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Research Article

On the Convergence of Multistep Iteration for Uniformly Continuous Φ-Hemicontractive Mappings

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It is shown that the convergence of the multistep iterative process with errors is obtained for uniformly continuous Φ-hemicontractive mappings in real Banach spaces. We also revise the problems of C. E. Chidume and C. O. Chidume (2005).

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. The single-valued-normalized duality mapping is denoted by j.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be strongly pseudocontractive if there is a constant $k \in (0,1)$, and for all $x, y \in D(T)$, $\exists j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \le k\|x-y\|^2. \tag{1.2}$$

The mapping T is called Φ-pseudocontractive if there exists a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|) \tag{1.3}$$

holds for all $x, y \in D(T)$. It is well known that the strongly pseudocontractive mapping must be the Φ-pseudocontractive mapping in the special case in which $\Phi(t) = (1-k)t^2$, but the converse is not true in general. That is, the class of strongly pseudocontractive mappings is a proper subclass of the class of Φ-pseudocontractive mappings. Let $F(T) = \{x \in D(T) : Tx = x\} \neq \emptyset$. If the inequalities (1.2) and (1.3) hold for any $x \in D(T)$ and $y \in F(T)$, then the corresponding mapping T is called strongly hemicontractive and Φ-hemicontractive, respectively.

Let $N(T) = \{x \in E : Tx = 0\} \neq \emptyset$. An operator $T : D(T) \subseteq E \to E$ is called strongly quasiaccretive, Φ-quasiaccretive if and only if $I - T$ is strongly hemicontractive, Φ-hemicontractive, respectively, where I denotes the identity mapping on E. That is, if T is Φ-quasi-accretive, then $N(T) \neq \emptyset$ and there exists a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq \Phi(\|x-y\|) \tag{1.4}$$

holds for all $x \in D(T)$ and $y \in N(T)$. Many authors have studied extensively the strongly convergence problems of the iterative algorithms for the class of operators.

In 2004, Rhoades and Soltuz [1] introduced the multistep iteration as follows.

Let D be a nonempty closed convex subset of real Banach space E and let $T : D \to D$ be a mapping. The multistep iteration $\{x_n\}$ is defined by

$$\begin{gathered}
x_0 \in D, \\
y_n^{p-1} = \left(1 - b_n^{p-1}\right)x_n + b_n^{p-1}Tx_n, \quad n \geq 0,\ p \geq 2, \\
y_n^k = \left(1 - b_n^k\right)x_n + b_n^k T y_n^{k+1}, \quad k = p-2,\ p-3, \ldots, 2, 1, \\
x_{n+1} = (1 - a_n)x_n + a_n T y_n^1, \quad n \geq 0,
\end{gathered} \tag{1.5}$$

where $\{a_n\}, \{b_n^k\}$ $(k = 1, 2, \ldots, p-1)$ in $[0,1]$ satisfy certain conditions. Obviously, the iteration defined above is generalization of Mann, Ishikawa, and Noor iterations.

Inspired and motivated by the work of Xu [2] and the iteration above, we discuss the following multistep iteration with errors:

$$\begin{gathered}
x_0 \in D, \\
y_n^{p-1} = \left(1 - b_n^{p-1} - d_n^{p-1}\right)x_n + b_n^{p-1}Tx_n + d_n^{p-1}w_n^{p-1}, \quad n \geq 0,\ p \geq 2, \\
y_n^k = \left(1 - b_n^k - d_n^k\right)x_n + b_n^k T y_n^{k+1} + d_n^k w_n^k, \quad k = p-2,\ p-3, \ldots, 2, 1, \\
x_{n+1} = (1 - a_n - c_n)x_n + a_n T y_n^1 + c_n u_n, \quad n \geq 0,
\end{gathered} \tag{1.6}$$

where $\{a_n\}, \{c_n\}, \{b_n^k\}, \{d_n^k\}$ $(k = 1, 2, \ldots, p-1)$ in $[0,1]$ with $a_n + c_n \leq 1$, $b_n^k + d_n^k \leq 1$, $\{u_n\}, \{w_n^k\}$ $(k = 1, 2, \ldots, p-1)$ are the bounded sequences of D.

In 2005, C. E. Chidume and C. O. Chidume [3] proved the convergence theorems for fixed points of uniformly continuous generalized Φ-hemicontractive mappings and published in [3]. However, there exists a gap in the proof course of their theorems.

