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## Research Article

# Strong Convergence Theorems for the Split Common Fixed Point Problem for Countable Family of Nonexpansive Operators

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We introduce a new iterative algorithm for solving the split common fixed point problem for countable family of nonexpansive operators. Under suitable assumptions, we prove that the iterative algorithm strongly converges to a solution of the problem.

## 1. Introduction

Let  $H_1$  and  $H_2$  be two real Hilbert spaces and let  $A : H_1 \rightarrow H_2$  be a bounded linear operator. The split feasibility problem (SFP), see [1], is to find a point  $x^*$  with the property:

$$x^* \in C, \quad Ax^* \in Q, \quad (1.1)$$

where  $C$  and  $Q$  are nonempty closed convex subsets of  $H_1$  and  $H_2$ , respectively. A more general form of the SFP is the so-called multiple-set split feasibility problem (MSSFP) which was recently introduced by Censor et al. [2]. Given integers  $p, r \geq 1$ , the MSSFP is to find a point  $x^*$  with the property:

$$x^* \in \bigcap_{i=1}^p C_i, \quad Ax^* \in \bigcap_{j=1}^r Q_j, \quad (1.2)$$

where  $\{C_i\}_{i=1}^p$  and  $\{Q_j\}_{j=1}^r$  are nonempty closed convex subsets of  $H_1$  and  $H_2$ , respectively. The SFP (1.1) and the MSSFP (1.2) serve as a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in

this operator's ranges. Recently, the SFP (1.1) and the MSSFP (1.2) are widely applied in the image reconstructions [1, 3], the intensity-modulated radiation therapy [4, 5], and many other areas. The problems have been investigated by many researchers, for instance, [6–13]. The SFP (1.1) can be viewed as a special case of the convex feasibility problem (CFP) since the SFP (1.1) can be rewritten as

$$x^* \in C, \quad x^* \in A^{-1}Q. \quad (1.3)$$

However, the methods for study the SFP (1.1) are actually different from those for the CFP in order to avoid the usage of the inverse  $A^{-1}$ . Byrne [6] introduced a so-called CQ algorithm:

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \quad n \geq 0, \quad (1.4)$$

where the operator  $A^{-1}$  is not relevant.

Censor and Segal in [14] firstly introduced the concept of the split common fixed point problem (SCFPP) in finite-dimensional Hilbert spaces. The SCFPP is a generalization of the convex feasibility problem (CFP) and the split feasibility problem (SFP). The SCFPP considers to find a common fixed point of a family of operators in  $H_1$  such that its image under a linear transformation  $A$  is a common fixed point of another family of operators in  $H_2$ . That is, the SCFPP is to find a point  $x^*$  with the property:

$$x^* \in \bigcap_{i=1}^p \text{Fix}(U_i), \quad Ax^* \in \bigcap_{j=1}^r \text{Fix}(T_j), \quad (1.5)$$

where  $U_i : H_1 \rightarrow H_1$  ( $i = 1, 2, \dots, p$ ) and  $T_j : H_2 \rightarrow H_2$  ( $j = 1, 2, \dots, r$ ) are nonlinear operators. If  $p = r = 1$ , the problem (1.5) deduces to the so-called two-set SCFPP, which is to find a point  $x^*$  such that

$$x^* \in \text{Fix}(U), \quad Ax^* \in \text{Fix}(T), \quad (1.6)$$

where  $U : H_1 \rightarrow H_1$  and  $T : H_2 \rightarrow H_2$  are nonlinear operators.

Censor and Segal in [14] considered the following iterative algorithm for the SCFPP (1.6) for Class- $\mathfrak{S}$  operators in finite-dimensional Hilbert spaces:

$$x_{n+1} = U(x_n - \gamma A^*(I - T)Ax_n), \quad n \geq 0, \quad (1.7)$$

where  $x_0 \in H_1$ ,  $0 < \gamma < 2/\|A\|^2$  and  $I$  is the identity operator.

Recently, in the infinite-dimensional Hilbert space, Wang and Xu [15] studied the SCFPP (1.5) and introduced the following iterative algorithm for Class- $\mathfrak{S}$  operators:

$$x_{n+1} = U_{[n]}(x_n - \gamma A^*(I - T_{[n]})Ax_n), \quad n \geq 0 \quad (1.8)$$

where  $[n] = n \bmod p$  and  $p = r$ . Under some mild conditions, they proved that  $\{x_n\}$  converges weakly to a solution of the SCFPP (1.5), extended and improved Censor and Segal's results.

