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Research Article

The Multiplication Operator from $F(p,q,s)$ Spaces to nth Weighted-Type Spaces on the Unit Disk

Yongmin Liu[1] and Yanyan Yu[2]

[1] *Department of Mathematics, Jiangsu Normal University, Xuzhou 221116, China*
[2] *School of Mathematics and Physics Science, Xuzhou Institute of Technology, Xuzhou 221111, China*

Correspondence should be addressed to Yongmin Liu, minliu@jsnu.edu.cn

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Let $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$ and $u \in H(\mathbb{D})$. The boundedness and compactness of the multiplication operator M_u from $F(p,q,s)$, (or $F_0(p,q,s)$) spaces to nth weighted-type spaces on the unit disk are investigated in this paper.

1. Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions in the open unit disc $\mathbb{D}$ of the finite-complex plane $\mathbb{C}$, $\partial\mathbb{D}$ the boundary of $\mathbb{D}$, $\mathbb{N}_0$ the set of all nonnegative integers and $\mathbb{N}$ the set of all positive integers. Let $\mu(z)$ be a positive continuous function on $\mathbb{D}$ (weight) such that $\mu(z) = \mu(|z|)$ and $n \in \mathbb{N}_0$. The nth weighted-type spaces on the unit disk $\mathbb{D}$, denoted by $\mathcal{W}_\mu^{(n)}(\mathbb{D})$ which were introduced in [1], consist of all $f \in H(\mathbb{D})$ such that

$$b_{\mathcal{W}_\mu^{(n)}(\mathbb{D})}(f) = \sup_{z\in\mathbb{D}} \mu(z)\left|f^{(n)}(z)\right| < \infty. \tag{1.1}$$

For $n = 0$, the space becomes the weighted-type space $H_\mu^\infty(\mathbb{D})$ (see, e.g., [2–4]), for $n = 1$ the Bloch-type space $\mathcal{B}_\mu(\mathbb{D})$ and for $n = 2$ the Zygmund-type space $\mathcal{Z}_\mu(\mathbb{D})$. For $\mu(z) = 1 - |z|^2$, we obtain correspondingly the classical weighted-type space, the Bloch space $\mathcal{B}(\mathbb{D}) = \mathcal{B}$, and the Zygmund space $\mathcal{Z}(\mathbb{D}) = \mathcal{Z}$. Some information on Zygmund-type spaces on the unit disk and some operators on them, for example, in [5–9] and on the unit ball, can be found, for example, in [10, 11]. From now on, we will assume that $n \in \mathbb{N}$. Set

$$\|f\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} = \sum_{j=0}^{n-1} \left| f^{(j)}(0) \right| + b_{\mathcal{W}_\mu^{(n)}(\mathbb{D})}(f). \tag{1.2}$$

With this norm, the nth weighted-type space becomes a Banach space.

The little nth weighted-type space, denoted by $\mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$, is a closed subspace of $\mathcal{W}_\mu^{(n)}(\mathbb{D})$ consisting of those f for which

$$\lim_{|z|\to 1} \mu(z) \left| f^{(n)}(z) \right| = 0. \tag{1.3}$$

A positive continuous function ϕ on $[0,1)$ is called a normal if there exist positive numbers $a, b,\ 0 < a < b$, and $t_0 \in [0,1)$, such that

$$\begin{gathered} \frac{\phi(t)}{(1-t^2)^a} \text{ decreases for } t_0 \le t < 1 \text{ and } \lim_{t\to 1^-} \frac{\phi(t)}{(1-t^2)^a} = 0, \\ \frac{\phi(t)}{(1-t^2)^b} \text{ increases for } t_0 \le t < 1 \text{ and } \lim_{t\to 1^-} \frac{\phi(t)}{(1-t^2)^b} = \infty \end{gathered} \tag{1.4}$$

(see [12]).

For $0 < p,\ s < \infty,\ -2 < q < \infty$, a function $f \in H(\mathbb{D})$ is said to belong to the general function space $F(p,q,s) = F(p,q,s)(\mathbb{D})$ if

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \left(1-|z|^2\right)^q \left(1-|\varphi_a(z)|^2\right)^s dA(z) < \infty, \tag{1.5}$$

where $\varphi_a(z) = (a-z)/(1-\bar{a}z), a \in \mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to $F_0(p,q,s) = F_0(p,q,s)(\mathbb{D})$ if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p \left(1-|z|^2\right)^q \left(1-|\varphi_a(z)|^2\right)^s dA(z) = 0. \tag{1.6}$$

The space $F(p,q,s)$ was introduced by Zhao in [13]. We can get many function spaces if we take some specific parameters of p, q, s; for example (see [13]), $F(p,q,s) = \mathcal{B}^{(2+q)/p}$ and $F_0(p,q,s) = \mathcal{B}_0^{(2+q)/p}$ for $s > 1$; $F(p,q,s) \subset \mathcal{B}^{(2+q)/p}$ and $F_0(p,q,s) \subset \mathcal{B}_0^{(2+q)/p}$ for $0 < s \le 1$; $F(2,0,s) = Q_s$ and $F_0(2,0,s) = Q_{s,0}$; $F(2,0,1) = \mathrm{BMOA}$ and $F_0(2,0,1) = \mathrm{VMOA}$. Since for $q + s \le -1$, $F(p,q,s)$ is the space of constant functions, we assume that $q + s > -1$.

The multiplication operator M_u is defined by $M_u f = uf$. It is interesting to provide a function theoretic characterization of when u induces a bounded or compact composition operator on various spaces (see, e.g., [14–20]). Yu and Liu in [21] studied the boundedness and compactness of the operator DM_u from mixed-norm spaces to the Bloch-type space. Stević in [22] studied the boundedness and compactness of the product of the differentiation and composition operator from the space of bounded analytic functions, the Bloch space and the little Bloch space to nth weighted-type spaces on the unit disk. Zhang and Xiao in [23] studied the bounded and compact-weighted composition operator uC_φ from the $F(p,q,s)$ space to the Bloch-type space in the unit disc. Zhang and Zeng in [24] studied the boundedness and compactness of weighted differentiation composition operators from weighted bergman space to nth weighted-type spaces on the unit disk. Ye in [25] studied the boundedness and compactness of the weighted composition operator uC_φ from $F(p,q,s)$ into the logarithmic Bloch space $\mathcal{B}_{\log}$ on the unit disk. Yang in [26] studied the boundedness and compactness of weighted differentiation composition operators from the $F(p,q,s)$ space to the Bloch-type space. Yang in [27] studied the boundedness and compactness of the composition operator from the $F(p,q,s)$ space to nth weighted-type spaces on the unit disk. Zhou and Chen in [28] studied the weighted composition operator from the $F(p,q,s)$ space to the Bloch-type space in the unit ball. Zhu in [29] studied the weighted composition operator from the $F(p,q,s)$ space to F_μ^∞ space in the unit ball. Stević in [30, 31] studied the boundedness and compactness of the integral operators between $F(p,q,s)$ spaces and Bloch-type spaces in the unit ball. This paper focuses on the boundedness and compactness of the operators M_u from $F(p,q,s)$ (or $F_0(p,q,s)$) to nth weighted-type spaces on the unit disk.

