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Research Article

Some Finite Sums Involving Generalized Fibonacci and Lucas Numbers

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By considering Melham's sums (Melham, 2004), we compute various more general nonalternating sums, alternating sums, and sums that alternate according to $(-1)^{\binom{n+1}{2}}$ involving the generalized Fibonacci and Lucas numbers.

1. Introduction

Let a, b , and p be assumed to be arbitrary nonzero complex numbers with $p(p^2+2)(p^2+4) \neq 0$. Define second-order linear recursion $\{W_n\}$ by

$$W_n = pW_{n-1} + W_{n-2}, \quad (1.1)$$

with $W_0 = a, W_1 = b$ for all integers n . Since $\Delta = p^2 + 4 \neq 0$, the roots α and β of $x^2 - px - 1 = 0$ are distinct.

Also define the sequence $\{X_n\}$ via the terms of sequence $\{W_n\}$ as $X_n = W_{n+1} + W_{n-1}$. The Binet formulas for the sequences $\{W_n\}$ and $\{X_n\}$ are

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \quad X_n = A\alpha^n + B\beta^n, \quad (1.2)$$

where $A = b - a\beta$ and $B = b - a\alpha$.

For $a = 0, b = 1$, we denote $W_n = U_n$ and so $X_n = V_n$, respectively. When $p = 1$, $U_n = F_n$ (n th Fibonacci number) and $V_n = L_n$ (n th Lucas number).

Inspired by the well-known identity

$$\sum_{n=1}^j F_n^2 = F_j F_{j+1}, \quad (1.3)$$

Clary and Hemenway [1] obtained factored closed-form expressions for all sums of the form $\sum_{n=1}^j F_{mn}^3$, where m is an integer. Motivated by the results in [1], Melham [2] computed all sums of the form $\sum_{n=1}^j (-1)^n F_{mn}^4$ and $\sum_{n=1}^j (-1)^n L_{mn}^4$. In [3], Melham computed various nonalternating sums, alternating sums, and sums that alternate according to $(-1)^{\binom{n+1}{2}}$ for sequences $\{W_n\}$ and $\{X_n\}$. The author gathers his sums in three sets. Here we recall one example from each set for the reader's convenience:

$$\sum_{n=i}^j W_n = \begin{cases} \frac{1}{p} V_{(j-i+1)/2} (W_{(j+i+1)/2} + W_{(j+i-1)/2}) & \text{if } j-i \equiv 1 \pmod{4}, \\ \frac{1}{p} U_{(j-i+1)/2} (X_{(j+i+1)/2} + X_{(j+i-1)/2}) & \text{if } j-i \equiv 3 \pmod{4}, \end{cases}$$

$$\sum_{n=4i}^{4j+3} (-1)^{\binom{n+1}{2}} W_{2n} = \frac{p}{\Delta - 2} U_{4j-4i+4} X_{4j+4i+3}, \quad (1.4)$$

$$\sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} U_n X_n = \frac{p}{\Delta - 2} V_{4j-4i+5} W_{4j+4i+3} + 2W_0.$$

We refer to [4] for general expansion formulas for sums of powers of Fibonacci and Lucas numbers, as considered by Melham, as well as some extensions such that

$$\sum_{k=0}^n F_{2k+\delta}^{2m+\epsilon}, \quad \sum_{k=0}^n L_{2k+\delta}^{2m+\epsilon}, \quad (1.5)$$

where $\delta, \epsilon \in \{0, 1\}$.

For alternating analogues of the results given by Prodinger, that is,

$$\sum_{k=0}^n (-1)^k F_{2k+\delta}^{2m+\epsilon}, \quad \sum_{k=0}^n (-1)^k L_{2k+\delta}^{2m+\epsilon}, \quad (1.6)$$

we refer to [5].

Hendel [6] gave the *factorization theorem* which exhibits factorizations of sums of the form $\sum_{j=i}^{n+i-1} F_{aj-b}$. The author also introduced a unified proof method based on formulae for the factorizations of $F_{q-d} + F_{q+d}$.