Moreover, they proved that the SCFPP (1.5) for the Class- $\mathfrak{S}$  operators is equivalent to a common fixed point problem. This is also a classical method. Many problems eventually converted to a common fixed point problem, see [16–18]. Very recently, the split common fixed point problems for various types of operators were studied in [19–21].

The above-mentioned results are about a finite number of operators; that is, the constraints are finite imposed on the solutions. In this paper, we consider the constraints are infinite, but countable. That is, we consider the generalized case of SCFPP for two countable families of operators (denoted GSCFPP), which is to find a point  $x^*$  such that

$$x^* \in \bigcap_{i=1}^{\infty} \text{Fix}(U_i), \quad Ax^* \in \bigcap_{j=1}^{\infty} \text{Fix}(T_j). \quad (1.9)$$

Of course, the GSCFPP is more general and widely used than the SCFPP. This is a novelty of this paper. At the same time, we consider the nonexpansive operator. The nonexpansive operator is important because it includes many types of nonlinear operator arising in applied mathematics. For instance, the projection and the identity operator are nonexpansive. We prove that the GSCFPP (1.9) for the nonexpansive operators is equivalent to a common fixed point problem. Very recently, Gu et al. [22] introduced a new iterative method for dealing with the countable family of operators. They studied the following iterative algorithm:

$$\begin{aligned} y_n &= P_C [\beta_n Sx_n + (1 - \beta_n)x_n], \\ x_{n+1} &= P_C \left[ \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i y_n \right], \end{aligned} \quad (1.10)$$

where  $S$  and  $\{T_i\}_{i=1}^{\infty}$  are nonexpansive,  $\alpha_0 = 1$ ,  $\{\alpha_n\}$  is strictly decreasing sequence in  $(0, 1)$ , and  $\{\beta_n\}$  is a sequence in  $(0, 1)$ . Under some certain conditions on parameters, they proved that the sequence  $\{x_n\}$  converges strongly to  $x^* \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$ . On the other hand, from weakly convergence to strongly convergence, the viscosity approximation method is also one of the classical methods, see [22–24].

Motivated and inspired by the above results, we introduce the following algorithm:

$$x_{n+1} = \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) U_i(x_n + \gamma A^*(T_i - I)Ax_n). \quad (1.11)$$

Under some certain conditions, we prove that the sequence  $\{x_n\}$  generated by (1.11) converges strongly to the solution of the GSCFPP (1.9).

## 2. Preliminaries

Throughout this paper, we write  $x_n \rightharpoonup x$  and  $x_n \rightarrow x$  to indicate that  $\{x_n\}$  converges weakly to  $x$  and converges strongly to  $x$ , respectively.

Let  $H$  be a real Hilbert space. An operator  $T : H \rightarrow H$  is said to be nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in H$ . The set of fixed points of  $T$  is denoted by  $\text{Fix}(T)$ . It is known that  $\text{Fix}(T)$  is closed and convex, see [25]. An operator  $f : H \rightarrow H$  is called

contraction if there exists a constant  $\rho \in [0, 1)$  such that  $\|f(x) - f(y)\| \leq \rho\|x - y\|$  for all  $x, y \in H$ . Let  $C$  be a nonempty closed convex subset of  $H$ . For each  $x \in H$ , there exists a unique nearest point in  $C$ , denoted by  $P_C x$ , such that  $\|x - P_C x\| \leq \|x - y\|$  for every  $y \in C$ .  $P_C$  is called the metric projection of  $H$  onto  $C$ . It is known that, for each  $x \in H$ ,

$$\langle x - P_C x, y - P_C x \rangle \leq 0 \quad (2.1)$$

for all  $y \in C$ .

In order to prove our main results, we collect the following lemmas in this section.

**Lemma 2.1** (see [26]). *Let  $H$  be a Hilbert space,  $C$  a closed convex subset of  $H$ , and  $T : C \rightarrow C$  a nonexpansive operator with  $\text{Fix}(T) \neq \emptyset$ . If  $\{x_n\}$  is a sequence in  $C$  weakly converging to  $x \in C$  and  $\{(I - T)x_n\}$  converges strongly to  $y \in C$ , then  $(I - T)x = y$ . In particular, if  $y = 0$ , then  $x \in \text{Fix}(T)$ .*

**Lemma 2.2** (see [23]). *Assume  $\{a_n\}$  is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0, \quad (2.2)$$

where  $\{\gamma_n\}$  is a sequence in  $(0, 1)$  and  $\{\delta_n\}$  is a sequence such that

- (i)  $\sum_{n=1}^{\infty} \gamma_n = \infty$ ,
- (ii)  $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$  or  $\sum_{n=1}^{\infty} |\delta_n| < \infty$ .