From now on, we will always assume that $0 < p,\ s < \infty$, $-2 < q < \infty$, $q + s > -1$, $\mu(z) = \mu(|z|)$ is normal and $n \in \mathbb{N}$. Further, for the sake of simplicity, C will always denote an independent constant, which can be different from one display to another.

2. Auxiliary Results

In this section we formulate some auxiliary results which will be used in the proofs of the main results.

Lemma 2.1 (see [13, 27]). *Assume that $f \in F(p,q,s)$. Then, for each $n \in \mathbb{N}$, there is a positive constant C, independent of f such that $\|f\|_{\mathcal{B}^{(2+q)/p}} \le C\|f\|_{F(p,q,s)}$ and*

$$\left|f^{(n)}(z)\right| \le \frac{C\|f\|_{F(p,q,s)}}{\left(1-|z|^2\right)^{((2+q-p)/p)+n}}, \quad z \in \mathbb{D}. \tag{2.1}$$

Moreover, if $f \in F_0(p,q,s)$, then $f \in \mathcal{B}_0^{(2+q)/p}$.

Lemma 2.2 (see [32, 33]). *Let $\alpha > 0$ and $f \in \mathcal{B}^\alpha$. Then,*

$$|f(z)| \le \begin{cases} C\|f\|_{\mathcal{B}^\alpha}, & \text{if } 0 < \alpha < 1, \\ C\log\dfrac{2}{1-|z|^2}\|f\|_{\mathcal{B}^\alpha}, & \text{if } \alpha = 1, \\ \dfrac{C}{(\alpha-1)\left(1-|z|^2\right)^{\alpha-1}}\|f\|_{\mathcal{B}^\alpha}, & \text{if } \alpha > 1. \end{cases} \tag{2.2}$$

Lemma 2.3 (see [1]). *Assume $a > 0$ and*

$$D_n(a) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & (a+1) & \cdots & (a+n-1) \\ a(a+1) & (a+1)(a+2) & \cdots & (a+n-1)(a+n) \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{j=0}^{n-2}(a+j) & \prod_{j=0}^{n-2}(a+j+1) & \cdots & \prod_{j=0}^{n-2}(a+j+n-1) \end{vmatrix}. \tag{2.3}$$

Then, $D_n(a) = \prod_{j=0}^{n-1} j!$.

By standard arguments (see, e.g., [34] or Lemma 3 in [35]) the following lemma follows.

Lemma 2.4. *Assume that $u \in H(\mathbb{D})$. Then, $M_u : F(p,q,s)(or\ F_0(p,q,s)) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact if and only if $M_u : F(p,q,s)(or\ F_0(p,q,s)) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded and for any bounded sequence $\{f_k\}$ in $F(p,q,s)(or\ F_0(p,q,s))$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, one has $\|M_u f_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \to 0$ as $k \to \infty$.*

3. The Boundedness and Compactness of M_u from $F(p,q,s)$ to $\mathcal{W}_\mu^{(n)}(\mathbb{D})$

In this section, we characterize the boundedness and compactness of $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$.

Theorem 3.1. *Assume that $u \in H(\mathbb{D})$ and μ is normal.*

(a) *If $2+q > p$, then $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{\mu(|z|)|u(z)|}{\left(1-|z|^2\right)^{((2+q-p)/p)+n}} < \infty. \tag{3.1}$$

(b) *If $2+q < p$, then $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded if and only if* (3.1) *holds and $u \in \mathcal{W}_\mu^{(n)}(\mathbb{D})$.*

(c) *If $2+q = p, s > 1$, then $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded if and only if* (3.1) *holds and*

$$\sup_{z\in\mathbb{D}} \mu(|z|)\left|u^{(n)}(z)\right| \log\frac{2}{1-|z|^2} < \infty. \tag{3.2}$$

Proof. Let $0 < p, s < \infty, -2 < q < \infty$, $q + s > -1$. Assume that conditions (3.1) holds. Then for all $z \in \mathbb{D}$

$$|u(z)| \leq C\frac{\left(1-|z|^2\right)^{((2+q-p)/p)+n}}{\mu(|z|)}. \tag{3.3}$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\left|e^{i\theta}-z\right|^2}d\theta = \frac{1}{1-|z|^2}, \quad \forall z \in \mathbb{D}, \tag{3.4}$$

let $\delta_z = (1+|z|)/2$, then we have $|z/\delta_z| = 2|z|/(1+|z|) < 1$, so

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\left|\delta_z e^{i\theta}-z\right|^2}d\theta = \frac{1}{\delta_z^2-|z|^2}. \tag{3.5}$$

By the Cauchy integral formula and (3.3), we obtain

$$\begin{aligned}
\left|u'(z)\right| &= \frac{1}{2\pi}\left|\int_0^{2\pi}\frac{u\left(\delta_z e^{i\theta}\right)}{\left(\delta_z e^{i\theta}-z\right)^2}\delta_z e^{i\theta}d\theta\right| \\
&\leq C\frac{\left(1-\delta_z^2\right)^{((2+q-p)/p)+n}}{\mu(\delta_z)}\frac{1}{2\pi}\int_0^{2\pi}\frac{\delta_z}{\left|\delta_z e^{i\theta}-z\right|^2}d\theta \\
&= C\frac{\left(1-\delta_z^2\right)^{((2+q-p)/p)+n}}{\mu(\delta_z)}\frac{\delta_z}{\delta_z^2-|z|^2} \\
&\leq C\frac{\left(1-\delta_z^2\right)^{((2+q-p)/p)+n}}{\mu(\delta_z)}\frac{1}{1-\delta_z}.
\end{aligned} \tag{3.6}$$

Note that

$$\frac{1}{2}(1-|z|) \leq 1-\delta_z^2 = (1+\delta_z)(1-\delta_z) \leq (1-|z|), \tag{3.7}$$

and μ are normal, we have

$$\left|u'(z)\right| \leq C\frac{\left(1-|z|^2\right)^{((2+q-p)/p)+n-1}}{\mu(|z|)}, \tag{3.8}$$

hence

$$\sup_{z\in\mathbb{D}} \frac{\mu(|z|)|u'(z)|}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-1}} < \infty. \tag{3.9}$$

Similarly, for $j \in \{2,3,\ldots,n\}$, we have that

$$\sup_{z\in\mathbb{D}} \frac{\mu(|z|)\left|u^{(j)}(z)\right|}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}} < \infty. \tag{3.10}$$