In [7], Curtin et al. derived formulae for the shifted summations

$$\sum_{j=0}^{d-1} F_{n+j} F_{m+j}, \quad \sum_{j=0}^{d-1} L_{n+j} L_{m+j}, \quad \sum_{j=0}^{d-1} F_{n+j} L_{m+j}, \quad (1.7)$$

and the shifted convolutions

$$\sum_{j=0}^{d-1} F_{n+j} F_{d-m-j}, \quad \sum_{j=0}^{d-1} L_{n+j} L_{d-m-j}, \quad \sum_{j=0}^{d-1} F_{n+j} L_{d-m-j} \quad (1.8)$$

for positive integers d and arbitrary integers n and m .

In this paper, our main purpose is to consider Melham's sums involving double products of terms of $\{W_n\}, \{X_n\}, \{U_n\}$, and $\{V_n\}$ given in [3] and then compute several more general nonalternating sums, alternating sums, and sums that alternate according to $(-1)^{\binom{n+1}{2}}$.

2. Certain Finite Sums of Double Products of Terms

In this section, we will investigate certain sums consisting of products of at most two terms of $\{W_n\}$: nonalternating sums, alternating sums and sums that alternate according to $(-1)^{\binom{n+1}{2}}$. From the Binet forms of $\{W_n\}$ and $\{X_n\}$, we give the following lemma for further use without proof.

Lemma 2.1. *Let a, b , and p be as in Section 1, and let $r = aW_2 - bW_1$. Then for all integers k ,*

$$\begin{aligned} b^2 U_{2k} + 2ab U_{2k-1} + a^2 U_{2k-2} &= W_k X_k, \\ b^2 U_{2k+1} + 2ab U_{2k} + a^2 U_{2k-1} &= W_{k+1} X_k + (-1)^k r, \\ b^2 V_{2k} + 2ab V_{2k-1} + a^2 V_{2k-2} &= X_k^2 + (-1)^k 2r, \\ b^2 V_{2k+1} + 2ab V_{2k} + a^2 V_{2k-1} &= X_k X_{k+1} + (-1)^k pr. \end{aligned} \quad (2.1)$$

Theorem 2.2. *Fix integers c, d , and m .*

(i) *If m is even, then for all integers $j > i$,*

$$\sum_{n=i}^j U_{mn+c} W_{mn+d} = \frac{U_{m(j-i+1)} X_{m(j+i)+c+d}}{\Delta U_m} - \frac{(-1)^c (j-i+1) X_{d-c}}{\Delta}. \quad (2.2)$$

(i) If m is odd, then for all integers $j > i$,

$$\sum_{n=i}^j U_{mn+c} W_{mn+d} = \begin{cases} \frac{U_{m(j-i+1)} W_{m(j+i)+c+d}}{V_m} & \text{if } j-i \equiv 1 \pmod{2}, \\ \frac{V_{m(j-i+1)} X_{m(j+i)+c+d}}{\Delta V_m} - \frac{(-1)^{c+j} X_{d-c}}{\Delta} & \text{if } j-i \equiv 0 \pmod{2}. \end{cases} \quad (2.3)$$

Proof. Using the Binet formulas, we compute

$$\begin{aligned} \sum_{n=i}^j U_{mn+c} W_{mn+d} &= \sum_{n=i}^j \left(\frac{\alpha^{mn+c} - \beta^{mn+c}}{\alpha - \beta} \right) \left(\frac{A\alpha^{mn+d} - B\beta^{mn+d}}{\alpha - \beta} \right) \\ &= \frac{1}{(\alpha - \beta)^2} \sum_{n=i}^j (A\alpha^{2mn+c+d} + B\beta^{2mn+c+d}) - \frac{(-1)^{mn+c}}{(\alpha - \beta)^2} (A\alpha^{d-c} + B\beta^{d-c}) \\ &= \frac{1}{\Delta} \sum_{n=i}^j X_{2mn+c+d} - \frac{(-1)^c}{\Delta} X_{d-c} \sum_{n=i}^j (-1)^{mn}. \end{aligned} \quad (2.4)$$

Since $X_n = W_{n-1} + W_{n+1}$, we can obtain that for even m

$$\sum_{n=i}^j X_{2mn+t} = \frac{U_{m(j-i+1)} X_{m(j+i)+t}}{U_m}. \quad (2.5)$$

The result follows. \square

For example, when $i = 2$, $m = 3$, $a = 0$, $b = c = p = 1$, and $d = 5$, we obtain

$$\sum_{n=2}^j F_{3n+1} F_{3n+5} = \frac{F_{3(j-1)} F_{3j+4}}{4}. \quad (2.6)$$

Theorem 2.3. Fix integers c, d , and m . Let $S = \sum_{n=i}^j (-1)^n U_{mn+c} W_{mn+d}$.