Then  $\lim_{n \rightarrow \infty} a_n = 0$ .

### 3. Main Results

Now we state and prove our main results of this paper.

**Theorem 3.1.** *Let  $\{U_n\}$  and  $\{T_n\}$  be sequences of nonexpansive operators on real Hilbert spaces  $H_1$  and  $H_2$ , respectively. Let  $f : H_1 \rightarrow H_1$  be a contraction with coefficient  $\rho \in [0, 1)$ . Suppose that the solution set  $\Omega$  of GSCFPP (1.9) is nonempty. Let  $x_1 \in H_1$  and  $0 < \gamma < 2/\|A\|^2$ . Set  $\alpha_0 = 1$ , and let  $\{\alpha_n\} \subset (0, 1]$  be a strictly decreasing sequence satisfying the following conditions:*

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;
- (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
- (iii)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ .

Then the sequence  $\{x_n\}$  generated by (1.11) converges strongly to  $w \in \Omega$ , where  $w = P_{\Omega} f(w)$ .

*Proof.* We proceed with the following steps.

*Step 1.* First show that there exists  $w \in \Omega$  such that  $w = P_{\Omega} f(w)$ .

In fact, since  $f$  is a contraction with coefficient  $\rho$ , we have

$$\|P_{\Omega} f(x) - P_{\Omega} f(y)\| \leq \|f(x) - f(y)\| \leq \rho\|x - y\| \quad (3.1)$$

for every  $x, y \in H_1$ . Hence  $P_{\Omega} f$  is also a contraction of  $H_1$  into itself. Therefore, there exists a unique  $w \in H_1$  such that  $w = P_{\Omega} f(w)$ . At the same time, we note that  $w \in \Omega$ .

Step 2. Now we show that  $\{x_n\}$  is bounded.

For simplicity, we set  $V_i = I + \gamma A^*(T_i - I)A$ . Then we can rewrite (1.11) to

$$x_{n+1} = \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) U_i V_i x_n. \quad (3.2)$$

Observe that

$$\begin{aligned} \|(T_i - I)Ax - (T_i - I)Ay\|^2 &= \|Ax - Ay\|^2 + \|T_i Ax - T_i Ay\|^2 - 2\langle Ax - Ay, T_i Ax - T_i Ay \rangle \\ &\leq 2\|Ax - Ay\|^2 - 2\langle Ax - Ay, T_i Ax - T_i Ay \rangle \\ &= -2\langle Ax - Ay, (T_i - I)Ax - (T_i - I)Ay \rangle \end{aligned} \quad (3.3)$$

for all  $x, y \in H_1$ . Thus it follows that

$$\begin{aligned} \|V_i x - V_i y\| &= \|(I + \gamma A^*(T_i - I)A)x - (I + \gamma A^*(T_i - I)A)y\|^2 \\ &\leq \|x - y\|^2 + \gamma^2 \|A\|^2 \|(T_i - I)Ax - (T_i - I)Ay\|^2 \\ &\quad + 2\gamma \langle Ax - Ay, (T_i - I)Ax - (T_i - I)Ay \rangle \\ &\leq \|x - y\|^2 + \gamma(\gamma \|A\|^2 - 1) \|(T_i - I)Ax - (T_i - I)Ay\|^2. \end{aligned} \quad (3.4)$$

For  $0 < \gamma < 1/\|A\|^2$ , we can immediately obtain that  $V_i$  is a nonexpansive operator for every  $i \in \mathbb{N}$ .

Let  $p \in \Omega$ , then  $U_i p = p$  and  $T_i A p = A p$  for every  $i \geq 1$ . Thus  $(T_i - I)A p = 0$ , which implies that  $V_i p = p$ . Since  $\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) = 1 - \alpha_n$ , we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|U_i V_i x_n - p\| \\ &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|V_i x_n - p\| \\ &\leq \alpha_n \rho \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| \\ &= (1 - \alpha_n(1 - \rho)) \|x_n - p\| + \alpha_n(1 - \rho) \frac{1}{1 - \rho} \|f(p) - p\| \\ &\leq \max \left\{ \|x_n - p\|, \frac{1}{1 - \rho} \|f(p) - p\| \right\}. \end{aligned} \quad (3.5)$$

Then it follows that

$$\|U_n V_n x_{n-1} - p\| \leq \|V_n x_{n-1} - p\| \leq \|x_{n-1} - p\| \leq \max \left\{ \|x_1 - p\|, \frac{1}{1 - \rho} \|f(p) - p\| \right\} \quad (3.6)$$

for every  $n \in \mathbb{N}$ . This shows that  $\{x_n\}$  and  $\{U_n V_n x_{n-1}\}$  is bounded. Hence,  $\{f(x_n)\}$  is also bounded.