Hence, if $2+q>p$, by Lemmas 2.1 and 2.2, the Leibnitz formula, (3.1), (3.9), and (3.10), we have that

$$\begin{aligned}
&\left|\mu(|z|)(M_u f)^{(n)}(z)\right| \\
&\quad = \mu(|z|)\left|(u(z)f(z))^{(n)}\right| \\
&\quad = \mu(|z|)\left|\sum_{j=0}^{n} C_n^j u^{(j)}(z) f^{(n-j)}(z)\right| \\
&\quad \le \mu(|z|)\left|u^{(n)}(z)f(z)\right| + C\sum_{j=0}^{n-1}\frac{\left|C_n^j\mu(|z|)u^{(j)}(z)\right|}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\|f\|_{F(p,q,s)} \\
&\quad \le C\frac{\mu(|z|)\left|u^{(n)}(z)\right|}{\left(1-|z|^2\right)^{(2+q-p)/p}}\|f\|_{\mathcal{B}^{(2+q)/p}} + C\sum_{j=0}^{n-1}\frac{\left|C_n^j\mu(|z|)u^{(j)}(z)\right|}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\|f\|_{F(p,q,s)} \\
&\quad \le C\|f\|_{F(p,q,s)},
\end{aligned} \tag{3.11}$$

for every $z\in\mathbb{D}$ and $f\in F(p,q,s)$, if $2+q<p$, using $u\in\mathcal{W}_\mu^{(n)}(\mathbb{D})$, we have that

$$\begin{aligned}
&\left|\mu(|z|)(M_u f)^{(n)}(z)\right| \\
&\quad \le C\mu(|z|)\left|u^{(n)}(z)\right|\|f\|_{\mathcal{B}^{(2+q)/p}} + C\sum_{j=0}^{n-1}\frac{\left|C_n^j\mu(|z|)u^{(j)}(z)\right|}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\|f\|_{F(p,q,s)} \\
&\quad \le C\|u\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})}\|f\|_{\mathcal{B}^{(2+q)/p}} + C\sum_{j=0}^{n-1}\frac{\left|C_n^j\mu(|z|)u^{(j)}(z)\right|}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\|f\|_{F(p,q,s)} \\
&\quad \le C\|f\|_{F(p,q,s)},
\end{aligned} \tag{3.12}$$

for every $z \in \mathbb{D}$ and $f \in F(p,q,s)$, if $2+q=p$, by (3.2), we have that

$$\begin{aligned} &\left|\mu(|z|)(M_u f)^{(n)}(z)\right| \\ &\quad \le C\mu(|z|)\left|u^{(n)}(z)\right| \log\frac{2}{1-|z|^2}\|f\|_{\mathcal{B}} + C\sum_{j=0}^{n-1}\frac{\left|C_n^j\mu(|z|)u^{(j)}(z)\right|}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\|f\|_{F(p,q,s)} \\ &\quad \le C\|f\|_{\mathcal{B}} + C\sum_{j=0}^{n-1}\frac{\left|C_n^j\mu(|z|)u^{(j)}(z)\right|}{\left(1-|z|^2\right)^{n-j}}\|f\|_{F(p,q,s)} \\ &\quad \le C\|f\|_{F(p,q,s)}, \end{aligned} \tag{3.13}$$

for every $z \in \mathbb{D}$ and $f \in F(p,q,s)$. We also have that

$$\left|(M_u f)(0)\right| = \left|u(0)f(0)\right| \le C\|f\|_{F(p,q,s)}, \tag{3.14}$$

and for each $s \in \{1,2,\ldots,n-1\}$,

$$\left|(M_u f)^{(s)}(0)\right| = \left|\sum_{j=0}^{s} C_s^j u^{(j)}(0) f^{(s-j)}(0)\right| \le C\sum_{j=0}^{s}\left|C_s^j u^{(j)}(0)\right|\|f\|_{F(p,q,s)}. \tag{3.15}$$

Using (3.11), (3.12), (3.13), (3.14), and (3.15) it follows that the operator $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded.

On the other hand, suppose that $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded, that is, there exists a constant C such that

$$\|M_u f\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \le C\|f\|_{F(p,q,s)} \tag{3.16}$$

for all $f \in F(p,q,s)$. Then, we can easily obtain $u \in \mathcal{W}_\mu^{(n)}(\mathbb{D})$ by taking $f(z)=1$. For a fixed $w \in \mathbb{D}$, and constants $c_1, c_2, \ldots, c_n$, set

$$f_w(z) = \sum_{j=1}^{n} c_j \frac{\left(1-|w|^2\right)^{j+1}}{(1-\overline{w}z)^{\alpha+j}}, \tag{3.17}$$

where $\alpha = (2+q)/p$. A straightforward calculation shows that, for $l \in \{1,\ldots,n\}$,

$$f_w^{(l)}(z) = \sum_{j=1}^{n} c_j \prod_{k=0}^{l-1}(\alpha+j+k)(\overline{w})^l \frac{\left(1-|w|^2\right)^{j+1}}{(1-\overline{w}z)^{\alpha+j+l}}, \tag{3.18}$$

so

$$f_w^{(l)}(w) = \frac{(\overline{w})^l}{\left(1-|w|^2\right)^{\alpha-1+l}} \sum_{j=1}^{n} c_j \prod_{k=0}^{l-1} (\alpha + j + k). \tag{3.19}$$

It is easy to see that $f_w \in F(p,q,s)$ for each $w \in \mathbb{D}$ and $\|f_w\|_{F(p,q,s)} \leq C$ by using the same methods in [23]. By Lemma 2.3, using the same method in [22, 36], we can choose $c_1, c_2, \ldots, c_n$, the corresponding function is denoted by f_w, such that

$$f_w^{(n)}(w) = \sum_{j=1}^{n} c_j \prod_{k=0}^{l-1} (\alpha + j + k) \frac{(\overline{w})^n}{\left(1-|w|^2\right)^{\alpha-1+n}}, \qquad f_w^{(l)}(w) = 0, \quad l \in \{0, 1, \ldots, n-1\}. \tag{3.20}$$

Therefore,

$$\begin{aligned} C &\geq \left\|M_u f_w\right\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \geq \mu(|w|)\left|u(w) f_w^{(n)}(w)\right| \\ &= \sum_{j=1}^{n} c_j \prod_{k=0}^{l-1} (\alpha + j + k) \frac{\mu(|w|)|w|^n |u(w)|}{\left(1-|w|^2\right)^{((2+q-p)/p)+n}}. \end{aligned} \tag{3.21}$$

From this, we obtain

$$\sup_{|w|>1/2} \frac{\mu(|w|)|u(w)|}{\left(1-|w|^2\right)^{((2+q-p)/p)+n}} \leq C. \tag{3.22}$$