The aim of this paper is to show the convergence of the multistep iteration with errors for fixed points of uniformly continuous Φ-hemicontractive mappings and revise the results of C. E. Chidume and C. O. Chidume [3]. For this, we need the following Lemmas.

Lemma 1.1 (see [4]). *Let E be a real Banach space and let $J : E \to 2^{E^*}$ be a normalized duality mapping. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \tag{1.7}$$

for all $x, y \in E$ and for all $j(x+y) \in J(x+y)$.

Lemma 1.2 (see [5]). *Let $\{\delta_n\}_{n=0}^{\infty}$, $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be three nonnegative real sequences and let $\Phi : [0,+\infty) \to [0,+\infty)$ be a strictly increasing and continuous function with $\Phi(0) = 0$ satisfying the following inequality:*

$$\delta_{n+1}^2 \leq \delta_n^2 - \lambda_n \Phi(\delta_{n+1}) + \gamma_n, \quad n \geq 0, \tag{1.8}$$

where $\lambda_n \in [0,1]$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\gamma_n = o(\lambda_n)$. Then $\delta_n \to 0$ as $n \to \infty$.

2. Main Results

Theorem 2.1. *Let E be an arbitrary real Banach space, D a nonempty closed convex subset of E, and $T : D \to D$ a uniformly continuous Φ-hemicontractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n^k\}, \{c_n\}, \{d_n^k\}$ be real sequences in $[0,1]$ and satisfy the conditions:*

(i) $a_n + c_n \leq 1,\ b_n^k + d_n^k \leq 1,\ k = 1, 2, \ldots, p-1$;

(ii) $a_n, c_n, b_n^k, d_n^k \to 0$ *as* $n \to \infty,\ k = 1, 2, \ldots, p-1$;

(iii) $c_n = o(a_n)$, $\sum_{n=0}^{\infty} a_n = \infty$.

For some $x_0 \in D$, let $\{u_n\}, \{w_n^1\}, \{w_n^2\}, \ldots, \{w_n^{p-1}\}$ be any bounded sequences of D, and let $\{x_n\}$ be the multistep iterative sequence with errors defined by (1.6). Then (1.6) converges strongly to the fixed point q of T.

Proof. Since $T : D \to D$ is Φ-hemicontractive mapping, then there exists a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Tq, j(x-q)\rangle \leq \|x-q\|^2 - \Phi(\|x-q\|), \tag{2.1}$$

for $x \in D$, $q \in F(T)$, that is

$$\langle Tx - x, j(x-q)\rangle \leq -\Phi(\|x-q\|). \tag{@}$$

Choose some $x_0 \in D$ and $x_0 \neq Tx_0$ such that $\|x_0 - Tx_0\| \cdot \|x_0 - q\| \in R(\Phi)$ and denote that $r_0 = \|x_0 - Tx_0\| \cdot \|x_0 - q\|$, $R(\Phi)$ is the range of Φ. Indeed, if $\Phi(r) \to +\infty$ as $r \to +\infty$, then

$r_0 \in R(\Phi)$; if $\sup\{\Phi(r) : r \in [0,+\infty)\} = r_1 < +\infty$ with $r_1 < r_0$ (here, we only give a example. If $r_0 = 2$, $\Phi(t) = t/(1+t)$, then $\sup\{\Phi(r) : r \in [0,+\infty)\} = r_1 = 1 < 2 = r_0$), then for $q \in D$, there exists a sequence $\{w_n\}$ in D such that $w_n \to q$ as $n \to \infty$ with $w_n \neq q$. Furthermore, we obtain that $Tw_n \to Tq$ as $n \to \infty$. So $\{w_n - Tw_n\}$ is the bounded sequence. Hence, there exists natural number n_0 such that $\|w_n - Tw_n\| \cdot \|w_n - q\| < r_1/2$ for $n \geq n_0$, then we redefine $x_0 = w_{n_0}$ and $\|x_0 - Tx_0\| \cdot \|x_0 - q\| \in R(\Phi)$. This is to ensure that $\Phi^{-1}(r_0)$ is defined well.

Step 1. We show that $\{x_n\}$ is a bounded sequence.

