \in [0,1]$. The relation "$\succeq$" on $F(\mathfrak{R})$ is a partial ordering.

Definition 3. Let A and B be two subsets of $F(\mathfrak{R})$. We write $A \preceq \tilde{x}$ if $\tilde{y} \preceq \tilde{x}$ for all $\tilde{y} \in A$. We write $A \preceq B$ if $A \preceq \tilde{x}$ for all $\tilde{x} \in B$.

Let $\tilde{f}$ and $\tilde{g}$ be two fuzzy-valued functions defined on the same real vector space V, and let X be a subset of V. Then,

$$\begin{aligned} \text{MIN}\left(\tilde{f}, X\right) &= \left\{\tilde{f}\left(x'\right) : \text{there exists no } x \in X \right.\\ &\qquad \left. \text{such that } \tilde{f}\left(x'\right) \succ \tilde{f}(x)\right\}, \\ \text{ARG-MIN }\left(\tilde{f}\left(x'\right), X\right) &= \left\{x \in X : \tilde{f}(x) \succeq \tilde{f}\left(x'\right)\right\}, \\ \text{MAX}\left(\tilde{g}, X\right) &= \left\{\tilde{g}\left(x'\right) : \text{there exists no } x \in X \right.\\ &\qquad \left. \text{such that } \tilde{g}\left(x'\right) \prec \tilde{g}(x)\right\}, \\ \text{ARG-MAX }\left(\tilde{g}\left(x'\right), X\right) &= \left\{x \in X : \tilde{g}(x) \preceq \tilde{g}\left(x'\right)\right\}. \end{aligned} \tag{2}$$

Definition 4. The primal fuzzy optimization problem (P) is defined as follows:

$$\begin{aligned} (P) \quad & \text{minimize} \quad \tilde{f}(x), \\ & \text{subject to} \quad \tilde{g}_i(x) \leq \tilde{0}, \quad \text{for } i = 1, \ldots, m,\ x \in X. \end{aligned} \tag{3}$$

The fuzzy-valued Lagrangian function for the primal problem (P) is defined as follows:

$$\tilde{\phi}(x,u) = \tilde{f}(x) \oplus \langle\langle u, \tilde{g}(x) \rangle\rangle \tag{4}$$

for all $x \in X$ and all $u = (u_1, \ldots, u_m) \in \mathfrak{R}^m_+$; that is, $u_i \geq 0$ for all $i = 1, \ldots, m$. We also write $u \geq 0$ if $u \in \mathfrak{R}^m_+$.

Definition 5. The dual fuzzy optimization problem (D) is defined as follows:

$$\begin{aligned} (D) \quad & \text{maximize} \quad \tilde{L}(u), \\ & \text{subject to} \quad u \geq 0, \end{aligned} \tag{5}$$

where the fuzzy-valued Lagrangian dual function is defined as

$$\begin{aligned} \tilde{L}(u) &= \text{MIN}\left(\tilde{\phi}(\cdot, u), X\right) \\ &= \left\{\tilde{\phi}(\tilde{x}, u) : \text{there exists no } x \in X \right.\\ &\qquad \left. \text{such that } \tilde{\phi}(\tilde{x}, u) \succ \tilde{\phi}(x, u)\right\}. \end{aligned} \tag{6}$$

3. Extend Duality Problems

In this section, we define continuous fuzzy nonlinear problem, discrete fuzzy nonlinear problem and mixed fuzzy nonlinear problem, and the extended dual problems, respectively. Let $\tilde{f}$, $\tilde{h}_i$ $(i = 1, \ldots, m)$ and $\tilde{g}_j$ $(j = 1, \ldots, r)$ be fuzzy-valued functions defined on the same real vector space V, and let X, Y be two subsets of V.

3.1. Continuous Fuzzy Nonlinear Programming Problems

Definition 6. A continuous fuzzy nonlinear programming problem (CFNP) is defined as

$$\begin{aligned} (P_c) \quad & \min_x \tilde{f}(x) \quad x = (x_1, x_2, \ldots, x_n)^T \in X, \\ & \text{Subject to} \quad \tilde{h}(x) = (\tilde{h}_1(x), \ldots, \tilde{h}_m(x)) = \tilde{0}, \\ & \qquad \tilde{g}(x) = (\tilde{g}_1(x), \ldots, \tilde{g}_r(x)) \preceq \tilde{0}, \end{aligned} \tag{7}$$

where x is a continuous variable and $\tilde{f}$ is a continuous and differentiable fuzzy-valued function.

Definition 7. Point x^* is a solution of P_c, if x^* is a feasible solution of P_c and there exists no feasible solution $x \in X$ such that $\tilde{f}(x^*) \succ \tilde{f}(x)$.

Let $X' = \{x \in X : \tilde{h}(x) = \tilde{0}, \tilde{g}(x) \preceq \tilde{0}\}$ be the feasible set of P_c and $\mathrm{OPM}_c(\tilde{f}, \tilde{h}, \tilde{g}, X)$ be the set of all solutions of P_c; then

$$\begin{aligned} \mathrm{MIN}_c(\tilde{f}, \tilde{h}, \tilde{g}, X) &= \mathrm{MIN}_c(\tilde{f}, X') \\ &= \{\tilde{f}(x^*) : \text{there exists no } x \in X' \\ &\qquad \text{such that } \tilde{f}(x^*) \succ \tilde{f}(x)\}, \\ \text{ARG-MIN}_c(\tilde{f}(x^*), \tilde{h}, \tilde{g}, X) & \\ &= \text{ARG-MIN}_c(\tilde{f}(x^*), X') \\ &= \{x \in X' : \tilde{f}(x) \succeq \tilde{f}(x^*)\}. \end{aligned} \tag{8}$$

Definition 8. The fuzzy-valued l_1^m-penalty function for P_c in (7) is defined as follows:

$$\begin{aligned} \tilde{\phi}_c(x, \alpha, \beta) = \tilde{f}(x) &\oplus \langle\langle \alpha, |\tilde{h}(x)| \rangle\rangle \\ &\oplus \langle\langle \beta, \max(\tilde{0}, \tilde{g}(x)) \rangle\rangle, \end{aligned} \tag{9}$$

where $|\tilde{h}(x)| = (|\tilde{h}_1(x)|, \ldots, |\tilde{h}_m(x)|)$ and $\max(\tilde{0}, \tilde{g}(x)) = (\max(\tilde{0}, \tilde{g}_1(x)), \ldots, \max(\tilde{0}, \tilde{g}_r(x)))$, and $\alpha \in \mathfrak{R}^m$ and $\beta \in \mathfrak{R}^r$ are penalty multipliers.

According to the fuzzy-valued l_1^m-penalty function for P_c, we define the fuzzy-valued extended dual function as follows.

Definition 9. The fuzzy-valued extended dual function for P_c is defined for $\alpha \in \mathfrak{R}^m$ and $\beta \in \mathfrak{R}^r$ as

$$\begin{aligned} \tilde{\varphi}_c(\alpha, \beta) &= \mathrm{MIN}_c(\tilde{\phi}_c(\cdot, \alpha, \beta), X) \\ &= \{\tilde{\phi}_c(x^*, \alpha, \beta) : \text{there exists no } x \in X \\ &\qquad \text{such that } \tilde{\phi}_c(x^*, \alpha, \beta) \succ \tilde{\phi}_c(x, \alpha, \beta)\}, \end{aligned} \tag{10}$$

where $\tilde{\varphi}_c$ is a point-to-set fuzzy-valued extended dual function; that is, for any fixed α and β, $\tilde{\varphi}_c(\alpha, \beta)$ is a subset of $F(\mathfrak{R})$.