Step 3. We show  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ .

From (3.2), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \left\| \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) U_i V_i x_n - \alpha_{n-1} f(x_{n-1}) - \sum_{i=1}^{n-1} (\alpha_{i-1} - \alpha_i) U_i V_i x_{n-1} \right\| \\
&\leq \alpha_n \|f(x_n) - f(x_{n-1})\| + (\alpha_{n-1} - \alpha_n) \|f(x_{n-1})\| \\
&\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|U_i V_i x_n - U_i V_i x_{n-1}\| + (\alpha_{n-1} - \alpha_n) \|U_n V_n x_{n-1}\| \\
&\leq \alpha_n \rho \|x_n - x_{n-1}\| + (\alpha_{n-1} - \alpha_n) (\|f(x_{n-1})\| + \|U_n V_n x_{n-1}\|) + (1 - \alpha_n) \|x_n - x_{n-1}\| \\
&\leq (1 - \alpha_n (1 - \rho)) \|x_n - x_{n-1}\| + (\alpha_{n-1} - \alpha_n) M,
\end{aligned} \tag{3.7}$$

where  $M$  is a constant such that

$$M = \sup_{n \geq 1} \{ \|f(x_{n-1})\| + \|U_n V_n x_{n-1}\| \}. \tag{3.8}$$

From (i), (ii), (iii), and Lemma 2.2, it follows that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$

Step 4. We show  $\lim_{n \rightarrow \infty} \|U_i x_n - x_n\| = 0$  and  $\lim_{n \rightarrow \infty} \|U_i V_i x_n - x_n\| = 0$  for  $i \in \mathbb{N}$ .

We first show  $\lim_{n \rightarrow \infty} \|U_i V_i x_n - x_n\| = 0$  for  $i \in \mathbb{N}$ . Since  $p \in \Omega$ , we note that

$$\begin{aligned}
\|x_n - p\|^2 &\geq \|V_i x_n - V_i p\|^2 \geq \|U_i V_i x_n - U_i V_i p\|^2 \\
&= \|U_i V_i x_n - p\|^2 = \|U_i V_i x_n - x_n + x_n - p\|^2 \\
&= \|U_i V_i x_n - x_n\|^2 + \|x_n - p\|^2 + 2\langle U_i V_i x_n - x_n, x_n - p \rangle,
\end{aligned} \tag{3.9}$$

which implies that

$$\frac{1}{2} \|U_i V_i x_n - x_n\|^2 \leq \langle x_n - U_i V_i x_n, x_n - p \rangle. \tag{3.10}$$

Using (3.2) and (3.10), we deduce

$$\begin{aligned}
\frac{1}{2} \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|U_i V_i x_n - x_n\|^2 &\leq \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle x_n - U_i V_i x_n, x_n - p \rangle \\
&= \left\langle (1 - \alpha_n) x_n - \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) U_i V_i x_n, x_n - p \right\rangle \\
&= \langle (1 - \alpha_n) x_n - x_{n+1} + \alpha_n f(x_n), x_n - p \rangle \\
&= \langle x_n - x_{n+1}, x_n - p \rangle + \alpha_n \langle f(x_n) - x_n, x_n - p \rangle \\
&\leq \|x_n - x_{n+1}\| \|x_n - p\| + \alpha_n \|f(x_n) - x_n\| \|x_n - p\|.
\end{aligned} \tag{3.11}$$

Noting that  $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = 0$  and  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , then we immediately obtain

$$\sum_{i=1}^{\infty} (\alpha_{i-1} - \alpha_i) \|U_i V_i x_n - x_n\|^2 = \lim_{n \rightarrow \infty} \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|U_i V_i x_n - x_n\|^2 = 0. \quad (3.12)$$

Since  $\{\alpha_n\}$  is strictly decreasing, it follows that

$$\lim_{n \rightarrow \infty} \|U_i V_i x_n - x_n\| = 0, \quad \text{for every } i \in \mathbb{N}. \quad (3.13)$$