(1) If m is odd, then S equals

$$S = \frac{(-1)^j U_{m(j-i+1)} X_{m(j+i)+c+d}}{\Delta U_m} - \frac{(-1)^c (j-i+1) X_{d-c}}{\Delta}. \quad (2.7)$$

(2) If m is odd and the parities of i and j are the same, then S equals

$$\frac{(-1)^j V_{m(j-i+1)} X_{m(j+i)+c+d}}{\Delta V_m} - \frac{\left((-1)^j + (-1)^i \right) (-1)^c X_{d-c}}{2\Delta}. \quad (2.8)$$

(3) If m is odd and the parities of i and j are the different, then S equals

$$\frac{(-1)^j U_{m(j-i+1)} W_{m(j+i)+c+d}}{V_m} - \frac{\left((-1)^j + (-1)^i\right) (-1)^c X_{d-c}}{2}. \quad (2.9)$$

Proof. Consider

$$\begin{aligned} \sum_{n=i}^j (-1)^n U_{mn+c} W_{mn+d} &= \sum_{n=i}^j \left(\frac{\alpha^{mn+c} - \beta^{mn+c}}{\alpha - \beta} \right) \left(\frac{A\alpha^{mn+d} - B\beta^{mn+d}}{\alpha - \beta} \right) \\ &= \frac{1}{(\alpha - \beta)^2} \sum_{n=i}^j (-1)^n \left(A\alpha^{2mn+c+d} + B\beta^{2mn+c+d} \right) - \frac{(-1)^{mn+c}}{(\alpha - \beta)^2} \left(A\alpha^{d-c} + B\beta^{d-c} \right) \\ &= \frac{1}{\Delta} \sum_{n=i}^j (-1)^n X_{2mn+c+d} - \frac{1}{\Delta} X_{d-c} \sum_{n=i}^j (-1)^{(m+1)n+c}. \end{aligned} \quad (2.10)$$

Since $X_n = W_{n-1} + W_{n+1}$, for odd m , we find

$$\sum_{n=i}^j (-1)^n X_{2mn+c} = \frac{(-1)^j U_{m(j-i+1)} X_{m(i+j)+c}}{U_m}. \quad (2.11)$$

The result is now obtained by considering the values of $\sum_{n=i}^j (-1)^{(m+1)n+c}$. \square

Theorem 2.4. Fix integers c, d , and m . For all integers $j > i$,

$$\sum_{n=4i+1}^{4j} (-1)^{\binom{n+1}{2}} U_{mn+c} W_{mn+d} = \frac{U_{4m(j-i)}}{V_{2m}} \begin{cases} V_m W_{s+m} & \text{if } m \text{ is even,} \\ U_m X_{s+m} & \text{if } m \text{ is odd,} \end{cases} \quad (2.12)$$

$$\begin{aligned} &\sum_{n=4i}^{4j+3} (-1)^{\binom{n+1}{2}} U_{mn+c} W_{mn+d} \\ &= \frac{U_{4m(j-i+1)}}{V_{2m}} \begin{cases} U_m X_{s+3m} & \text{if } m \text{ is even,} \\ V_m W_{s+3m} & \text{if } m \text{ is odd,} \end{cases} \end{aligned}$$