Definition 10. The extended dual continuous fuzzy nonlinear programming problem (EDCFNP) is defined as follows:

$$\begin{aligned} (\mathrm{ED}_c) \quad & \text{maximize} \quad \tilde{\varphi}_c(\alpha, \beta), \\ & \text{Subject to} \quad \alpha \geq 0, \quad \beta \geq 0. \end{aligned} \tag{11}$$

Definition 11. Point (α^*, β^*) is a solution of ED_c, if there exists a $f(\tilde{x}) \in \tilde{\varphi}_c(\alpha^*, \beta^*)$ such that $f(\tilde{x}) \succeq \tilde{\varphi}_c(\alpha, \beta)$ for all $\alpha \neq \alpha^*$, $\beta \neq \beta^*$, $\alpha \geq 0$, and $\beta \geq 0$.

Let $\mathrm{OPM}_{\mathrm{ED}_c}(\tilde{\varphi}_c, \mathfrak{R}_+^m, \mathfrak{R}_+^r)$ denote the set of all solutions of extended dual continuous fuzzy nonlinear programming problem ED_c, $\mathrm{MAX}_{\mathrm{ED}_c}(\tilde{\varphi}_c, \mathfrak{R}_+^m, \mathfrak{R}_+^r) = \{\tilde{\varphi}_c(\alpha, \beta) : (\alpha, \beta)$ is a solution of $\mathrm{ED}_c\}$. If α and β are fixed, then $\text{ARG-MAX}_{\mathrm{ED}_c}(\tilde{\varphi}_c(x^*, \alpha, \beta), X) = \{x \in X : \tilde{\varphi}_c(x, \alpha, \beta) \preceq \tilde{\varphi}_c(x^*, \alpha, \beta)\}$.

3.2. Discrete Fuzzy Nonlinear Programming Problems

Definition 12. A discrete fuzzy nonlinear programming problem (DFNP) is defined as

$$\begin{aligned} (P_d) \quad & \min_y \tilde{f}(y), \quad y = (y_1, y_2, \ldots, y_n)^T \in Y, \\ & \text{Subject to} \quad \tilde{h}(y) = (\tilde{h}_1(y), \ldots, \tilde{h}_m(y)) = \tilde{0}, \\ & \qquad \tilde{g}(y) = (\tilde{g}_1(y), \ldots, \tilde{g}_r(y)) \preceq \tilde{0}, \end{aligned} \tag{12}$$

where y is a discrete variable.

Definition 13. Point y^* is a solution of P_d, if y^* is a feasible solution of P_d and there exists no feasible solution $y \in Y$ such that $\tilde{f}(y^*) \succ \tilde{f}(y)$.

Let $Y' = \{y \in Y : \tilde{h}(y) = \tilde{0}, \tilde{g}(y) \preceq \tilde{0}\}$ be the feasible set and let $\text{OPM}_d(\tilde{f}, \tilde{h}, \tilde{g}, Y)$ be the set of all solutions of P_d; then,

$$\begin{aligned} \text{MIN}_d\left(\tilde{f}, \tilde{h}, \tilde{g}, Y\right) &= \text{MIN}_d\left(\tilde{f}, Y'\right) \\ &= \left\{\tilde{f}\left(y^*\right) : \text{there exists no } y \in Y' \right. \\ &\qquad \left. \text{such that } \tilde{f}\left(y^*\right) \succ \tilde{f}\left(y\right)\right\}, \\ \text{ARG-MIN}_d\left(\tilde{f}\left(y^*\right), \tilde{h}, \tilde{g}, Y\right) &= \text{ARG-MIN}_d\left(\tilde{f}\left(y^*\right), Y'\right) \\ &= \left\{y \in Y' : \tilde{f}\left(y\right) \succeq \tilde{f}\left(y^*\right)\right\}. \end{aligned} \tag{13}$$

Definition 14. The fuzzy-valued l_1^m-penalty function for P_d in (12) is defined as follows:

$$\begin{aligned} \tilde{\phi}_d\left(y, \alpha, \beta\right) = \tilde{f}\left(y\right) &\oplus \left\langle\left\langle \alpha, \left|\tilde{h}\left(y\right)\right|\right\rangle\right\rangle \\ &\oplus \left\langle\left\langle \beta, \max\left(\tilde{0}, \tilde{g}\left(y\right)\right)\right\rangle\right\rangle, \end{aligned} \tag{14}$$

where $|\tilde{h}(y)| = (|\tilde{h}_1(y)|, \ldots, |\tilde{h}_m(y)|)$ and $\max(\tilde{0}, \tilde{g}(y)) = (\max(\tilde{0}, \tilde{g}_1(y)), \ldots, \max(\tilde{0}, \tilde{g}_r(y)))$, and $\alpha \in \mathfrak{R}^m$ and $\beta \in \mathfrak{R}^r$ are penalty multipliers.

Definition 15. The fuzzy-valued extended dual function for P_d is defined for $\alpha \in \mathfrak{R}^m$ and $\beta \in \mathfrak{R}^r$ as

$$\begin{aligned} &\tilde{\varphi}_d\left(\alpha, \beta\right) \\ &\quad = \text{MIN}_d\left(\tilde{\phi}_d\left(\cdot, \alpha, \beta\right), Y\right) \\ &\quad = \left\{\tilde{\phi}_d\left(y^*, \alpha, \beta\right) : \text{there exists no } y \in Y \right. \\ &\qquad\qquad \left. \text{such that } \tilde{\phi}_d\left(y^*, \alpha, \beta\right) \succ \tilde{\phi}_d\left(y, \alpha, \beta\right)\right\}. \end{aligned} \tag{15}$$

Similarly, $\tilde{\varphi}_d$ is a point-to-set fuzzy-valued extended dual function for P_d; that is, for any fixed α and β, $\tilde{\varphi}_d(\alpha, \beta)$ is a subset of $F(\mathfrak{R})$.

Definition 16. The extended dual discrete fuzzy nonlinear programming problem (EDDFNP) is defined as follows:

$$\begin{aligned} (\text{ED}_d) \quad & \text{maximize} \quad \tilde{\varphi}_d\left(\alpha, \beta\right) \\ & \text{Subject to} \quad \alpha \geq 0, \quad \beta \geq 0. \end{aligned} \tag{16}$$

Definition 17. Point (α^*, β^*) is a solution of ED_d, if there exists a $f(\bar{y}) \in \tilde{\varphi}_d(\alpha^*, \beta^*)$ such that $f(\bar{y}) \succeq \tilde{\varphi}_d(\alpha, \beta)$ for all $\alpha \neq \alpha^*$, $\beta \neq \beta^*$, $\alpha \geq 0$, and $\beta \geq 0$.