Set $R = \Phi^{-1}(r_0)$, then from above formula (@), we obtain that $\|x_0 - q\| \leq R$. Denote

$$B_1 = \{x \in D : \|x - q\| \leq R\}, \qquad B_2 = \{x \in D : \|x - q\| \leq 2R\}. \tag{2.2}$$

Since T is the uniformly continuous, so T is a bounded mapping. We let

$$\begin{aligned} M = &\sup_{x\in B_2}\{\|Tx - q\| + 1\} \\ &+ \max\left\{\sup_n\left\{\left\|w_n^1 - q\right\|\right\}, \sup_n\left\{\left\|w_n^2 - q\right\|\right\}, \dots, \sup_n\left\{\left\|w_n^{p-1} - q\right\|\right\}, \sup_n\{\|u_n - q\|\}\right\}. \end{aligned} \tag{2.3}$$

Next, we want to prove that $x_n \in B_1$. If $n = 0$, then $x_0 \in B_1$. Now, assume that it holds for some n, that is, $x_n \in B_1$. We prove that $x_{n+1} \in B_1$. Suppose that it is not the case, then $\|x_{n+1} - q\| > R > R/2$. Since T is uniformly continuous, then for $\epsilon_0 = \Phi(R/2)/8R$, there exists $\delta > 0$ such that $\|Tx - Ty\| < \epsilon_0$ when $\|x - y\| < \delta$. Denote

$$\tau_0 = \min\left\{1, \frac{R}{M}, \frac{\Phi(R/2)}{8R(M+2R)}, \frac{\delta}{2M+4R}\right\}. \tag{2.4}$$

Since $a_n, b_n^k, c_n, d_n^k \to 0$ as $n \to \infty$ for $k = 1, 2, \dots, p-1$. Without loss of generality, we assume that $0 \leq a_n, b_n^k, c_n, d_n^k \leq \tau_0$ for any $n \geq 0$. Since $c_n = o(a_n)$, let $c_n < a_n\tau_0$. Now, estimate $\|y_n^k - q\|$ for $k = 1, 2, \dots, p-1$. By using (1.6), we have

$$\begin{aligned} \left\|y_n^{p-1} - q\right\| &\leq \left(1 - b_n^{p-1} - d_n^{p-1}\right)\|x_n - q\| + b_n^{p-1}\|Tx_n - q\| + d_n^{p-1}\left\|w_n^{p-1} - q\right\| \\ &\leq R + \tau_0 M \\ &\leq 2R, \end{aligned} \tag{2.5}$$

then $y_n^{p-1} \in B_2$. Similarly, we have

$$\begin{aligned} \left\|y_n^{p-2} - q\right\| &\leq \left(1 - b_n^{p-2} - d_n^{p-2}\right)\|x_n - q\| + b_n^{p-2}\left\|Ty_n^{p-1} - q\right\| + d_n^{p-2}\left\|w_n^{p-2} - q\right\| \\ &\leq R + \tau_0 M \\ &\leq 2R, \end{aligned} \tag{2.6}$$

then $y_n^{p-2} \in B_2, \ldots$. We have

$$\begin{aligned} \left\|y_n^1 - q\right\| &\leq \left(1 - b_n^1 - d_n^1\right)\|x_n - q\| + b_n^1\left\|Ty_n^2 - q\right\| + d_n^1\left\|w_n^1 - q\right\| \\ &\leq R + \tau_0 M \\ &\leq 2R, \end{aligned} \tag{2.7}$$

then $y_n^1 \in B_2$. Therefore, we get

$$\begin{aligned} \|x_{n+1} - q\| &\leq (1 - a_n - c_n)\|x_n - q\| + a_n\left\|Ty_n^1 - q\right\| + c_n\|u_n - q\| \\ &\leq R + \tau_0 M \\ &\leq 2R. \end{aligned} \tag{2.8}$$