$$\begin{aligned}
& \sum_{n=4i+3}^{4j} (-1)^{\binom{n+1}{2}} U_{mn+c} W_{mn+d} \\
&= \begin{cases} \frac{V_m V_{2m(2(j-i)-1)} X_{s+3m}}{\Delta V_{2m}} - \frac{2(-1)^c X_{d-c}}{\Delta} & \text{if } m \text{ is even,} \\ \frac{U_m V_{2m(2(j-i)-1)} W_{s+3m}}{V_{2m}} & \text{if } m \text{ is odd,} \end{cases} \\
& \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} U_{mn+c} W_{mn+d} \\
&= \begin{cases} \frac{U_m V_{2m(2(j-i)+1)} W_{s+5m}}{V_{2m}} & \text{if } m \text{ is even,} \\ \frac{V_m V_{m(4(j-i+1)-2)} X_{s+5m}}{\Delta V_{2m}} + \frac{2(-1)^c X_{d-c}}{\Delta} & \text{if } m \text{ is odd,} \end{cases}
\end{aligned} \tag{2.13}$$

where $s = m(4(j+i)) + c + d$.

Proof. Consider

$$\begin{aligned}
& \sum_{n=4i+1}^{4j} (-1)^{\binom{n+1}{2}} U_{mn+c} W_{mn+d} \\
&= \sum_{n=4i+1}^{4j} (-1)^{\binom{n+1}{2}} \left(\frac{\alpha^{mn+c} - \beta^{mn+c}}{\alpha - \beta} \right) \left(\frac{A\alpha^{mn+d} - B\beta^{mn+d}}{\alpha - \beta} \right) \\
&= \frac{1}{(\alpha - \beta)^2} \sum_{n=4i+1}^{4j} (-1)^{\binom{n+1}{2}} (A\alpha^{2mn+c+d} + B\beta^{2mn+c+d}) - \frac{(-1)^{mn+c}}{(\alpha - \beta)^2} (A\alpha^{d-c} + B\beta^{d-c}) \\
&= \frac{1}{\Delta} \sum_{n=4i+1}^{4j} (-1)^{\binom{n+1}{2}} X_{2mn+c+d} - \frac{1}{\Delta} X_{d-c} (-1)^c \sum_{n=4i+1}^{4j} (-1)^{\binom{n+1}{2}}.
\end{aligned} \tag{2.14}$$

Here we have that $\sum_{n=4i+1}^{4j} (-1)^{\binom{n+1}{2}} = 0$ and, by $X_n = W_{n-1} + W_{n+1}$,

$$\sum_{n=4i+1}^{4j} (-1)^{\binom{n+1}{2}} X_{2mn+c+d} = \frac{\Delta V_m U_{4m(j-i)} W_{m(4(j+i)+1)+c+d}}{V_{2m}}, \tag{2.15}$$

for even m . Now formula (2.12) follows. The remaining formulas are proven in a similar manner. \square

Notice that in (2.12)-(2.13), one limit of summation is even while the other is odd. Accordingly we have observed that each of (2.12)-(2.13) has a dual sum that is obtained with

the use of the rule below. We highlight this rule since it also applies to get certain groups of sums in Section 2.

From [3], we recall the rule for the formation of the dual sum.

- (1) Replace the even limit by the even limit corresponding to the other residue class modulo 4 and the odd limit by the odd limit corresponding to the other residue class modulo 4.
- (2) Calculate the subscripts on the right in accordance with the paragraph following (2.13).
- (3) Multiply the right side by -1 .

For example, for odd integer m , the dual of (2.13) is

$$\sum_{n=4i}^{4j+1} (-1)^{\binom{n+1}{2}} U_{mn+c} W_{mn+d} = -\frac{1}{\Delta} \left(\frac{V_m V_{2m(2(j-i)+1)} X_{s+m}}{V_{2m}} + 2(-1)^c X_{d-c} \right), \quad (2.16)$$

where s is defined as before.

Theorem 2.5. Fix integers c, d , and m .