Next we show  $\lim_{n \rightarrow \infty} \|T_i A x_n - A x_n\| = 0$ , for every  $i \in \mathbb{N}$ . Note for every  $i \in \mathbb{N}$ ,

$$\begin{aligned} \|A x_n - A p\|^2 &\geq \|T_i A x_n - T_i A p\|^2 = \|T_i A x_n - A p\|^2 \\ &= \|T_i A x_n - A x_n + A x_n - A p\|^2 \\ &= \|T_i A x_n - A x_n\|^2 + \|A x_n - A p\|^2 + 2\langle T_i A x_n - A x_n, A x_n - A p \rangle, \end{aligned} \quad (3.14)$$

which follows that

$$\langle T_i A x_n - A x_n, A x_n - A p \rangle \leq -\frac{1}{2} \|T_i A x_n - A x_n\|^2, \quad (3.15)$$

for every  $i \in \mathbb{N}$ . From (3.2), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \left\| \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) U_i V_i x_n - p \right\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|U_i V_i x_n - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|x_n + \gamma A^*(T_i - I) A x_n - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \left[ \|x_n - p\|^2 + \gamma^2 \|A\|^2 \|T_i A x_n - A x_n\|^2 \right. \\ &\quad \left. + 2\gamma \langle A x_n - A p, T_i A x_n - A x_n \rangle \right]. \end{aligned} \quad (3.16)$$

By (3.15), it follows that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 \\ &\quad + \gamma (\gamma \|A\|^2 - 1) \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|T_i A x_n - A x_n\|^2. \end{aligned} \quad (3.17)$$

Thus,

$$\begin{aligned}
& \gamma(1 - \gamma\|A\|^2) \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|T_i A x_n - A x_n\|^2 \\
& \leq \alpha_n (\|f(x_n) - p\|^2 - \|x_n - p\|^2) + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
& = \alpha_n (\|f(x_n) - p\|^2 - \|x_n - p\|^2) + (\|x_n - p\| + \|x_{n+1} - p\|) (\|x_n - p\| - \|x_{n+1} - p\|) \\
& \leq \alpha_n (\|f(x_n) - p\|^2 - \|x_n - p\|^2) + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\|.
\end{aligned} \tag{3.18}$$

Using  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ , we have

$$\lim_{n \rightarrow \infty} \gamma(1 - \gamma\|A\|^2) \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|T_i A x_n - A x_n\|^2 = 0. \tag{3.19}$$

By  $0 < \gamma < 1/\|A\|^2$ , there holds

$$\sum_{i=1}^{\infty} (\alpha_{i-1} - \alpha_i) \|T_i A x_n - A x_n\|^2 = \lim_{n \rightarrow \infty} \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|T_i A x_n - A x_n\|^2 = 0. \tag{3.20}$$

Since  $\{\alpha_n\}$  is strictly decreasing, we obtain

$$\lim_{n \rightarrow \infty} \|T_i A x_n - A x_n\| = 0, \quad \forall i \in \mathbb{N}. \tag{3.21}$$

Last we show  $\lim_{n \rightarrow \infty} \|U_i x_n - x_n\| = 0$  for every  $i \in \mathbb{N}$ . In fact, we note that for every  $i \in \mathbb{N}$ ,

$$\begin{aligned}
\|U_i x_n - x_n\| & \leq \|U_i x_n - U_i V_i x_n\| + \|U_i V_i x_n - x_n\| \\
& \leq \|x_n - V_i x_n\| + \|U_i V_i x_n - x_n\| \\
& = \|x_n - x_n - \gamma A^*(T_i - I) A x_n\| + \|U_i V_i x_n - x_n\| \\
& \leq \gamma \|A\| \|(T_i - I) A x_n\| + \|U_i V_i x_n - x_n\|.
\end{aligned} \tag{3.22}$$

Then by (3.13) and (3.21), we obtain

$$\lim_{n \rightarrow \infty} \|U_i x_n - x_n\| = 0, \quad \forall i \in \mathbb{N}. \tag{3.23}$$

*Step 5.* Show  $\limsup_{n \rightarrow \infty} \langle f(w) - w, x_n - w \rangle \leq 0$ , where  $w = P_{\Omega} f(w)$ .