And we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq a_n\left\|Ty_n^1 - x_n\right\| + c_n\|u_n - x_n\| \\ &\leq a_n\left(\left\|Ty_n^1 - q\right\| + \|x_n - q\|\right) + c_n(\|u_n - q\| + \|x_n - q\|) \\ &\leq \tau_0\left[\left(\left\|Ty_n^1 - q\right\| + \|u_n - q\|\right) + 2\|x_n - q\|\right] \\ &\leq \tau_0(M + 2R) \\ &\leq \frac{\Phi(R/2)}{8R}, \\ \left\|x_{n+1} - y_n^1\right\| &\leq a_n\left\|Ty_n^1 - x_n\right\| + c_n\|u_n - x_n\| + b_n^1\left\|Ty_n^2 - x_n\right\| + d_n^1\left\|w_n^1 - x_n\right\| \\ &\leq a_n\left(\left\|Ty_n^1 - q\right\| + \|x_n - q\|\right) + c_n(\|u_n - q\| + \|x_n - q\|) \\ &\quad + b_n^1\left(\left\|Ty_n^2 - q\right\| + \|x_n - q\|\right) + d_n^1\left(\left\|w_n^1 - q\right\| + \|x_n - q\|\right) \\ &\leq \tau_0\left[\left(\left\|Ty_n^1 - q\right\| + \|u_n - q\| + 2\|x_n - q\|\right)\right. \\ &\qquad \left. + \left(\left\|Ty_n^2 - q\right\| + \left\|w_n^1 - q\right\| + 2\|x_n - q\|\right)\right] \\ &\leq \tau_0(2M + 4R) \\ &\leq \delta. \end{aligned} \tag{2.9}$$

Further, by using uniform continuity of T, we have

$$\left\|Tx_{n+1} - Ty_n^1\right\| < \frac{\Phi(R/2)}{8R}. \tag{2.10}$$

In view of Lemma 1.1 and the above formulas, we obtain

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&\quad = \left\|(x_n-q)+a_n\left(Ty_n^1-x_n\right)+c_n(u_n-x_n)\right\|^2 \\
&\quad \le \|x_n-q\|^2+2a_n\left\langle Ty_n^1-x_n, j(x_{n+1}-q)\right\rangle+2c_n\langle u_n-x_n, j(x_{n+1}-q)\rangle \\
&\quad \le \|x_n-q\|^2+2a_n\left\langle Tx_{n+1}-x_{n+1}+x_{n+1}-x_n-Tx_{n+1}+Ty_n^1, j(x_{n+1}-q)\right\rangle \\
&\qquad +2c_n\|u_n-x_n\|\cdot\|x_{n+1}-q\| \\
&\quad \le \|x_n-q\|^2-2a_n\Phi(\|x_{n+1}-q\|)+2a_n\|x_{n+1}-x_n\|\cdot\|x_{n+1}-q\| \\
&\qquad +2a_n\left\|Tx_{n+1}-Ty_n^1\right\|\cdot\|x_{n+1}-q\|+2c_n(\|u_n-q\|+\|x_n-q\|)\|x_{n+1}-q\| \\
&\quad \le \|x_n-q\|^2-2a_n\Phi\left(\frac{R}{2}\right)+2a_n\frac{\Phi(R/2)}{8R}\cdot 2R+2a_n\frac{\Phi(R/2)}{8R}\cdot 2R+2a_n\tau_0(R+M)2R \\
&\quad \le \|x_n-q\|^2-a_n\Phi\left(\frac{R}{2}\right)+2a_n\frac{\Phi(R/2)}{8R(M+2R)}(R+M)2R \\
&\quad \le \|x_n-q\|^2-\frac{a_n}{2}\Phi\left(\frac{R}{2}\right)\le R^2,
\end{aligned} \tag{2.11}$$

which is a contradiction. Hence, $x_{n+1}\in B_1$, that is, $\{x_n\}$ is a bounded sequence; it leads to that $\{y_n^1\},\{y_n^2\},\dots,\{y_n^{p-1}\}$ are all bounded sequences as well.

Step 2. We want to prove $\|x_n-q\|\to 0$ as $n\to\infty$.

Since $a_n, b_n^k, c_n, d_n^k\to 0$ as $n\to\infty$ and $\{x_n\},\{y_n^1\}$ are bounded. From (2.9), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0,\qquad \lim_{n\to\infty}\left\|x_{n+1}-y_n^1\right\|=0,\qquad \lim_{n\to\infty}\left\|Tx_{n+1}-Ty_n^1\right\|=0. \tag{2.12}$$