- (i) If c and d have the same parities, then

$$\begin{aligned} & \sum_{n=4i+1}^{4j} (-1)^{\binom{n+1}{2}} W_{(2m+1)n+c} W_{(2m+1)n+d} \\ &= \frac{U_{2m+1} U_{4(2m+1)(j-i)}}{V_{2(2m+1)}} \times \left(X_{2(2m+1)(j+i)+t} X_{2(2m+1)(j+i)+t+1} + pr(-1)^t \right), \\ & \sum_{n=4i+1}^{4j} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} \\ &= \frac{V_{2m} U_{8m(j-i)}}{V_{4m}} \times \left(W_{m(4j+4i)+t} X_{m(4j+4i)+t} \right), \\ & \sum_{n=4i+3}^{4j} (-1)^{\binom{n+1}{2}} W_{(2m+1)n+c} W_{(2m+1)n+d} \\ &= \frac{U_{2m+1} V_{2(2m+1)(2(j-i)-1)}}{V_{2(2m+1)}} \times \left(W_{(2m+1)(2(j+i)+1)+t+1} X_{(2m+1)(2(j+i)+1)+t} - r(-1)^t \right), \end{aligned}$$

$$\begin{aligned}
& \sum_{n=4i+3}^{4j} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} \\
&= \frac{V_{2m} V_{4m(2(j-i)-1)}}{\Delta V_{4m}} \times \left(X_{m(4(j+i)+2)+t}^2 + 2r(-1)^t \right) + \frac{2r(-1)^c V_{d-c}}{\Delta}, \\
& \sum_{n=4i}^{4j+3} (-1)^{\binom{n+1}{2}} W_{(2m+1)n+c} W_{(2m+1)n+d} \\
&= \frac{V_{2m+1} U_{4(2m+1)(j-i+1)}}{V_{2(2m+1)}} \times \left(W_{(2m+1)(2(j+i)+1)+t+1} X_{(2m+1)(2(j+i)+1)+t} - r(-1)^t \right), \\
& \sum_{n=4i}^{4j+3} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} \\
&= \frac{U_{2m} U_{8m(j-i+1)}}{V_{4m}} \times \left(X_{m(4(j+i)+2)+t}^2 + 2r(-1)^t \right), \\
& \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} W_{(2m+1)n+c} W_{(2m+1)n+d} \\
&= \frac{V_{2m+1} V_{2(2m+1)(2(j-i)+1)}}{\Delta V_{2(2m+1)}} \times \left(X_{2(2m+1)(j+i+1)+t+1} X_{2(2m+1)(j+i+1)+t} + pr(-1)^t \right) - \frac{2r(-1)^c V_{d-c}}{\Delta}, \\
& \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} \\
&= \frac{1}{V_{4m}} \left(U_{2m} V_{4m(2(j-i)+1)} W_{4m(j+i+1)+t} X_{4m(j+i+1)+t} \right),
\end{aligned} \tag{2.17}$$

where $t = (c + d)/2 + m$.

(ii) If c and d have different parities, then

$$\begin{aligned}
& \sum_{n=4i+1}^{4j} (-1)^{\binom{n+1}{2}} W_{(2m+1)n+c} W_{(2m+1)n+d} \\
&= \frac{U_{2m+1} U_{4(2m+1)(j-i)}}{V_{2(2m+1)}} \left(X_{2(2m+1)(j+i)+v}^2 + 2r(-1)^v \right),
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=4i+1}^{4j} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} \\
&= \frac{V_{2m} U_{8m(j-i)}}{V_{4m}} (X_{4m(j+i)+v-1} W_{4m(j+i)+v} - r(-1)^v), \\
& \sum_{n=4i+3}^{4j} (-1)^{\binom{n+1}{2}} W_{(2m+1)n+c} W_{(2m+1)n+d} \\
&= \frac{1}{V_{2(2m+1)}} \times U_{2m+1} V_{2(2m+1)(2(j-i)-1)} W_{(2m+1)(2(j+i)+1)+v} X_{(2m+1)(2(j+i)+1)+v}, \\
& \sum_{n=4i+3}^{4j} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} \\
&= \frac{V_{2m} V_{4m(2(j-i)-1)}}{\Delta V_{4m}} \times (X_{m(4(j+i)+2)+v} X_{m(4(j+i)+2)+v-1} + pr(-1)^v) + \frac{2r(-1)^c V_{d-c}}{\Delta}, \\
& \sum_{n=4i}^{4j+3} (-1)^{\binom{n+1}{2}} W_{(2m+1)n+c} W_{(2m+1)n+d} \\
&= \frac{V_{2m+1} U_{4(2m+1)(j-i+1)}}{V_{2(2m+1)}} \times X_{(2m+1)(2(j+i)+1)+v} W_{(2m+1)(2(j+i)+1)+v}, \\
& \sum_{n=4i}^{4j+3} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} \\
&= \frac{U_{2m} U_{8m(j-i+1)}}{V_{4m}} \times (X_{m(4(j+i)+2)+v} X_{m(4(j+i)+2)+v-1} - pr(-1)^v), \\
& \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} W_{(2m+1)n+c} W_{(2m+1)n+d} \\
&= -\frac{2r(-1)^c V_{d-c}}{\Delta} + \frac{V_{2m+1} V_{2(2m+1)(2(j-i)+1)}}{\Delta V_{2(2m+1)}} (X_{2(2m+1)(j+i+1)+v}^2 + 2r(-1)^v), \\
& \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} \\
&= \frac{U_{2m} V_{4m(2(j-i)+1)}}{V_{4m}} \times (W_{4m(j+i+1)+v} X_{4m(j+i+1)+v-1} - r(-1)^v),
\end{aligned} \tag{2.18}$$