Since  $\{x_n\}$  is bounded, there exist a point  $v \in H_1$  and a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that

$$\limsup_{n \rightarrow \infty} \langle f(w) - w, x_n - w \rangle = \lim_{k \rightarrow \infty} \langle f(w) - w, x_{n_k} - w \rangle \quad (3.24)$$

and  $x_{n_k} \rightharpoonup v$ . Since  $A$  is a bounded linear operator, we have  $Ax_{n_k} \rightharpoonup Av$ . Now applying (3.21), (3.23), and Lemma 2.1, we conclude that  $v \in \text{Fix}(U_i)$  and  $Av \in \text{Fix}(T_i)$  for every  $i$ . Hence,  $v \in \Omega$ . Since  $\Omega$  is closed and convex, by (2.1), we get

$$\limsup_{n \rightarrow \infty} \langle f(w) - w, x_n - w \rangle = \lim_{k \rightarrow \infty} \langle f(w) - w, x_{n_k} - w \rangle = \langle f(w) - w, v - w \rangle \leq 0. \quad (3.25)$$

*Step 6.* Show  $x_n \rightarrow w = P_\Omega f(w)$ .

Since  $w \in \Omega$ , we have  $U_i w = w$  and  $T_i A w = A w$  for every  $i \in \mathbb{N}$ . It follows that  $V_i w = w$ . Using (3.2), we have

$$\begin{aligned} \|x_{n+1} - w\|^2 &= \left\langle \alpha_n (f(x_n) - w) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (U_i V_i x_n - w), x_{n+1} - w \right\rangle \\ &= \alpha_n \langle f(x_n) - f(w), x_{n+1} - w \rangle + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle \\ &\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle U_i V_i x_n - w, x_{n+1} - w \rangle \\ &\leq \alpha_n \rho \|x_n - w\| \|x_{n+1} - w\| + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle \\ &\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|U_i V_i x_n - w\| \|x_{n+1} - w\| \\ &\leq \frac{1}{2} \alpha_n \rho (\|x_n - w\|^2 + \|x_{n+1} - w\|^2) + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle \\ &\quad + \frac{1}{2} (1 - \alpha_n) (\|x_n - w\|^2 + \|x_{n+1} - w\|^2) \\ &\leq \frac{1}{2} [1 - \alpha_n (1 - \rho)] \|x_n - w\|^2 + \frac{1}{2} \|x_{n+1} - w\|^2 + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle, \end{aligned} \quad (3.26)$$

which implies that

$$\|x_{n+1} - w\|^2 \leq [1 - \alpha_n (1 - \rho)] \|x_n - w\|^2 + 2\alpha_n (1 - \rho) \frac{1}{1 - \rho} \langle f(w) - w, x_{n+1} - w \rangle, \quad (3.27)$$

for every  $n \in \mathbb{N}$ . Consequently, according to (3.25),  $\rho \in [0, 1)$ , and Lemma 2.2, we deduce that  $\{x_n\}$  converges strongly to  $w = P_\Omega(w)$ . This completes the proof.  $\square$

*Remark 3.2.* If we set  $\alpha_n = 1/n$  and  $f(x) = u$  for all  $x \in H_1$ , where  $u$  is an arbitrary point in  $H_1$ , it is easily seen that our conditions are satisfied.

**Corollary 3.3.** Let  $U : H_1 \rightarrow H_1$  and  $T : H_2 \rightarrow H_2$  be nonexpansive operators. Let  $f : H_1 \rightarrow H_1$  be a contraction with coefficient  $\rho \in [0, 1)$ . Suppose that the solution set  $\Omega$  of SCFPP (1.6) is nonempty. Let  $x_1 \in H_1$  and define a sequence  $\{x_n\}$  by the following algorithm:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)U(x_n + \gamma A^*(T - I)Ax_n), \quad (3.28)$$

where  $0 < \gamma < 1/\|A\|^2$ ,  $\alpha_0 = 1$  and  $\{\alpha_n\} \subset (0, 1]$  is a strictly decreasing sequence satisfying the following conditions:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;
- (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
- (iii)  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ .

Then  $\{x_n\}$  converges strongly to  $w \in \Omega$ , where  $w = P_{\Omega}f(w)$ .

*Proof.* Set  $\{U_n\}$  and  $\{T_n\}$  to be sequences of operators defined by  $U_n = U$  and  $T_n = T$  for all  $n \in \mathbb{N}$  in Theorem 3.1. Then by Theorem 3.1 we obtain the desired result.  $\square$

*Remark 3.4.* By adding more operators to the families  $\{U_n\}$  and  $\{T_n\}$  by setting  $U_i = I$  for  $i \geq p+1$  and  $T_j = I$  for  $j \geq r+1$ , the SCFPP (1.5) can be viewed as a special case of the GSCFPP (1.9).

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