By (2.11), we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \left\|(x_n-q)+a_n\left(Ty_n^1-x_n\right)+c_n(u_n-x_n)\right\|^2 \\
&\le \|x_n-q\|^2+2a_n\left\langle Ty_n^1-x_n, j(x_{n+1}-q)\right\rangle+2c_n\langle u_n-x_n, j(x_{n+1}-q)\rangle \\
&\le \|x_n-q\|^2+2a_n\left\langle Tx_{n+1}-x_{n+1}+x_{n+1}-x_n-Tx_{n+1}+Ty_n^1, j(x_{n+1}-q)\right\rangle \\
&\quad +2c_n\|u_n-x_n\|\cdot\|x_{n+1}-q\|
\end{aligned}$$

$$\begin{aligned}
&\leq \left\|x_n - q\right\|^2 - 2a_n\Phi\left(\left\|x_{n+1} - q\right\|\right) + 2a_n\|x_{n+1} - x_n\| \cdot \left\|x_{n+1} - q\right\| \\
&\quad + 2a_n\left\|Tx_{n+1} - Ty_n^1\right\| \cdot \left\|x_{n+1} - q\right\| + 2c_n\|u_n - x_n\| \cdot \left\|x_{n+1} - q\right\| \\
&= \left\|x_n - q\right\|^2 - 2a_n\Phi\left(\left\|x_{n+1} - q\right\|\right) + o(a_n),
\end{aligned} \tag{2.13}$$

where $2a_n\|x_{n+1}-x_n\|\cdot\|x_{n+1}-q\|+2a_n\|Tx_{n+1}-Ty_n^1\|\cdot\|x_{n+1}-q\|+2c_n\|u_n-x_n\|\cdot\|x_{n+1}-q\| = o(a_n)$. By Lemma 1.2, we obtain that $\lim_{n\to\infty}\|x_n - q\| = 0$. □

Theorem 2.2. *Let E be an arbitrary real Banach space and let $T : E \to E$ be a uniformly continuous Φ-quasi-accretive operator with $q \in N(T) \neq \emptyset$. Let $\{a_n\}, \{b_n^k\}, \{c_n\}, \{d_n^k\}$ be real sequences in $[0,1]$ and satisfy the conditions:*

(i) $a_n + c_n \leq 1,\ b_n^k + d_n^k \leq 1,\ k = 1,2,\dots,p-1$;

(ii) $a_n, c_n, b_n^k, d_n^k \to 0$ *as* $n \to \infty,\ k = 1,2,\dots,p-1$;

(iii) $c_n = o(a_n),\ \sum_{n=0}^{\infty} a_n = \infty$.

For some $x_0 \in E$, let $\{u_n\}, \{w_n^1\}, \{w_n^2\}, \dots, \{w_n^{p-1}\}$ be any bounded sequences of E, and let $\{x_n\}$ be the multistep iterative sequence with errors defined by

$$\begin{gathered}
x_0 \in D, \\
y_n^{p-1} = \left(1 - b_n^{p-1} - d_n^{p-1}\right)x_n + b_n^{p-1}Sx_n + d_n^{p-1}w_n^{p-1}, \quad n \geq 0,\ p \geq 2, \\
y_n^k = \left(1 - b_n^k - d_n^k\right)x_n + b_n^kSy_n^{k+1} + d_n^kw_n^k, \quad k = p-2,\ p-3,\dots,2,1, \\
x_{n+1} = (1 - a_n - c_n)x_n + a_nSy_n^1 + c_nu_n, \quad n \geq 0,
\end{gathered} \tag{2.14}$$

where $S : E \to E$ is defined by $Sx = x - Tx$ for all $x \in E$. Then (2.14) *converges strongly to the fixed point q of S.*

Proof. We find easily that S is a uniformly continuous Φ-hemicontractive. Then the conclusion of Theorem 2.2 is obtained directly by Theorem 2.1. □

Remark 2.3. In Theorems 2.1 and 2.2, if $b_n^k = d_n^k = 0$ ($k = p-1,\ p-2,\dots,2,1$), then, the conclusions are as follows.

Corollary 2.4. *Let E be an arbitrary real Banach space, D a nonempty closed convex subset of E, and $T : D \to D$ a uniformly continuous Φ-hemicontractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}, \{c_n\}$ be real sequences in $[0,1]$ and satisfy the conditions* (i) $a_n + c_n \leq 1$; (ii) $a_n, c_n \to 0$ *as* $n \to \infty$; (iii) $c_n = o(a_n)$ *and* $\sum_{n=0}^{\infty} a_n = \infty$. *For some $x_0 \in D$, let $\{u_n\}$ be any bounded sequence of D, and let $\{x_n\}$ be Mann iterative sequence with errors defined by $x_{n+1} = (1-a_n-c_n)x_n + a_nTx_n + c_nu_n,\ n \geq 0$. Then $\{x_n\}$ converges strongly to the fixed point q of T.*