where r is defined as before and $v = (c + d + 1)/2 + m$.

Proof. Suppose that c and d have the same parities. Consider

$$\begin{aligned}
& \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} \\
&= \frac{1}{\Delta} \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} \left(A^2 \alpha^{4mn+c+d} + B^2 \beta^{4mn+c+d} - AB \alpha^{2mn+c} \beta^{2mn+d} - AB \beta^{2mn+c} \alpha^{2mn+d} \right) \\
&= \frac{1}{\Delta} \left(b^2 \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} V_{4mn+c+d} + 2ab \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} V_{4mn+c+d-1} \right. \\
&\quad \left. + a^2 \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} V_{4mn+c+d-2} \right) \\
&\quad + \frac{1}{\Delta} (-1)^c r V_{d-c} \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}}.
\end{aligned} \tag{2.19}$$

From the definition of $\{V_n\}$, we obtain

$$\sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} V_{4mn+c} = \frac{\Delta U_{2m} V_{4m(2(j-i)+1)} U_{2m(4(j+i+1)+1)+c}}{V_{4m}}. \tag{2.20}$$

Since

$$\sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} = 0, \tag{2.21}$$

we get

$$\begin{aligned}
& \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} \\
&= \frac{U_{2m} V_{4m(2(j-i)+1)}}{V_{4m}} \left(b^2 U_{2m(4(j+i+1)+1)+c+d} + 2ab U_{2m(4(j+i+1)+1)+c+d-1} + a^2 U_{2m(4(j+i+1)+1)+c+d-2} \right)
\end{aligned} \tag{2.22}$$

Taking $2k = 2m(4(j+i+1)+1)+c+d$ in Lemma 2.1, we write

$$\begin{aligned}
& \sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} \\
&= \frac{U_{2m} V_{4m(2(j-i)+1)} W_{m(4(j+i+1)+1)+(d+c)/2} X_{m(4(j+i+1)+1)+(d+c)/2}}{\Delta V_{4m}}.
\end{aligned} \tag{2.23}$$

Thus the result follows. Similar arguments yield the remaining formulas, where we must consider the parities of c, d . \square

For example, the dual of (2.17) is given by if c and d have the same parities,

$$\sum_{n=4i}^{4j+1} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} = -\frac{U_{2m} V_{4m(2(j-i)+1)} W_{4m(j+i)+t} X_{4m(j+i)+t}}{V_{4m}}, \quad (2.24)$$

and the dual of (2.18) is given by if c and d have different parities,

$$\begin{aligned} \sum_{n=4i}^{4j+1} (-1)^{\binom{n+1}{2}} W_{2mn+c} W_{2mn+d} \\ = -\frac{U_{2m} V_{4m(2(j-i)+1)}}{V_{4m}} \times (W_{4m(j+i)+v} X_{4m(j+i)+v-1} - r(-1)^v), \end{aligned} \quad (2.25)$$

where t and v are defined as before.

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