Let $\text{OPM}_{\text{ED}_d}(\tilde{\varphi}_d, \mathfrak{R}_+^m, \mathfrak{R}_+^r)$ denote the set of all solutions of extended dual discrete fuzzy nonlinear programming problem ED_d, $\text{MAX}_{\text{ED}_d}(\tilde{\varphi}_d, \mathfrak{R}_+^m, \mathfrak{R}_+^r) = \{\tilde{\varphi}_d(\alpha, \beta)$: (α, β) is a solution of $\text{ED}_d\}$. If α and β are fixed, then

$$\begin{aligned} &\text{ARG-MAX}_{\text{ED}_d}\left(\tilde{\varphi}_d\left(y^*, \alpha, \beta\right), Y\right) \\ &\quad = \left\{y \in Y : \tilde{\varphi}_d\left(y, \alpha, \beta\right) \preceq \tilde{\varphi}_d\left(y^*, \alpha, \beta\right)\right\}. \end{aligned} \tag{17}$$

Definition 18. A mixed fuzzy nonlinear programming problem (MFNP) is defined as

$$\begin{aligned} (P_m) \quad \min_{x,y} \tilde{f}\left(x, y\right), \quad & x = \left(x_1, x_2, \ldots, x_n\right)^T \in X, \\ & x \text{ is a continuous variable} \\ & y = \left(y_1, y_2, \ldots, y_n\right)^T \in Y, \\ & y \text{ is a discrete varible} \\ \text{Subject to} \quad & \tilde{h}\left(x, y\right) = \left(\tilde{h}_1\left(x, y\right), \ldots, \tilde{h}_m\left(x, y\right)\right) \\ & \quad = \tilde{0}, \\ & \tilde{g}\left(x, y\right) = \left(\tilde{g}_1\left(x, y\right), \ldots, \tilde{g}_r\left(x, y\right)\right) \\ & \quad \preceq \tilde{0}. \end{aligned} \tag{18}$$

Definition 19. Point (x^*, y^*) is a solution of P_m, if (x^*, y^*) is a feasible solution of P_m and there exists no feasible solution $(x, y) \in (X, Y)$ such that $\tilde{f}(x^*, y^*) \succ \tilde{f}(x, y)$.

Similarly, $(X, Y)' = \{(x, y) \in (X, Y)$: $\tilde{h}(x, y) = \tilde{0}$, $\tilde{g}(x, y) \preceq \tilde{0}\}$ denotes the feasible set of mixed fuzzy nonlinear programming problem P_m, $\text{OPM}_m(\tilde{f}, \tilde{h}, \tilde{g}, (X, Y))$ denotes the set of all solutions of mixed fuzzy nonlinear programming problem P_m, $\text{MIN}_m(\tilde{f}, \tilde{h}, \tilde{g}, (X, Y)) = \text{MIN}_m(\tilde{f}, (X, Y)') = \{\tilde{f}(x^*, y^*)$: there exists no $(x, y) \in (X, Y)'$ such that $\tilde{f}(x^*, y^*) \succ \tilde{f}(x, y)\}$, and $\text{ARG-MIN}_m(\tilde{f}(x^*, y^*), \tilde{h}, \tilde{g}, (X, Y)) = \text{ARG-MIN}_m(\tilde{f}(x^*, y^*), (X, Y)') = \{(x, y) \in (X, Y)' : \tilde{f}(x, y) \succeq \tilde{f}(x^*, y^*)\}$.

Definition 20. The fuzzy-valued l_1^m-penalty function for P_m in (18) is defined as follows:

$$\begin{aligned} \tilde{\phi}_m\left(x, y, \alpha, \beta\right) = \tilde{f}\left(x, y\right) &\oplus \left\langle\left\langle \alpha, \left|\tilde{h}\left(x, y\right)\right|\right\rangle\right\rangle \\ &\oplus \left\langle\left\langle \beta, \max\left(\tilde{0}, \tilde{g}\left(x, y\right)\right)\right\rangle\right\rangle, \end{aligned} \tag{19}$$

where $|\tilde{h}(x, y)| = (|\tilde{h}_1(x, y)|, \ldots, |\tilde{h}_m(x, y)|)$ and $\max(\tilde{0}, \tilde{g}(x, y)) = (\max(\tilde{0}, \tilde{g}_1(x, y)), \ldots, \max(\tilde{0}, \tilde{g}_r(x, y)))$, and $\alpha \in \mathfrak{R}^m$ and $\beta \in \mathfrak{R}^r$ are penalty multipliers.

Definition 21. The fuzzy-valued extended dual function for P_m is defined for $\alpha \in \mathfrak{R}^m$ and $\beta \in \mathfrak{R}^r$ as

$$\begin{aligned}\tilde{\varphi}_m(\alpha,\beta) &= \mathrm{MIN}_m\left(\tilde{\phi}_m(\cdot,\alpha,\beta),(X,Y)\right) \\ &= \left\{\tilde{\phi}_m(x^*,y^*,\alpha,\beta) : \text{there exists no } (x,y)\in(X,Y) \text{ such that } \tilde{\phi}_m(x^*,y^*,\alpha,\beta) \succ \tilde{\phi}_m(x,y,\alpha,\beta)\right\}.\end{aligned} \tag{20}$$

$\tilde{\varphi}_m$ is a point-to-set fuzzy-valued extended dual function for P_m; that is, for any fixed α and β, $\tilde{\varphi}_m(\alpha,\beta)$ is a subset of $F(\mathfrak{R})$.

Definition 22. The extended dual mixed fuzzy nonlinear programming problem (EDMFNP) is defined as follows:

$$\begin{aligned}(\mathrm{ED}_m) \quad & \text{maximize} \quad \tilde{\varphi}_m(\alpha,\beta) \\ & \text{Subject to} \quad \alpha \ge 0, \quad \beta \ge 0.\end{aligned} \tag{21}$$

Definition 23. Point (α^*,β^*) is a solution of ED_m, if there exists a $f(\tilde{z}) \in \tilde{\varphi}_m(\alpha^*,\beta^*)$ such that $f(\tilde{z}) \succeq \tilde{\varphi}_m(\alpha,\beta)$ for all $\alpha \neq \alpha^*$, $\beta \neq \beta^*$, $\alpha \ge 0$, and $\beta \ge 0$.

Let $\mathrm{OPM}_{\mathrm{ED}_m}(\tilde{\varphi}_m,\mathfrak{R}^m_+,\mathfrak{R}^r_+)$ denote the set of all solutions of extended dual mixed fuzzy nonlinear programming problem ED_m, $\mathrm{MAX}_{\mathrm{ED}_m}(\tilde{\varphi}_m,\mathfrak{R}^m_+,\mathfrak{R}^r_+) = \{\tilde{\varphi}_m(\alpha,\beta) : (\alpha,\beta)$ is a solution of $\mathrm{ED}_m\}$. If α and β are fixed, then

$$\begin{aligned}&\text{ARG-MAX}_{\mathrm{ED}_m}\left(\tilde{\varphi}_m(x^*,y^*,\alpha,\beta),(X,Y)\right) \\ &\quad = \left\{(x,y)\in(X,Y) : \tilde{\varphi}_m(x,y,\alpha,\beta) \preceq \tilde{\varphi}_m(x^*,y^*,\alpha,\beta)\right\}.\end{aligned} \tag{22}$$

4. Extended Duality Theorems

Duality theorem is an important approach for fuzzy optimization problems. However, the duality gap is generally nonzero for nonconvex fuzzy optimization problems. In this section, we prove the weak and strong extended duality theorems and show there is no duality gap between original problem and extended dual problem for fuzzy nonlinear problem with continuous or discrete, or mixed variables.