Since μ is normal and $u \in H(\mathbb{D})$, we get

$$\sup_{|w|\leq 1/2} \frac{\mu(|w|)|u(w)|}{\left(1-|w|^2\right)^{((2+q-p)/p)+n}} \leq C \sup_{|w|\leq 1/2} \mu(|w|)|u(w)| \leq C, \tag{3.23}$$

which along with (3.22) implies that (3.1) is necessary for all case. Let $2+q = p, s > 1$, and $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ be bounded. To prove (3.2), we set

$$g_w(z) = A \log \frac{2}{1-\overline{w}z} + B \frac{\left(\log(2/(1-\overline{w}z))\right)^2}{\log\left(2/\left(1-|w|^2\right)\right)}, \quad z, w \in \mathbb{D}. \tag{3.24}$$

By a direct calculation, we get

$$\left|g_w'(z)\right| \leq \frac{|A|}{|1-\overline{w}z|} + \frac{2\left|B \log(2/(1-\overline{w}z))\right|}{\left|\log\left(2/\left(1-|w|^2\right)\right)\right| |1-\overline{w}z|} \leq \frac{C}{|1-\overline{w}z|}, \tag{3.25}$$

thus we have $g_w \in F(p,q,s)$ and $\sup_{w\in\mathbb{D}}\|g_w\|_{F(p,q,s)} \le C < \infty$ (see [19, 30]). On the other hand, we have that

$$\begin{aligned}
g_w'(z) &= A\frac{\overline{w}}{1-\overline{w}z} + 2B\frac{\overline{w}\log(2/(1-\overline{w}z))}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)(1-\overline{w}z)},\\
g_w''(z) &= A\frac{(\overline{w})^2}{(1-\overline{w}z)^2} + 2B\frac{(\overline{w})^2}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)(1-\overline{w}z)^2}\\
&\quad + 2B\frac{(\overline{w})^2\log(2/(1-\overline{w}z))}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)(1-\overline{w}z)^2},\\
g_w'''(z) &= 2A\frac{(\overline{w})^3}{(1-\overline{w}z)^3} + 6B\frac{(\overline{w})^3}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)(1-\overline{w}z)^3}\\
&\quad + 2\cdot 2B\frac{(\overline{w})^3\log(2/(1-\overline{w}z))}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)(1-\overline{w}z)^3},\\
g_w^{(4)}(z) &= 3\cdot 2A\frac{(\overline{w})^4}{(1-\overline{w}z)^4} + 22B\frac{(\overline{w})^4}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)(1-\overline{w}z)^4}\\
&\quad + 3!2B\frac{(\overline{w})^4\log(2/(1-\overline{w}z))}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)(1-\overline{w}z)^4},\\
g_w^{(5)}(z) &= 4!A\frac{(\overline{w})^5}{(1-\overline{w}z)^5} + 100B\frac{(\overline{w})^5}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)(1-\overline{w}z)^5}\\
&\quad + 4!2B\frac{(\overline{w})^5\log(2/(1-\overline{w}z))}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)(1-\overline{w}z)^5},\\
&\;\;\vdots\\
g_w^{(m)}(z) &= (m-1)!A\frac{(\overline{w})^m}{(1-\overline{w}z)^m} + A_mB\frac{(\overline{w})^m}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)(1-\overline{w}z)^m}\\
&\quad + 2(m-1)!B\frac{(\overline{w})^m\log(2/(1-\overline{w}z))}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)(1-\overline{w}z)^m}.
\end{aligned}\tag{3.26}$$

Moreover, we have that

$$g_w(w) = (A+B)\log\frac{2}{1-|w|^2},$$

$$
\begin{aligned}
g_w'(w) &= (A+2B)\frac{\overline{w}}{1-|w|^2},\\
g_w''(w) &= (A+2B)\frac{(\overline{w})^2}{\left(1-|w|^2\right)^2} + 2B\frac{(\overline{w})^2}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)\left(1-|w|^2\right)^2},\\
g_w'''(w) &= 2(A+2B)\frac{(\overline{w})^3}{\left(1-|w|^2\right)^3} + 6B\frac{(\overline{w})^3}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)\left(1-|w|^2\right)^3},\\
g_w^{(4)}(w) &= 6(A+2B)\frac{(\overline{w})^4}{\left(1-|w|^2\right)^4} + 22B\frac{(\overline{w})^4}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)\left(1-|w|^2\right)^4},\\
g_w^{(5)}(w) &= 24(A+2B)\frac{(\overline{w})^5}{\left(1-|w|^2\right)^5} + 100B\frac{(\overline{w})^5}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)\left(1-|w|^2\right)^5},\\
&\ \vdots\\
g_w^{(m)}(w) &= (m-1)!(A+2B)\frac{(\overline{w})^m}{\left(1-|w|^2\right)^m} + A_m B\frac{(\overline{w})^m}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)\left(1-|w|^2\right)^m}.
\end{aligned}
\tag{3.27}
$$

Taking $A = 2$, $B = -1$, if $n = 1$, we have

$$
\mu(|w|)\left|u'(w)\log\frac{2}{1-|w|^2}\right| \le \left\|M_u g_w\right\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \le C\|M_u\| < \infty, \tag{3.28}
$$

if $n \ne 1$, we have

$$
\begin{aligned}
\left\|M_u g_w\right\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} &\ge \mu(|w|)\left|\sum_{j=0}^{n} C_n^j u^{(j)}(w) g_w^{(n-j)}(w)\right|\\
&\ge \mu(|w|)\left|u^{(n)}(w)g_w(w)\right| - \mu(|w|)\left|\sum_{j=0}^{n-1} C_n^j u^{(j)}(w) g_w^{(n-j)}(w)\right|\\
&\ge \mu(|w|)\left|u^{(n)}(w)g_w(w)\right| - \mu(|w|)\\
&\quad\times\left|\sum_{j=0}^{n-1} C_n^j u^{(j)}(w) A_{n-j} B\frac{(\overline{w})^{n-j}}{\left(\log\left(2/\left(1-|w|^2\right)\right)\right)\left(1-|w|^2\right)^{n-j}}\right|.
\end{aligned}
\tag{3.29}
$$

From (3.29), (3.1), (3.9), and (3.10) we have

$$\mu(|w|)\left|u^{(n)}(w)\log\frac{2}{1-|w|^2}\right| \le \|M_u g_w\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} + C\left|\sum_{j=0}^{n-1}\frac{\mu(|w|)u^{(j)}(w)}{\left(1-|w|^2\right)^{n-j}}\right| \le C\|M_u\| + C < \infty. \tag{3.30}$$

Using (3.28) and (3.30), it is easy to get that (3.2) holds, finishing the proof of the theorem. □

Theorem 3.2. *Assume that $u \in H(\mathbb{D})$ and μ is normal.*

(a) *If $2+q>p$, then $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{\mu(|z|)|u(z)|}{\left(1-|z|^2\right)^{((2+q-p)/p)+n}} = 0. \tag{3.31}$$