Corollary 2.5. *Let E be an arbitrary real Banach space and let $T : E \to E$ be a uniformly continuous Φ-quasi-accretive operator with $q \in N(T) \neq \emptyset$. Let $\{a_n\}, \{c_n\}$ be real sequences in $[0,1]$ and satisfy the conditions* (i) $a_n + c_n \leq 1$; (ii) $a_n, c_n \to 0$ *as* $n \to \infty$; (iii) $c_n = o(a_n)$ *and* $\sum_{n=0}^{\infty} a_n = \infty$. *For* *some $x_0 \in E$, let $\{u_n\}$ be any bounded sequence of E, and let $\{x_n\}$ be Mann iterative sequence with errors defined by $x_{n+1} = (1 - a_n - c_n)x_n + a_n S x_n + c_n u_n,\ n \geq 0$. where $S : E \to E$ is defined by $Sx = x - Tx$ for all $x \in E$. Then $\{x_n\}$ converges strongly to the fixed point q of S.*

Remark 2.6. It is mentioned to notice that there exists a serious shortcoming in the proof process of Theorem 2.3 of [3]. That is, $M_1 c_n \leq (\Phi(\epsilon)/4)\alpha_n$ does not hold in line 15 of Claim 2 of page 552. The reason is that the conditions $\sum_{n=0}^{\infty} c_n < +\infty$ and $\sum_{n=0}^{\infty} b_n = +\infty$, $b_n \to 0$ as $n \to \infty$ can not obtain $c_n = o(b_n)$.

Counterexample, let the iteration parameters be $a_n = 1 - b_n - c_n, b_n, c_n$ in the following:

$$\begin{gathered}\{b_n\} : b_0 = b_1 = 0,\ b_n = \frac{1}{n}, \quad n \geq 2, \\ \{c_n\} : 0, \frac{1}{1}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4}, \frac{1}{5^2}, \frac{1}{6^2}, \frac{1}{7^2}, \frac{1}{8^2}, \frac{1}{9}, \frac{1}{10^2}, \frac{1}{11^2}, \frac{1}{12^2}, \frac{1}{13^2}, \frac{1}{14^2}, \frac{1}{15^2}, \frac{1}{16}, \\ \frac{1}{17^2}, \frac{1}{18^2}, \ldots, \frac{1}{23^2}, \frac{1}{24^2}, \frac{1}{25}, \frac{1}{26^2}, \ldots, \frac{1}{35^2}, \frac{1}{36}, \frac{1}{37^2}, \ldots\end{gathered} \tag{2.15}$$

Then, $\sum_{n=0}^{\infty} b_n = +\infty$, $\sum_{n=0}^{\infty} c_n < 2\sum_{n=1}^{\infty}(1/n^2) < +\infty$, but $c_n \neq o(b_n)$.

Application 1. Let $E = R$ be a real number space with the usual norm and $D = [0, +\infty)$. Define $T : D \to D$ by

$$Tx = \frac{x^3}{1 + x^2} \tag{2.16}$$

for all $x \in D$. Then T is uniformly continuous with $F(T) = \{0\}$. Define $\Phi : [0, +\infty) \to [0, +\infty)$ by

$$\Phi(t) = \frac{t^2}{1 + t^2}, \tag{2.17}$$

then Φ is a strictly increasing function with $\Phi(0) = 0$. For all $x \in D,\ q \in F(T)$, we obtain that

$$\begin{aligned}\langle Tx - Tq, j(x - q)\rangle &= \left\langle \frac{x^3}{1 + x^2} - 0, j(x - 0) \right\rangle \\ &= \left\langle \frac{x^3}{1 + x^2}, x \right\rangle \\ &= \frac{x^4}{1 + x^2} \\ &= |x - q|^2 - \frac{|x - q|^2}{1 + |x - q|^2} \\ &= |x - q|^2 - \Phi(|x - q|).\end{aligned} \tag{2.18}$$

Therefore, T is a Φ-hemicontractive mapping. Set

$$a_n = \frac{1}{n+2}, \quad c_n = \frac{1}{(n+2)^2}, \quad b_n^k = d_n^k = \frac{1}{n+2}, \quad k = 1, 2, \ldots, p-1 \tag{2.19}$$

for all $n \geq 0$.

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