4.1. Extended Duality Theorem for CFNPs

Theorem 24. *Suppose x and (α,β) are feasible solution of problem P_c and ED_c, respectively; moreover*

$$x \in \bigcap_{\{x'\in X:\tilde{\phi}_c(x',\alpha,\beta)\in\tilde{\varphi}_c(\alpha,\beta)\}} \text{ARG-MIN}_c\left(\tilde{\phi}_c(x',\alpha,\beta),X\right). \tag{23}$$

Then we have $\tilde{\varphi}_c(\alpha,\beta) \preceq \tilde{f}(x)$.

Proof. Let $\tilde{\phi}_c(x',\alpha,\beta) \in \tilde{\varphi}_c(\alpha,\beta)$. For any $x \in \text{ARG-MIN}_c(\tilde{\phi}_c(x',\alpha,\beta),X)$, we have

$$\begin{aligned}\tilde{\phi}_c(x',\alpha,\beta) &\preceq \tilde{\phi}_c(x,\alpha,\beta) \\ &= \tilde{f}(x) \oplus \left\langle\left\langle \alpha, \left|\tilde{h}(x)\right|\right\rangle\right\rangle \\ &\quad \oplus \left\langle\left\langle \beta, \max\left(\tilde{0},\tilde{g}(x)\right)\right\rangle\right\rangle.\end{aligned} \tag{24}$$

Since x is a feasible solution of problem P_c, we obtain $|\tilde{h}(x)| = \tilde{0}$, $\max(\tilde{0},\tilde{g}(x)) = \tilde{0}$. Thus,

$$\tilde{\phi}_c(x',\alpha,\beta) \preceq \tilde{f}(x). \tag{25}$$

This inequality is satisfied for all $\tilde{\phi}_c(x',\alpha,\beta) \in \tilde{\varphi}_c(\alpha,\beta)$. According to Definition 3, therefore we have

$$\tilde{\varphi}_c(\alpha,\beta) \preceq \tilde{f}(x). \tag{26}$$

□

Theorem 25 (weak extended duality theorem for CFNPs). *Suppose that*

$$OPM_c\left(\tilde{f},\tilde{h},\tilde{g},X\right) \subseteq \bigcap_{\{\alpha\in\mathfrak{R}^m,\beta\in\mathfrak{R}^r:(\alpha,\beta)\in OPM_{ED_c}(\tilde{\varphi}_c,\mathfrak{R}^m_+,\mathfrak{R}^r_+)\}} \bigcap_{\{x'\in X:\tilde{\phi}_c(x',\alpha,\beta)\in\tilde{\varphi}_c(\alpha,\beta)\}} \text{ARG-MIN}_c\left(\tilde{\phi}_c(x',\alpha,\beta),X\right). \tag{27}$$

Then $\mathrm{MAX}_{\mathrm{ED}_c}(\tilde{\varphi}_c,\mathfrak{R}^m_+,\mathfrak{R}^r_+) \preceq \mathrm{MIN}_c(\tilde{f},\tilde{h},\tilde{g},X)$.

Proof. If $x \in \mathrm{OPM}_c(\tilde{f},\tilde{h},\tilde{g},X)$, then $\tilde{f}(x) \in \mathrm{MIN}_c(\tilde{f},\tilde{h},\tilde{g},X)$. According to Theorem 24, we have $\tilde{\varphi}_c(\alpha,\beta) \preceq \tilde{f}(x)$ if x satisfies formula (23). Therefore, from Definition 3, if

$$\begin{aligned}&\mathrm{OPM}_c\left(\tilde{f},\tilde{h},\tilde{g},X\right) \\ &\quad \subseteq \bigcap_{\{x'\in X:\ \tilde{\phi}_c(x',\alpha,\beta)\in\tilde{\varphi}_c(\alpha,\beta)\}} \text{ARG-MIN}_c\left(\tilde{\phi}_c(x',\alpha,\beta),X\right),\end{aligned} \tag{28}$$

then

$$\tilde{\varphi}_c(\alpha,\beta) \preceq \mathrm{MIN}_c\left(\tilde{f},\tilde{h},\tilde{g},X\right). \tag{29}$$

Moreover, if $(\alpha,\beta) \in \mathrm{OPM}_{\mathrm{ED}_c}(\tilde{\varphi}_c,\mathfrak{R}^m_+,\mathfrak{R}^r_+)$, then

$$\tilde{\varphi}_c(\alpha,\beta) \in \mathrm{MAX}_{\mathrm{ED}_c}\left(\tilde{\varphi}_c,\mathfrak{R}^m_+,\mathfrak{R}^r_+\right). \tag{30}$$

Therefore, according to Definition 3, we have

$$\mathrm{MAX}_{\mathrm{ED}_c}\left(\tilde{\varphi}_c,\mathfrak{R}^m_+,\mathfrak{R}^r_+\right) \preceq \mathrm{MIN}_c\left(\tilde{f},\tilde{h},\tilde{g},X\right). \tag{31}$$

□

Definition 26. Let P_c be a continuous fuzzy nonlinear programming problem and let ED_c be an extended dual continuous fuzzy nonlinear programming problem. There is no duality gap between P_c and ED_c if there exist $\tilde{f}(x^*) \in \mathrm{MIN}_c(\tilde{f}, \tilde{h}, \tilde{g}, X)$ and $\tilde{\varphi}_c(\alpha^*, \beta^*) \in \mathrm{MAX}_{\mathrm{ED}_c}(\tilde{\varphi}_c, \mathfrak{R}_+^m, \mathfrak{R}_+^r)$, such that $\tilde{f}(x^*) \in \tilde{\varphi}_c(\alpha^*, \beta^*)$.

Theorem 27. *Suppose $x^* \in X$ is a solution of continuous fuzzy nonlinear programming problem P_c; then there exist finite $\alpha^* \geq 0$ and $\beta^* \geq 0$ such that*

$$\tilde{f}(x^*) \in \mathrm{MIN}_c\left(\tilde{\phi}_c(\cdot, \alpha^{**}, \beta^{**}), X\right), \quad \text{for any } \alpha^{**} > \alpha^*, \quad \beta^{**} > \beta^*. \tag{32}$$

Proof. Since x^* is a solution of problem P_c, we have $\tilde{h}(x^*) = \tilde{0}$, $\tilde{g}(x^*) \preceq \tilde{0}$, and there exists no $x \in X$ such that $\tilde{f}(x^*) \succ \tilde{f}(x)$. We set the following α^* and β^*:

$$\alpha_i^* = \max_{x \in X, |\tilde{h}_i(x)| \succ \tilde{0}} \left\{ \frac{\tilde{f}_\alpha^L(x^*) - \tilde{f}_\alpha^L(x)}{|\tilde{h}_{i\alpha}^L(x)|}, \frac{\tilde{f}_\alpha^U(x^*) - \tilde{f}_\alpha^U(x)}{|\tilde{h}_{i\alpha}^U(x)|} \right\}, \quad i = 1, \ldots, m.$$

$$\beta_j^* = \max_{x \in X, \tilde{g}_j(x) \succ \tilde{0}} \left\{ \frac{\tilde{f}_\alpha^L(x^*) - \tilde{f}_\alpha^L(x)}{\tilde{g}_{j\alpha}^L(x)}, \frac{\tilde{f}_\alpha^U(x^*) - \tilde{f}_\alpha^U(x)}{\tilde{g}_{j\alpha}^U(x)} \right\}, \quad j = 1, \ldots, r. \tag{33}$$

Suppose X' be the set of feasible solutions of P_c.