(b) *If $2+q<p$, then $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact if and only if* (3.31) *holds and $u \in \mathcal{W}_\mu^{(n)}(\mathbb{D})$.*

(c) *If $2+q=p, s>1$, then $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact if and only if* (3.31) *holds and*

$$\lim_{|z|\to 1}\mu(|z|)\left|u^{(n)}(z)\right|\log\frac{2}{1-|z|^2} = 0. \tag{3.32}$$

Proof. Assume that conditions (3.31) hold. By Theorem 3.1, $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded. For any bounded sequence $\{f_k\}$ in $F(p,q,s)$ with $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ establish the assertion, it suffices, in view of Lemma 2.4, to show that

$$\|M_u f_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.33}$$

We assume that $\|f_k\|_{F(p,q,s)} \le 1$. From (3.31) and (3.32), given $\epsilon > 0$, there exists a $\delta \in (0,1)$, when $\delta < |z| < 1$, we have

$$\frac{\mu(|z|)|u(z)|}{\left(1-|z|^2\right)^{((2+q-p)/p)+n}} < \epsilon, \tag{3.34}$$

$$\mu(|z|)\left|u^{(n)}(z)\right|\log\frac{2}{1-|z|^2} < \epsilon. \tag{3.35}$$

By (3.34) and the Cauchy integral formula, we obtain

$$\begin{aligned}
|u'(z)| &= \frac{1}{2\pi}\left|\int_0^{2\pi}\frac{u(\delta_z e^{i\theta})}{(\delta_z e^{i\theta}-z)^2}\delta_z e^{i\theta}d\theta\right| \\
&< \epsilon\frac{(1-\delta_z^2)^{((2+q-p)/p)+n}}{\mu(\delta_z)}\frac{1}{2\pi}\int_0^{2\pi}\frac{\delta_z}{|\delta_z e^{i\theta}-z|^2}d\theta \\
&= \epsilon\frac{(1-\delta_z^2)^{((2+q-p)/p)+n}}{\mu(\delta_z)}\frac{\delta_z}{\delta_z^2-|z|^2} \\
&\le \epsilon C\frac{\left(1-|z|^2\right)^{((2+q-p)/p)+n-1}}{\mu(|z|)},
\end{aligned} \tag{3.36}$$

when $\delta < |z| < 1$. Hence,

$$\frac{\mu(|z|)|u'(z)|}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-1}} < C\epsilon, \tag{3.37}$$

when $\tau < |z| < 1$. Similarly, for $j \in \{2,\ldots,n\}$, we have that

$$\frac{\mu(|z|)\left|u^{(j)}(z)\right|}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}} < C\epsilon, \tag{3.38}$$

when $\delta < |z| < 1$. Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate gives that $f_k^{(j)}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ for each $j \in \{1,\ldots,n\}$, there exists a $K_0 \in \mathbb{N}$ such that $k > K_0$ implies that

$$\sum_{j=0}^{n-1}\sum_{m=0}^{j-1}C_j^m\left|u^{(m)}(0)f_k^{(j-m)}(0)\right| + \sup_{|z|\le\delta}\mu(|z|)\left|\sum_{j=0}^{n}C_n^j u^{(j)}(z)f_k^{(n-j)}(z)\right| < C\epsilon. \tag{3.39}$$

If $2+q>p$, from (3.34), (3.38), (3.39), and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&\left\|M_u f_k\right\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \\
&= \sum_{j=0}^{n-1}\left|(uf_k)^{(j)}(0)\right| + \sup_{z\in\mathbb{D}}\mu(|z|)\left|(uf_k)^{(n)}(z)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=0}^{n-1}\sum_{m=0}^{j-1} C_j^m \left|u^{(m)}(0) f_k^{(j-m)}(0)\right| + \sup_{|z|\leq\delta}\mu(|z|)\left|\sum_{j=0}^{n} C_n^j u^{(j)}(z) f_k^{(n-j)}(z)\right| \\
&\quad + \sup_{|z|>\delta}\mu(|z|)\left|\sum_{j=0}^{n} C_n^j u^{(j)}(z) f_k^{(n-j)}(z)\right| \\
&< C\epsilon + C\sup_{|z|>\delta}\left|\mu(|z|) u^{(n)}(z) f_k(z)\right| + C\sup_{|z|>\delta}\sum_{j=0}^{n-1}\left|\frac{C_n^j \mu(|z|) u^{(j)}(z)}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\right| \left\|f_k\right\|_{F(p,q,s)} \\
&\leq C\epsilon + C\sup_{|z|>\delta}\left|\frac{\mu(|z|) u^{(n)}(z)}{\left(1-|z|^2\right)^{(2+q-p)/p}}\right| \left\|f_k\right\|_{\mathcal{B}^{(2+q)/p}} + C\sup_{|z|>\delta}\sum_{j=0}^{n-1}\left|\frac{C_n^j \mu(|z|) u^{(j)}(z)}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\right| \\
&\leq C\epsilon + C\sup_{|z|>\delta}\left|\frac{\mu(|z|) u^{(n)}(z)}{\left(1-|z|^2\right)^{(2+q-p)/p}}\right| + C\sup_{|z|>\delta}\sum_{j=0}^{n-1}\left|\frac{C_n^j \mu(|z|) u^{(j)}(z)}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\right| \\
&< \left(2 + C\sum_{j=0}^{n} C_n^j\right) C\epsilon,
\end{aligned} \tag{3.40}$$

when $k > K_0$. If $2 + q < p$, then $f_k \in F(p,q,s) \subset \mathcal{B}^{(2+q)/p}$, $k \in \mathbb{N}$. By [37, Lemma 3.2], we have

$$\lim_{k\to\infty}\sup_{z\in\mathbb{D}}\left|f_k(z)\right| = 0. \tag{3.41}$$

By (3.38), (3.39), Lemma 2.1, and $u \in \mathcal{W}_\mu^{(n)}(\mathbb{D})$, we get

$$\begin{aligned}
&\left\|M_u f_k\right\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \\
&= \sum_{j=0}^{n-1}\left|\left(u f_k\right)^{(j)}(0)\right| + \sup_{z\in\mathbb{D}}\mu(|z|)\left|\left(u f_k\right)^{(n)}(z)\right| \\
&< C\epsilon + C\sup_{|z|>\delta}\left|\mu(|z|) u^{(n)}(z) f_k(z)\right| + C\sup_{|z|>\delta}\sum_{j=0}^{n-1}\left|\frac{C_n^j \mu(|z|) u^{(j)}(z)}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\right| \left\|f_k\right\|_{F(p,q,s)} \\
&\leq C\epsilon + C\sup_{z\in\mathbb{D}}\left|f_k(z)\right| + C\sup_{|z|>\delta}\sum_{j=0}^{n-1}\left|\frac{C_n^j \mu(|z|) u^{(j)}(z)}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\right|
\end{aligned}$$