(1) For any $x \in X'$, that is to say that x is a feasible solution of P_c, then $\tilde{h}(x) = \tilde{0}$, $\tilde{g}(x) \preceq \tilde{0}$. Thus we have

$$\begin{aligned} &\tilde{\phi}_c(x, \alpha^{**}, \beta^{**}) \\ &\quad = \tilde{f}(x) \oplus \left\langle \left\langle \alpha^{**}, |\tilde{h}(x)| \right\rangle \right\rangle \oplus \left\langle \left\langle \beta^{**}, \max\left(\tilde{0}, \tilde{g}(x)\right) \right\rangle \right\rangle \\ &\quad = \tilde{f}(x). \end{aligned} \tag{34}$$

Therefore there exists no $x \in X'$ such that $\tilde{f}(x^*) \succ \tilde{f}(x) = \tilde{\phi}_c(x, \alpha^{**}, \beta^{**})$.

(2) For any $x \in X$ but $x \notin X'$, that is to say that x is an infeasible solution of P_c. Assume x violates an equality constraint $\tilde{h}_i(\cdot)$ (the case with an inequality constraint function is similar), so $|\tilde{h}_i(x)| \neq \tilde{0}$. We also have $|\tilde{h}_{i\alpha}^L(x)| \neq 0$ and $|\tilde{h}_{i\alpha}^U(x)| \neq 0$, for all $\alpha \in [0, 1]$

$$\begin{aligned} &\left(\tilde{\phi}_c(x, \alpha^{**}, \beta^{**})\right)_\alpha^L \\ &\quad = \left(\tilde{f}(x) \oplus \left\langle \left\langle \alpha^{**}, |\tilde{h}(x)| \right\rangle \right\rangle \oplus \left\langle \left\langle \beta^{**}, \max\left(\tilde{0}, \tilde{g}(x)\right) \right\rangle \right\rangle \right)_\alpha^L \\ &\quad = \tilde{f}_\alpha^L(x) + \sum_{i=1}^m \alpha_i^{**} \times |\tilde{h}_{i\alpha}^L(x)| + \sum_{j=1}^r \beta_j^{**} \times \max\left(0, \tilde{g}_{i\alpha}^L(x)\right) \\ &\quad \geq \tilde{f}_\alpha^L(x) + \alpha_i^{**} \times |\tilde{h}_{i\alpha}^L(x)| \\ &\quad > \tilde{f}_\alpha^L(x) + \left(\frac{\tilde{f}_\alpha^L(x^*) - \tilde{f}_\alpha^L(x)}{|\tilde{h}_{i\alpha}^L(x)|} \right) \times |\tilde{h}_{i\alpha}^L(x)| = \tilde{f}_\alpha^L(x^*), \\ &\left(\tilde{\phi}_c(x, \alpha^{**}, \beta^{**})\right)_\alpha^U \\ &\quad = \left(\tilde{f}(x) \oplus \left\langle \left\langle \alpha^{**}, |\tilde{h}(x)| \right\rangle \right\rangle \oplus \left\langle \left\langle \beta^{**}, \max\left(\tilde{0}, \tilde{g}(x)\right) \right\rangle \right\rangle \right)_\alpha^U \\ &\quad = \tilde{f}_\alpha^U(x) + \sum_{i=1}^m \alpha_i^{**} \times |\tilde{h}_{i\alpha}^U(x)| + \sum_{j=1}^r \beta_j^{**} \times \max\left(0, \tilde{g}_{i\alpha}^U(x)\right) \\ &\quad \geq \tilde{f}_\alpha^U(x) + \alpha_i^{**} \times |\tilde{h}_{i\alpha}^U(x)| \\ &\quad > \tilde{f}_\alpha^U(x) + \left(\frac{\tilde{f}_\alpha^U(x^*) - \tilde{f}_\alpha^U(x)}{|\tilde{h}_{i\alpha}^U(x)|} \right) \times |\tilde{h}_{i\alpha}^U(x)| = \tilde{f}_\alpha^U(x^*). \end{aligned} \tag{35}$$

Thus,

$$\tilde{\phi}_c(x, \alpha^{**}, \beta^{**}) = \tilde{f}(x) \oplus \left\langle \left\langle \alpha^{**}, |\tilde{h}(x)| \right\rangle \right\rangle \oplus \left\langle \left\langle \beta^{**}, \max\left(\tilde{0}, \tilde{g}(x)\right) \right\rangle \right\rangle \succ \tilde{f}(x^*). \tag{36}$$

Therefore,

$$\tilde{f}(x^*) \in \mathrm{MIN}_c\left(\tilde{\phi}_c(\cdot, \alpha^{**}, \beta^{**}), X\right). \tag{37}$$

□

Theorem 28 (strong extended duality theorem for CFNPs). *Under the assumptions and results in Theorem 27, we further assume*

$$x^* \in \bigcap_{\{\alpha \in \mathfrak{R}^m, \beta \in \mathfrak{R}^r : \alpha \neq \alpha^{**}, \beta \neq \beta^{**}\}} \bigcap_{\{x \in X : \tilde{\phi}_c(x, \alpha, \beta) \in \tilde{\varphi}_c(\alpha, \beta)\}} \text{ARG-MIN}_c\left(\tilde{\phi}_c(x, \alpha, \beta), X\right). \tag{38}$$

Then there is no duality gap between the problem P_c and ED_c.

Proof. According to Theorem 27, there exist finite $\alpha^* \geq 0$ and $\beta^* \geq 0$ such that

$$\tilde{f}(x^*) \in \text{MIN}_c\left(\tilde{\phi}_c(\cdot, \alpha^{**}, \beta^{**}), X\right), \quad \text{for any } \alpha^{**} > \alpha^*, \quad \beta^{**} > \beta^*. \tag{39}$$

Then we have $\tilde{f}(x^*) \in \tilde{\varphi}_c(\alpha^{**}, \beta^{**})$. From Theorem 24, we have $\tilde{f}(x^*) \succeq \tilde{\varphi}_c(\alpha, \beta)$ if

$$x^* \in \bigcap_{\{x \in X : \tilde{\phi}_c(x,\alpha,\beta) \in \tilde{\varphi}_c(\alpha,\beta)\}} \text{ARG-MIN}_c\left(\tilde{\phi}_c(x, \alpha, \beta), X\right). \tag{40}$$

Thus, according to the known condition, we have $\tilde{f}(x^*) \succeq \tilde{\varphi}_c(\alpha, \beta)$ for all $\alpha \neq \alpha^{**}$, $\beta \neq \beta^{**}$, $\alpha \geq 0$, and $\beta \geq 0$.

Therefore, $(\alpha^{**}, \beta^{**})$ is a solution of extended dual continuous fuzzy nonlinear programming problem ED_c; that is, $\tilde{\varphi}_c(\alpha^{**}, \beta^{**}) \in \text{MAX}_{\text{ED}_c}(\tilde{\varphi}_c, \mathfrak{R}^m_+, \mathfrak{R}^r_+)$. This shows that there is no duality gap between the problem P_c and ED_c. □

4.2. Extended Duality Theorem for DFNPs

Theorem 29. *Suppose y and (α, β) are feasible solution of problem P_d and ED_d, respectively; moreover*

$$y \in \bigcap_{\{y' \in Y : \tilde{\phi}_d(y',\alpha,\beta) \in \tilde{\varphi}_d(\alpha,\beta)\}} \text{ARG-MIN}_d\left(\tilde{\phi}_d(y', \alpha, \beta), Y\right). \tag{41}$$

Then we have $\tilde{\varphi}_d(\alpha, \beta) \preceq \tilde{f}(y)$.

Proof. Let $\tilde{\phi}_d(y', \alpha, \beta) \in \tilde{\varphi}_d(\alpha, \beta)$. For any $y \in \text{ARG-MIN}_d(\tilde{\phi}_d(y', \alpha, \beta), Y)$, we have

$$\tilde{\phi}_d(y', \alpha, \beta) \preceq \tilde{\phi}_d(y, \alpha, \beta) = \tilde{f}(y) \oplus \left\langle\left\langle \alpha, \left|\tilde{h}(y)\right| \right\rangle\right\rangle \oplus \left\langle\left\langle \beta, \max\left(\tilde{0}, \tilde{g}(y)\right)\right\rangle\right\rangle. \tag{42}$$

Since y is a feasible solution of problem P_d, we obtain $|\tilde{h}(y)| = \tilde{0}$, $\max(\tilde{0}, \tilde{g}(y)) = \tilde{0}$. Thus,

$$\tilde{\phi}_d(y', \alpha, \beta) \preceq \tilde{f}(y). \tag{43}$$

This inequality is satisfied for all $\tilde{\phi}_d(y', \alpha, \beta) \in \tilde{\varphi}_d(\alpha, \beta)$. According to Definition 3, therefore we have

$$\tilde{\varphi}_d(\alpha, \beta) \preceq \tilde{f}(y). \tag{44}$$

□

Theorem 30 (weak extended duality theorem for DFNPs). *Suppose that*

$$OPM_d\left(\tilde{f}, \tilde{h}, \tilde{g}, Y\right) \subseteq \bigcap_{\{\alpha \in \mathfrak{R}^m, \beta \in \mathfrak{R}^r : (\alpha,\beta) \in OPM_{ED_d}(\tilde{\varphi}_d, \mathfrak{R}^m_+, \mathfrak{R}^r_+)\}} \bigcap_{\{y' \in Y : \tilde{\phi}_d(y',\alpha,\beta) \in \tilde{\varphi}_d(\alpha,\beta)\}} \text{ARG-MIN}_d\left(\tilde{\phi}_d(y', \alpha, \beta), Y\right). \tag{45}$$

Then, $\text{MAX}_{ED_d}(\tilde{\varphi}_d, \mathfrak{R}^m_+, \mathfrak{R}^r_+) \preceq \text{MIN}_d(\tilde{f}, \tilde{h}, \tilde{g}, Y)$.

Proof. The proof is similar to the proof of Theorem 25. □

Definition 31. Let P_d be a discrete fuzzy nonlinear programming problem and let ED_d be an extended dual discrete fuzzy nonlinear programming problem. There is no duality gap between P_d and ED_d if there exist $\tilde{f}(y^*) \in \text{MIN}_d(\tilde{f}, \tilde{h}, \tilde{g}, Y)$ and $\tilde{\varphi}_d(\alpha^*, \beta^*) \in \text{MAX}_{\text{ED}_d}(\tilde{\varphi}_d, \mathfrak{R}^m_+, \mathfrak{R}^r_+)$, such that $\tilde{f}(y^*) \in \tilde{\varphi}_d(\alpha^*, \beta^*)$.

Theorem 32. *Suppose $y^* \in Y$ is a solution of discrete fuzzy nonlinear programming problem P_d; then there exist finite $\alpha^* \geq 0$ and $\beta^* \geq 0$ such that*

$$\tilde{f}(y^*) \in \text{MIN}_d\left(\tilde{\phi}_d(\cdot, \alpha^{**}, \beta^{**}), Y\right), \quad \text{for any } \alpha^{**} > \alpha^*, \quad \beta^{**} > \beta^*. \tag{46}$$

The proof of Theorem 32 is similar to Theorem 27, so we do not repeat it again.

Theorem 33 (strong extended duality theorem for DNLPs). *Under the assumptions and results in Theorem 32, assume*

$$y^* \in \bigcap_{\{\alpha \in \mathfrak{R}^m, \beta \in \mathfrak{R}^r : \alpha \neq \alpha^{**}, \beta \neq \beta^{**}\}} \bigcap_{\{y \in Y : \tilde{\phi}_d(y,\alpha,\beta) \in \tilde{\varphi}_d(\alpha,\beta)\}} \text{ARG-MIN}_d\left(\tilde{\phi}_d(y, \alpha, \beta), Y\right). \tag{47}$$

Then there is no duality gap between the problem P_d and ED_d.

Proof. The proof is similar to the proof of Theorem 28. □

4.3. Extended Duality Theorem for MFNPs

Theorem 34. *Suppose (x, y) and (α, β) are feasible solution of problem P_m and ED_m, respectively; moreover*

$$(x, y) \in \bigcap_{\{(x', y') \in (X,Y):\ \tilde{\phi}_m(x', y', \alpha, \beta) \in \tilde{\varphi}_m(\alpha, \beta)\}} \text{ARG-MIN}_m \left(\tilde{\phi}_m \left(x', y', \alpha, \beta\right), (X, Y)\right). \quad (48)$$

Then we have $\tilde{\varphi}_m(\alpha, \beta) \preceq \tilde{f}(x, y)$.

Proof. Let $\tilde{\phi}_m(x', y', \alpha, \beta) \in \tilde{\varphi}_m(\alpha, \beta)$. For any $(x, y) \in \text{ARG-MIN}_m(\tilde{\phi}_m(x', y', \alpha, \beta), (X, Y))$, we have

$$\begin{aligned}\tilde{\phi}_m \left(x', y', \alpha, \beta\right) &\preceq \tilde{\phi}_m (x, y, \alpha, \beta) \\ &= \tilde{f}(x, y) \oplus \left\langle\left\langle \alpha, \left|\tilde{h}(x, y)\right| \right\rangle\right\rangle \\ &\quad \oplus \left\langle\left\langle \beta, \max\left(\tilde{0}, \tilde{g}(x, y)\right)\right\rangle\right\rangle.\end{aligned} \quad (49)$$

Since (x, y) is a feasible solution of problem P_m, we obtain $|\tilde{h}(x, y)| = \tilde{0}$, $\max(\tilde{0}, \tilde{g}(x, y)) = \tilde{0}$. Thus,

$$\tilde{\phi}_m \left(x', y', \alpha, \beta\right) \preceq \tilde{f}(x, y). \quad (50)$$