$$\leq C\epsilon + C\epsilon + C\sup_{|z|>\delta}\sum_{j=0}^{n-1}\left|\frac{C_n^j\mu(|z|)u^{(j)}(z)}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\right|$$
$$< \left(2 + C\sum_{j=0}^{n}C_n^j\right)C\epsilon, \tag{3.42}$$

when $k > K_0$. If $2 + q = p$, from (3.35), (3.38), (3.39), and Lemmas 2.1 and 2.2, we get

$$\begin{aligned}
&\|M_u f_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \\
&= \sum_{j=0}^{n-1}\left|(uf_k)^{(j)}(0)\right| + \sup_{z\in\mathbb{D}}\mu(|z|)\left|(uf_k)^{(n)}(z)\right| \\
&< C\epsilon + C\sup_{|z|>\delta}\left|\mu(|z|)u^{(n)}(z)f_k(z)\right| + C\sup_{|z|>\delta}\sum_{j=0}^{n-1}\left|\frac{C_n^j\mu(|z|)u^{(j)}(z)}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\right|\|f_k\|_{F(p,q,s)} \\
&< C\epsilon + C\sup_{|z|>\delta}\left|\mu(|z|)u^{(n)}(z)\right|\log\frac{2}{1-|z|^2}\|f_k\|_{\mathcal{B}} + C\sup_{|z|>\delta}\sum_{j=0}^{n-1}\left|\frac{C_n^j\mu(|z|)u^{(j)}(z)}{\left(1-|z|^2\right)^{((2+q-p)/p)+n-j}}\right| \\
&\leq C\epsilon + C\sup_{|z|>\delta}\left|\mu(|z|)u^{(n)}(z)\right|\log\frac{2}{1-|z|^2}\|f_k\|_{F(p,q,s)} + C\sup_{|z|>\delta}\sum_{j=0}^{n-1}\left|\frac{C_n^j\mu(|z|)u^{(j)}(z)}{\left(1-|z|^2\right)^{n-j}}\right| \\
&< \left(2 + C\sum_{j=0}^{n}C_n^j\right)C\epsilon,
\end{aligned} \tag{3.43}$$

when $k > K_0$. It follows that the operator $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact.

Conversely, assume that $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact. Taking $f = 1$, we get $u \in \mathcal{W}_\mu^{(n)}(\mathbb{D})$. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|z_k| \to 1$ as $k \to \infty$. Taking the test functions h_{z_k}, where h_{z_k} is defined by (3.17), we will write

$$h_k(z) = f_{z_k}(z). \tag{3.44}$$

We obtain that $h_k \in F(p,q,s)$,

$$\sup_{k\in\mathbb{N}}\|h_k\|_{F(p,q,s)} \leq C, \tag{3.45}$$

and for $t \in \{1, \ldots, n-1\}$,

$$h_k^{(n)}(z_k) = \sum_{j=1}^{n} c_j \prod_{k=0}^{l-1} (\alpha + j + k) \frac{(\overline{z_k})^n}{\left(1 - |z_k|^2\right)^{((2+q-p)/p)+n}}, \quad h_k^{(t)}(z_k) = 0. \tag{3.46}$$

For $|z| = r < 1$, we have

$$|h_k(z)| = \left| \sum_{j=1}^{n} c_j \frac{\left(1 - |z_k|^2\right)^{j+1}}{(1 - \overline{z_k} z)^{\alpha+j}} \right| \le \sum_{j=1}^{n} c_j \frac{\left(1 - |z_k|^2\right)^{j+1}}{(1 - r)^{\alpha+j}} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{3.47}$$

that is, h_k converges to 0 uniformly on compact subsets of $\mathbb{D}$, using the compactness of $M_u : F(p, q, s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$, we obtain

$$\frac{\mu(z_k)|u(z_k)||z_k|^n}{\left(1 - |z_k|^2\right)^{((2+q-p)/p)+n}} \le C\|M_u h_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{3.48}$$

and consequently (3.31) holds.

Assume that $2 + q = p, s > 1$ and $M_u : F(p, q, s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact, we only need to prove (3.32) holds. To do this, let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|z_k| \to 1$ as $k \to \infty$. Set

$$\begin{aligned} a_k &= \log \frac{2}{1 - |z_k|^2}, \\ J_k(z) &= \frac{E}{a_k} \left(\log \frac{2}{1 - \overline{z_k} z} \right)^2 + \frac{F}{a_k^2} \left(\log \frac{2}{1 - \overline{z_k} z} \right)^3. \end{aligned} \tag{3.49}$$

Since

$$\left| J_k'(z) \right| \le \left| \frac{2E}{a_k} \left(\log \frac{2}{1 - \overline{z_k} z} \right) \frac{\overline{z_k}}{1 - \overline{z_k} z} \right| + \left| \frac{3F}{a_k^2} \left(\log \frac{2}{1 - \overline{z_k} z} \right)^2 \frac{\overline{z_k}}{1 - \overline{z_k} z} \right| \le \frac{C}{|1 - \overline{z_k} z|}, \tag{3.50}$$

we have $J_k \in F(p, q, s)$ and $\sup_{k \in \mathbb{N}} \|J_k\|_{F(p,q,s)} \le C$ (see [19, 30]), and $J_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since $M_u : F(p, q, s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is compact, we have