This inequality is satisfied for all $\tilde{\phi}_m(x', y', \alpha, \beta) \in \tilde{\varphi}_m(\alpha, \beta)$. According to Definition 3, therefore we obtain

$$\tilde{\varphi}_m (\alpha, \beta) \preceq \tilde{f}(x, y). \quad (51)$$

□

Theorem 35 (weak extended duality theorem for MFNPs). *Suppose that*

$$\begin{aligned}&OPM_m \left(\tilde{f}, \tilde{h}, \tilde{g}, (X, Y)\right) \\ &\subseteq \bigcap_{\{\alpha \in \mathfrak{R}^m, \beta \in \mathfrak{R}^r : (\alpha, \beta) \in OPM_{ED_m}(\tilde{\varphi}_m, \mathfrak{R}^m_+, \mathfrak{R}^r_+)\}} \ \bigcap_{\{(x', y') \in (X,Y): \tilde{\phi}_m(x', y', \alpha, \beta) \in \tilde{\varphi}_m(\alpha, \beta)\}} \text{ARG-MIN}_m \left(\tilde{\phi}_m \left(x', y', \alpha, \beta\right), (X, Y)\right).\end{aligned} \quad (52)$$

Then $\text{MAX}_{\text{ED}_m}(\tilde{\varphi}_m, \mathfrak{R}^m_+, \mathfrak{R}^r_+) \preceq \text{MIN}_m(\tilde{f}, \tilde{h}, \tilde{g}, (X, Y))$.

Proof. If $(x, y) \in \text{OPM}_m(\tilde{f}, \tilde{h}, \tilde{g}, (X, Y))$, then $\tilde{f}(x, y) \in \text{MIN}_m(\tilde{f}, \tilde{h}, \tilde{g}, (X, Y))$. According to Theorem 34, we have $\tilde{\varphi}_m(\alpha, \beta) \preceq \tilde{f}(x, y)$ if x satisfies formula (48). Therefore, from Definition 3, if

$$\text{OPM}_m \left(\tilde{f}, \tilde{h}, \tilde{g}, (X, Y)\right) \subseteq \bigcap_{\{(x', y') \in (X,Y):\ \tilde{\phi}_m(x', y', \alpha, \beta) \in \tilde{\varphi}_m(\alpha, \beta)\}} \text{ARG-MIN}_m \left(\tilde{\phi}_m \left(x', y', \alpha, \beta\right), (X, Y)\right), \quad (53)$$

then

$$\tilde{\varphi}_m (\alpha, \beta) \preceq \text{MIN}_m \left(\tilde{f}, \tilde{h}, \tilde{g}, (X, Y)\right). \quad (54)$$

Moreover, if $(\alpha, \beta) \in \text{OPM}_{\text{ED}_m}(\tilde{\varphi}_m, \mathfrak{R}^m_+, \mathfrak{R}^r_+)$, then

$$\tilde{\varphi}_m (\alpha, \beta) \in \text{MAX}_{\text{ED}_m} \left(\tilde{\varphi}_m, \mathfrak{R}^m_+, \mathfrak{R}^r_+\right). \quad (55)$$

Therefore, according to Definition 3, we have

$$\text{MAX}_{\text{ED}_m} \left(\tilde{\varphi}_m, \mathfrak{R}^m_+, \mathfrak{R}^r_+\right) \preceq \text{MIN}_m \left(\tilde{f}, \tilde{h}, \tilde{g}, (X, Y)\right). \quad (56)$$

□

Definition 36. Let P_m be a mixed fuzzy nonlinear programming problem and let ED_m be an extended dual mixed fuzzy nonlinear programming problem. There is no duality gap between P_m and ED_m if there exist $\tilde{f}(x^*, y^*) \in \text{MIN}_m(\tilde{f}, \tilde{h}, \tilde{g}, (X, Y))$ and $\tilde{\varphi}_m(\alpha^*, \beta^*) \in \text{MAX}_{\text{ED}_m}(\tilde{\varphi}_m, \mathfrak{R}^m_+, \mathfrak{R}^r_+)$, such that $\tilde{f}(x^*, y^*) \in \tilde{\varphi}_m(\alpha^*, \beta^*)$.

Theorem 37. *Suppose $(x^*, y^*) \in (X, Y)$ is a solution of continuous fuzzy nonlinear programming problem P_m; then there exist finite $\alpha^* \geq 0$ and $\beta^* \geq 0$ such that*

$$\begin{aligned}&\tilde{f}(x^*, y^*) \in \text{MIN}_m \left(\tilde{\phi}_m (\cdot, \alpha^{**}, \beta^{**}), (X, Y)\right), \\ &\qquad \textit{for any } \alpha^{**} > \alpha^*, \quad \beta^{**} > \beta^*.\end{aligned} \quad (57)$$

Proof. Since (x^*, y^*) is a solution of problem P_m, we have $\tilde{h}(x^*, y^*) = \tilde{0}$, $\tilde{g}(x^*, y^*) \preceq \tilde{0}$, and there exist no $(x, y) \in (X, Y)$ such that $\tilde{f}(x^*, y^*) \succ \tilde{f}(x, y)$. We set the following α^* and β^*:

$$\alpha_i^* = \max_{(x,y)\in(X,Y),|\tilde{h}_i(x,y)|\succ\tilde{0}} \left\{ \frac{\tilde{f}_\alpha^L(x^*, y^*) - \tilde{f}_\alpha^L(x, y)}{\left|\tilde{h}_{i\alpha}^L(x, y)\right|}, \frac{\tilde{f}_\alpha^U(x^*, y^*) - \tilde{f}_\alpha^U(x, y)}{\left|\tilde{h}_{i\alpha}^U(x, y)\right|} \right\}, \quad i = 1, \ldots, m.$$

$$\beta_j^* = \max_{(x,y)\in(X,Y),\tilde{g}_j(x,y)\succ\tilde{0}} \left\{ \frac{\tilde{f}_\alpha^L(x^*, y^*) - \tilde{f}_\alpha^L(x, y)}{\tilde{g}_{j\alpha}^L(x, y)}, \frac{\tilde{f}_\alpha^U(x^*, y^*) - \tilde{f}_\alpha^U(x, y)}{\tilde{g}_{j\alpha}^U(x, y)} \right\}, \quad j = 1, \ldots, r. \tag{58}$$

Suppose (X', Y') be the set of feasible solutions of P_m.

(1) For any $(x, y) \in (X', Y')$, that is to say that (x, y) is a feasible solution of P_m, then $\tilde{h}(x, y) = \tilde{0}$, $\tilde{g}(x, y) \preceq \tilde{0}$. Thus we have

$$\begin{aligned} \tilde{\phi}_m(x, y, \alpha^{**}, \beta^{**}) &= \tilde{f}(x, y) \oplus \left\langle\left\langle \alpha^{**}, \left|\tilde{h}(x, y)\right| \right\rangle\right\rangle \\ &\quad \oplus \left\langle\left\langle \beta^{**}, \max\left(\tilde{0}, \tilde{g}(x, y)\right) \right\rangle\right\rangle = \tilde{f}(x, y). \end{aligned} \tag{59}$$

Therefore there exists no $(x, y) \in (X', Y')$ such that $\tilde{f}(x^*, y^*) \succ \tilde{f}(x, y) = \tilde{\phi}_m(x, y, \alpha^{**}, \beta^{**})$.

(2) For any $(x, y) \in (X, Y)$ but $(x, y) \notin (X', Y')$, that is to say that (x, y) is an infeasible solution of P_m. Assume (x, y) violates an equality constraint $\tilde{h}_i(\cdot)$ (the case with an inequality constraint function is similar), so $|\tilde{h}_i(x, y)| \neq \tilde{0}$. We also have $|\tilde{h}_{i\alpha}^L(x, y)| \neq 0$ and $|\tilde{h}_{i\alpha}^U(x, y)| \neq 0$, for all $\alpha \in [0, 1]$

$$\begin{aligned} \left(\tilde{\phi}_m(x, y, \alpha^{**}, \beta^{**})\right)_\alpha^L &= \left(\tilde{f}(x, y) \oplus \left\langle\left\langle \alpha^{**}, \left|\tilde{h}(x, y)\right| \right\rangle\right\rangle \oplus \left\langle\left\langle \beta^{**}, \max\left(\tilde{0}, \tilde{g}(x, y)\right) \right\rangle\right\rangle\right)_\alpha^L \\ &= \tilde{f}_\alpha^L(x, y) + \sum_{i=1}^m \alpha_i^{**} \times \left|\tilde{h}_{i\alpha}^L(x, y)\right| + \sum_{j=1}^r \beta_j^{**} \times \max\left(0, \tilde{g}_{i\alpha}^L(x, y)\right) \\ &\geq \tilde{f}_\alpha^L(x, y) + \alpha_i^{**} \times \left|\tilde{h}_{i\alpha}^L(x, y)\right| \\ &> \tilde{f}_\alpha^L(x, y) + \left(\frac{\tilde{f}_\alpha^L(x^*, y^*) - \tilde{f}_\alpha^L(x, y)}{\left|\tilde{h}_{i\alpha}^L(x, y)\right|}\right) \times \left|\tilde{h}_{i\alpha}^L(x, y)\right| = \tilde{f}_\alpha^L(x^*, y^*), \\ \left(\tilde{\phi}_m(x, y, \alpha^{**}, \beta^{**})\right)_\alpha^U &= \left(\tilde{f}(x, y) \oplus \left\langle\left\langle \alpha^{**}, \left|\tilde{h}(x, y)\right| \right\rangle\right\rangle \oplus \left\langle\left\langle \beta^{**}, \max\left(\tilde{0}, \tilde{g}(x, y)\right) \right\rangle\right\rangle\right)_\alpha^U \\ &= \tilde{f}_\alpha^U(x, y) + \sum_{i=1}^m \alpha_i^{**} \times \left|\tilde{h}_{i\alpha}^U(x, y)\right| + \sum_{j=1}^r \beta_j^{**} \times \max\left(0, \tilde{g}_{i\alpha}^U(x, y)\right) \\ &\geq \tilde{f}_\alpha^U(x, y) + \alpha_i^{**} \times \left|\tilde{h}_{i\alpha}^U(x, y)\right| \\ &> \tilde{f}_\alpha^U(x, y) + \left(\frac{\tilde{f}_\alpha^U(x^*, y^*) - \tilde{f}_\alpha^U(x, y)}{\left|\tilde{h}_{i\alpha}^U(x, y)\right|}\right) \times \left|\tilde{h}_{i\alpha}^U(x, y)\right| = \tilde{f}_\alpha^U(x^*, y_*). \end{aligned} \tag{60}$$

Thus,

$$\begin{aligned} \tilde{\phi}_m(x, y, \alpha^{**}, \beta^{**}) &= \tilde{f}(x, y) \oplus \left\langle\left\langle \alpha^{**}, \left|\tilde{h}(x, y)\right| \right\rangle\right\rangle \\ &\quad \oplus \left\langle\left\langle \beta^{**}, \max\left(\tilde{0}, \tilde{g}(x, y)\right) \right\rangle\right\rangle \succ \tilde{f}(x^*, y^*). \end{aligned} \tag{61}$$

Therefore,

$$\tilde{f}(x^*, y^*) \in \underset{m}{\mathrm{MIN}}\left(\tilde{\phi}_m(\cdot, \alpha^{**}, \beta^{**}), (X, Y)\right). \tag{62}$$

□

Theorem 38 (strong extended duality theorem for MFNPs). *Under the assumptions and results in Theorem 37, assume*

$$(x^*, y^*) \in \bigcap_{\{\alpha\in\mathfrak{R}^m,\beta\in\mathfrak{R}^r:\alpha\neq\alpha^{**},\beta\neq\beta^{**}\}} \bigcap_{\{(x,y)\in(X,Y):\tilde{\phi}_m(x,y,\alpha,\beta)\in\tilde{\varphi}_m(\alpha,\beta)\}} \text{ARG-MIN}_m\left(\tilde{\phi}_m(x,y,\alpha,\beta),(X,Y)\right). \quad (63)$$

Then there is no duality gap between the problem P_m and ED_m.

Proof. According to Theorem 37, there exist finite $\alpha^* \geq 0$ and $\beta^* \geq 0$ such that

$$\tilde{f}(x^*, y^*) \in \text{MIN}_m\left(\tilde{\phi}_m(\cdot,\alpha^{**},\beta^{**}),(X,Y)\right), \quad \text{for any } \alpha^{**} > \alpha^*, \quad \beta^{**} > \beta^*. \quad (64)$$

Then we have $\tilde{f}(x^*, y^*) \in \tilde{\varphi}_m(\alpha^{**}, \beta^{**})$. From Theorem 34, we have $\tilde{f}(x^*, y^*) \succeq \tilde{\varphi}_m(\alpha, \beta)$ if

$$(x^*, y^*) \in \bigcap_{\{(x,y)\in(X,Y):\tilde{\phi}_m(x,y,\alpha,\beta)\in\tilde{\varphi}_m(\alpha,\beta)\}} \text{ARG-MIN}_m\left(\tilde{\phi}_m(x,y,\alpha,\beta),(X,Y)\right). \quad (65)$$

Thus, according to the known condition, we have $\tilde{f}(x^*, y^*) \succeq \tilde{\varphi}_m(\alpha, \beta)$ for all $\alpha \neq \alpha^{**}$, $\beta \neq \beta^{**}$, $\alpha \geq 0$, and $\beta \geq 0$.

Therefore, $(\alpha^{**}, \beta^{**})$ is a solution of extended dual mixed fuzzy nonlinear programming problem ED_m; that is, $\tilde{\varphi}_m(\alpha^{**}, \beta^{**}) \in \text{MAX}_{\text{ED}_m}(\tilde{\varphi}_m, \mathfrak{R}^m_+, \mathfrak{R}^r_+)$. This shows that there is no duality gap between the problem P_m and ED_m. □

5. Conclusions

In this paper, we proposed the fuzzy optimization problems in continuous, discrete, and mixed spaces and defined the extended duality problems, respectively. Moreover, we presented the theory of extended duality for fuzzy nonlinear programming problems. This approach overcomes the limitation of conventional duality theory by providing a duality condition that leads to no duality gap for general nonconvex fuzzy optimization problems in continuous, discrete, and mixed spaces. Based on the definition of new penalty function, our theory first transforms the fuzzy nonlinear programming to an equivalent extended dual problem and then solves the dual problem and finally shows no duality gap between the original problem and the extended dual problem.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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