$$\|M_u J_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.51}$$

We also have that

$$
\begin{aligned}
J_k'(z) &= \frac{2E}{a_k}\left(\log\frac{2}{1-\overline{z_k}z}\right)\frac{\overline{z_k}}{1-\overline{z_k}z} + \frac{3F}{a_k^2}\left(\log\frac{2}{1-\overline{z_k}z}\right)^2\frac{\overline{z_k}}{1-\overline{z_k}z}, \\
J_k''(z) &= \frac{2E}{a_k}\frac{(\overline{z_k})^2}{(1-\overline{z_k}z)^2} + \frac{2E}{a_k}\left(\log\frac{2}{1-\overline{z_k}z}\right)\frac{(\overline{z_k})^2}{(1-\overline{z_k}z)^2} \\
&\quad + \frac{3\cdot 2F}{a_k^2}\left(\log\frac{2}{1-\overline{z_k}z}\right)\frac{(\overline{z_k})^2}{(1-\overline{z_k}z)^2} + \frac{3F}{a_k^2}\left(\log\frac{2}{1-\overline{z_k}z}\right)^2\frac{(\overline{z_k})^2}{(1-\overline{z_k}z)^2}, \\
J_k'''(z) &= \frac{2\cdot 2E}{a_k}\frac{(\overline{z_k})^3}{(1-\overline{z_k}z)^3} + \frac{2E}{a_k}\frac{(\overline{z_k})^3}{(1-\overline{z_k}z)^3} + \frac{4E}{a_k}\left(\log\frac{2}{1-\overline{z_k}z}\right)\frac{(\overline{z_k})^3}{(1-\overline{z_k}z)^3} \\
&\quad + \frac{3\cdot 2F}{a_k^2}\frac{(\overline{z_k})^3}{(1-\overline{z_k}z)^3} + \frac{3\cdot 2\cdot 2F}{a_k^2}\left(\log\frac{2}{1-\overline{z_k}z}\right)\frac{(\overline{z_k})^3}{(1-\overline{z_k}z)^3} \\
&\quad + \frac{3\cdot 2F}{a_k^2}\left(\log\frac{2}{1-\overline{z_k}z}\right)\frac{(\overline{z_k})^3}{(1-\overline{z_k}z)^3} + \frac{6F}{a_k^2}\left(\log\frac{2}{1-\overline{z_k}z}\right)^2\frac{(\overline{z_k})^3}{(1-\overline{z_k}z)^3}, \\
&= \frac{6E}{a_k}\frac{(\overline{z_k})^3}{(1-\overline{z_k}z)^3} + \frac{4E}{a_k}\left(\log\frac{2}{1-\overline{z_k}z}\right)\frac{(\overline{z_k})^3}{(1-\overline{z_k}z)^3} + \frac{6F}{a_k^2}\frac{(\overline{z_k})^3}{(1-\overline{z_k}z)^3} \\
&\quad + \frac{18F}{a_k^2}\left(\log\frac{2}{1-\overline{z_k}z}\right)\frac{(\overline{z_k})^3}{(1-\overline{z_k}z)^3} + \frac{6F}{a_k^2}\left(\log\frac{2}{1-\overline{z_k}z}\right)^2\frac{(\overline{z_k})^3}{(1-\overline{z_k}z)^3}, \\
J_k^{(4)}(z) &= \frac{22E}{a_k}\frac{(\overline{z_k})^4}{(1-\overline{z_k}z)^4} + \frac{3!2E}{a_k}\left(\log\frac{2}{1-\overline{z_k}z}\right)\frac{(\overline{z_k})^4}{(1-\overline{z_k}z)^4} + \frac{36F}{a_k^2}\frac{(\overline{z_k})^4}{(1-\overline{z_k}z)^4} \\
&\quad + \frac{66F}{a_k^2}\left(\log\frac{2}{1-\overline{z_k}z}\right)\frac{(\overline{z_k})^4}{(1-\overline{z_k}z)^4} + \frac{3!3F}{a_k^2}\left(\log\frac{2}{1-\overline{z_k}z}\right)^2\frac{(\overline{z_k})^4}{(1-\overline{z_k}z)^4}, \\
&\ \ \vdots \\
J_k^{(m)}(z) &= \frac{b_mE}{a_k}\frac{(\overline{z_k})^m}{(1-\overline{z_k}z)^m} + \frac{(m-1)!2E}{a_k}\left(\log\frac{2}{1-\overline{z_k}z}\right)\frac{(\overline{z_k})^m}{(1-\overline{z_k}z)^m} + \frac{d_mF}{a_k^2}\frac{(\overline{z_k})^m}{(1-\overline{z_k}z)^m} \\
&\quad + \frac{c_mF}{a_k^2}\left(\log\frac{2}{1-\overline{z_k}z}\right)\frac{(\overline{z_k})^m}{(1-\overline{z_k}z)^m} + \frac{(m-1)!3F}{a_k^2}\left(\log\frac{2}{1-\overline{z_k}z}\right)^2\frac{(\overline{z_k})^m}{(1-\overline{z_k}z)^m}.
\end{aligned}
\tag{3.52}
$$

Therefore,

$$
\begin{aligned}
J_k(z_k) &= (E+F)a_k, \\
J_k'(z_k) &= (2E+3F)\frac{\overline{z_k}}{1-|z_k|^2},
\end{aligned}
$$

$$J_k''(z_k) = (2E+3F)\frac{(\overline{z_k})^2}{\left(1-|z_k|^2\right)^2} + \frac{2(E+3F)}{a_k}\frac{(\overline{z_k})^2}{\left(1-|z_k|^2\right)^2},$$

$$J_k'''(z_k) = 2(2E+3F)\frac{(\overline{z_k})^3}{\left(1-|z_k|^2\right)^3} + \frac{6(E+3F)}{a_k}\frac{(\overline{z_k})^3}{\left(1-|z_k|^2\right)^3} + \frac{6F}{a_k^2}\frac{(\overline{z_k})^3}{\left(1-|z_k|^2\right)^3},$$

$$J_k^{(4)}(z_k) = 3!(2E+3F)\frac{(\overline{z_k})^4}{\left(1-|z_k|^2\right)^4} + \frac{22(E+3F)}{a_k}\frac{(\overline{z_k})^4}{\left(1-|z_k|^2\right)^4} + \frac{36F}{a_k^2}\frac{(\overline{z_k})^4}{\left(1-|z_k|^2\right)^4},$$

$$\vdots$$

$$J_k^{(m)}(z_k) = (m-1)!(2E+3F)\frac{(\overline{z_k})^m}{\left(1-|z_k|^2\right)^m} + \frac{b_mE+c_mF}{a_k}\frac{(\overline{z_k})^m}{\left(1-|z_k|^2\right)^m} + \frac{d_mF}{a_k^2}\frac{(\overline{z_k})^m}{\left(1-|z_k|^2\right)^m}. \tag{3.53}$$

Taking $E=3$ and $F=-2$, if $n=1$, we have

$$\mu(|z_k|)\left|u'(z_k)\log\frac{2}{1-|z_k|^2}\right| \le \|M_uJ_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \longrightarrow 0 \quad \text{as } k\longrightarrow\infty, \tag{3.54}$$

if $n\neq 1$, we have

$$\begin{aligned}\|M_uJ_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} &\ge \mu(|z_k|)\left|\sum_{j=0}^{n}C_n^ju^{(j)}(z_k)J_k^{(n-j)}(z_k)\right|\\ &\ge \mu(|z_k|)\left|u^{(n)}(z_k)J_k(z_k)\right| - \mu(|z_k|)\left|\sum_{j=0}^{n-1}C_n^ju^{(j)}(z_k)J_k^{(n-j)}(z_k)\right|\\ &\ge \mu(|z_k|)\left|u^{(n)}(z_k)\log\frac{2}{1-|z_k|^2}\right| - \mu(|z_k|)\left|\sum_{j=0}^{n-1}C_n^ju^{(j)}(z_k)J_k^{(n-j)}(z_k)\right|.\end{aligned} \tag{3.55}$$

From (3.31), (3.35), (3.37), and (3.38), we have that for sufficiently large k:

$$\begin{aligned}&\mu(|z_k|)\left|u^{(n)}(z_k)\log\frac{2}{1-|z_k|^2}\right|\\ &\quad\le \|M_uJ_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} + \mu(|z_k|)\left|\sum_{j=0}^{n-1}C_n^ju^{(j)}(z_k)J_k^{(n-j)}(z_k)\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \|M_u J_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} + C\sum_{j=0}^{n-1}\left(\frac{\mu(|z_k|)\left|u^{(j)}(z_k)\right|}{a_k\left(1-|z_k|^2\right)^{n-j}} + \frac{\mu(|z_k|)\left|u^{(j)}(z_k)\right|}{a_k^2\left(1-|z_k|^2\right)^{n-j}}\right) \\
&\leq \|M_u J_k\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} + C\sum_{j=0}^{n-1}\frac{\mu(|z_k|)\left|u^{(j)}(z_k)\right|}{\left(1-|z_k|^2\right)^{n-j}} \\
&< C\epsilon.
\end{aligned}
\tag{3.56}
$$

Using (3.54) and (3.56), it is easy to get that (3.32) holds. From which we obtain the desired result. □

4. The Boundedness of M_u from $F(p,q,s)$ (or $F_0(p,q,s)$) to $\mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$

In this section, we characterize the boundedness of $M_u : F(p,q,s)$ (or $F_0(p,q,s)$) $\to \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$.

Theorem 4.1. *Assume that $u \in H(\mathbb{D})$ and μ is normal. Then,*

(1) *$M_u : F(p,q,s) \to \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$ is bounded if and only if $M_u : F(p,q,s) \to \mathcal{W}_{\mu}^{(n)}(\mathbb{D})$ is bounded and*

$$
\lim_{|z|\to 1}\mu(z)\left|u^{(n)}(z)\right| = 0. \tag{4.1}
$$

(2) *$M_u : F_0(p,q,s) \to \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$ is bounded if and only if $M_u : F_0(p,q,s) \to \mathcal{W}_{\mu}^{(n)}(\mathbb{D})$ is bounded and* (4.1) *holds.*

Proof. (1) Assume that $M_u : F(p,q,s) \to \mathcal{W}_{\mu}^{(n)}(\mathbb{D})$ is bounded and condition (4.1) holds. Since

$$
u(z) - u(0) = \int_0^1 xu'(zx)\,dx, \tag{4.2}
$$

it follows that

$$
\begin{aligned}
\mu(z)|u(z)| &\leq \mu(z)|u(0)| + \mu(z)\left|\int_0^1 xu'(zx)\,dx\right| \\
&\leq \mu(z)|u(0)| + \mu(z)\left|\int_0^{1/2} xu'(zx)\,dx\right| + \mu(z)\left|\int_{1/2}^1 xu'(zx)\,dx\right| \\
&\leq \mu(z)|u(0)| + \mu(z)\max_{|z|\leq 1/2}\left|u'(z)\right| + \mu(z)\left|\int_{1/2}^1 xu'(zx)\,dx\right|.
\end{aligned}
\tag{4.3}
$$

Since μ is normal, by the monotonicity of $\mu(t)/(1-t^2)^a$, for $t_0 \le t_1 < t < 1$, we have

$$\mu(t) = \left(1-t^2\right)^a \frac{\mu(t)}{(1-t^2)^a} \le \left(1-t^2\right)^a \frac{\mu(t_1)}{(1-t_1^2)^a} < \mu(t_1), \tag{4.4}$$

that is, μ is decreasing on $[t_0, 1)$, and for any $\epsilon > 0$, there is a $\tau > 0$ such that

$$0 < \mu(|z|) < \epsilon\left(1-|z|^2\right)^a, \quad (\tau < |z| < 1), \tag{4.5}$$

which implies $\lim_{|z|\to 1}\mu(|z|) = 0$. Since for $1/2 < x < 1$ and $2t_0 < |z| < 1$, we have $\mu(|z|) \le \mu(x|z|)$. From (4.3), it follows that

$$\mu(|z|)|u(|z|)| \le \mu(|z|)|u(0)| + \mu(|z|)\max_{|z|\le 1/2}\left|u'(z)\right| + \int_{1/2}^{1}\mu(x|z|)\left|u'(zx)\right|dx. \tag{4.6}$$

For $j \in \{2, 3, \dots, n\}$, by applying formula (4.6) to the function $u^{(j-1)}$, we get

$$\mu(|z|)\left|u^{(j-1)}(z)\right| \le \mu(|z|)\left|u^{(j-1)}(0)\right| + \mu(|z|)\max_{|z|\le 1/2}\left|u^{(j)}(z)\right| + \int_{1/2}^{1}\mu(|z|x)\left|u^{(j)}(zx)\right|dx, \tag{4.7}$$

when $2t_0 < |z| < 1$. It follows from (4.1), (4.6), and (4.7) that $\lim_{|z|\to 1}\mu(|z|)|u^{(j-1)}(z)| = 0$ for $j \in \{1, 3, \dots, n\}$. Since, for each polynomial p, we have

$$\begin{aligned}\left|\mu(|z|)(M_u p)^{(n)}(z)\right| &= \mu(|z|)\left|(u(z)p(z))^{(n)}\right| \\ &= \mu(|z|)\left|\sum_{j=0}^{n} C_n^j u^{(j)}(z) p^{(n-j)}(z)\right| \\ &\le C\sum_{j=0}^{n}\left|C_n^j \mu(|z|) u^{(j)}(z)\right|\left\|p^{(n-j)}\right\|_\infty,\end{aligned} \tag{4.8}$$

hence, $M_u p \in \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$. Since the set of all polynomials is dense in $F(p,q,s)$, we have that for every $f \in F(p,q,s)$ there is a sequence of polynomials $\{p_k\}$ such that

$$\lim_{k\to\infty}\|p_k - f\|_{F(p,q,s)} = 0. \tag{4.9}$$

From this and since the operator $M_u : F(p,q,s) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded, we have that

$$\|M_u p_k - M_u f\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \le \|M_u\|\|p_k - f\|_{F(p,q,s)} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.10}$$

Since $\mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$ is a closed subspace of $\mathcal{W}_\mu^{(n)}(\mathbb{D})$, therefore, we have $M_u(F(p,q,s) \subset \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$, from which the boundedness of $M_u : F(p,q,s) \to \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$ follows.

On the other hand, assume that $M_u : F(p,q,s) \to \mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$ is bounded, then $M_u : F(p,q,s) \to \mathcal{W}_{\mu}^{(n)}(\mathbb{D})$ is bounded. By taking the function given by $f(z) = 1$, we obtain

$$\mu(|z|)\left|u^{(n)}(z)\right| = \left|\mu(|z|)\left(M_u f\right)^{(n)}(z)\right| \longrightarrow 0 \quad (\text{as } |z| \longrightarrow 1), \tag{4.11}$$

as desired.

(2) The proof is similar to that of the case (1). We leave the details to the interested reader. □

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