Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 968065, 12 pages
http://dx.doi.org/10.1155/2014/968065

Research Article

Valuation of Credit Derivatives with Multiple Time Scales in the Intensity Model

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Received 30 April 2014; Accepted 5 August 2014; Published 21 August 2014

Academic Editor: Allan C. Peterson

We propose approximate solutions for pricing zero-coupon defaultable bonds, credit default swap rates, and bond options based on the averaging principle of stochastic differential equations. We consider the intensity-based defaultable bond, where the volatility of the default intensity is driven by multiple time scales. Small corrections are computed using regular and singular perturbations to the intensity of default. The effectiveness of these corrections is tested on the bond price and yield curve by investigating the behavior of the time scales with respect to the relevant parameters.

1. Introduction

It is well-known that the methodology for modeling a credit risk can be split into two primary models that attempt to describe default processes: the structural model and the intensity-based model. Structural models assume the market has complete information with respect to the underlying firm's value process and knows the details of the underlying firm's capital structure. In contrast, intensity-based (or reduced-form) models have been developed under the assumption that the default is the first jump of an exogenously given jump process. Hence, the underlying firm's default time is inaccessible and driven by a default intensity function of some latent state variables. Because we are concerned with modeling the default time, we adopt an intensity-based model with the fractional recovery assumption made by Duffie and Singleton [1], which can be written as follows:

$$\mathscr{Z}(t,T) = E^{*}\left[e^{-\int_t^T \{r_s+(1-R)\lambda_s\}ds} \mid \mathscr{F}_t\right], \tag{1}$$

under a risk-neutral probability P^*, where t is the current time, T is the time to maturity, $R \in [0,1]$ is the recovery rate, and $\mathscr{F}_t$ is the filtration generated by the joint process of the interest rate and intensity, denoted by r_s and λ_s, respectively.

Since the initial contribution to intensity-based modeling, given by Jarrow and Turnbull [2], who considered a constant Poisson intensity, there have been many mathematical studies on credit risk. Among them, Lando [3] used a default term described by a Cox process, whereas Schönbucher [4] developed the term-structure model of defaultable interest rates using the Heath-Jarrow-Morton approach. There are also specific models for interest rate and intensity processes, depending on the macroeconomic environment. For example, Duffee [5] performed the estimation of an intensity-based model using an extended Kalman filter approach and Ma and Kim [6] provided a pricing formula for the credit default swap (CDS) by modeling the intensities as jump-diffusion processes. They provided affine processes for the interest rate and intensity steps, supposing a zero correlation between the two processes. However, several empirical papers have found a nonzero correlation for the interest rate and intensity processes. To model this, Tchuindjo [7] provided closed-form solutions for pricing zero-coupon defaultable bonds.

The aim of this paper is to investigate the effect of multiple time scales of default in pricing defaultable bonds. This is based on modeling the volatility of the default intensity via fast and slow time scales. Papageorgiou and Sircar [8] studied the pricing of defaultable derivatives, such as bonds, bond options, and CDS rates, by intensity-based models under a two-factor diffusion model for the default intensity. Their study was based on multiple time scales, as developed by Fouque et al. [9], where the evolution of the stochastic default rate is subject to fast and slow scale variations and showed empirical evidence for the existence of multiple time scales to price defaultable bonds. However, they do not assume a specific process for the default intensity to derive approximate solutions for the pricing of defaultable derivatives. Also, the correlated Hull and White model, developed by Tchuindjo, did not produce a hump-shaped yield curve that matches a typical yield curve for defaultable bonds, as in the Merton model [10]. Thus, we mathematically supplement Papageorgiou and Sircar's model with the specific process for the default intensity and expand Tchuindjo's model by modifying multiple time scales in the stochastic volatility of the intensity process. For the analytic tractability of multiple time scales, we use an asymptotic analysis, developed by Fouque et al. [11] for stochastic volatility in equity models and obtain approximations of the pricing functions of zero-coupon defaultable bonds, CDS rates, and bond options. Numerical examples indicate that the multiple time scales have both quantitative and qualitative effects and that the zero-coupon defaultable bond with a stochastic default intensity tends to be mispriced in terms of the relevant parameters. In addition, our results are compared to those in Fouque et al. [12], who studied the price of defaultable bonds with stochastic volatility using the structural model and a method to relax the drawbacks of the affine processes.

The remainder of this paper is structured as follows. In Section 2, we obtain a partial differential equation (PDE) with multiple time scales to price zero-coupon defaultable bonds. In Section 3, approximate solutions for zero-coupon defaultable bond prices are derived using an asymptotic analysis that includes the behavior of multiple time scales in terms of the relevant parameters and some numerical examples are provided. Section 4 applies the results of Section 3 to CDS rates and bond options, respectively. We present our concluding remarks in Section 5.

2. Pricing Equation for Zero-Coupon Defaultable Bonds

Alizadeh et al. [13] and Masoliver and Perelló [14] have shown that two volatility factors, which not only control the persistence of volatility but also revert rapidly to the mean and contribute to the volatility of volatility, as pointed out by the empirical findings, are in operation. Fouque et al. [11] also attempted to balance within a sing model involved in different time scales. That is, empirical evidence suggests that there are two volatility factors. To focus on the intensity process of intensity-based defaultable bonds, we replace the constant volatility of the default intensity with a stochastic term, denoted by $\tilde{\sigma}_t$. The value of $\tilde{\sigma}_t$ is given by a bounded, smooth, and strictly positive function f such that $\tilde{\sigma}_t = f(y_t, z_t)$, with a fast scale-factor process y_t and a slow scale-factor process z_t. Note that leaving the choice of f free affords sufficient flexibility, while our pricing functions are unchanged by different choices. Refer to Fouque et al. [15]. To model this, we take the first factor y_t driving the volatility $\tilde{\sigma}_t$ as a fast mean-reverting process. For the analytic goal, we choose the following Ornstein-Uhlenbeck (OU) process:

$$dy_t = \alpha\left(m - y_t\right)dt + \beta\, dB_t^{(y)*}, \tag{2}$$

where $B_t^{(y)*}$ is a standard Brownian motion in the risk-neutral probability P^*. It is well-known that the solution of (2) is a Gaussian process given by

$$y_t = m + \left(y_0 - m\right)e^{-\alpha t} + \beta\int_0^t e^{-\alpha(t-s)}dB_s^{(y)*} \tag{3}$$

and $y_t \sim N(m + (y_0 - m)e^{-\alpha t}, (\beta^2/2\alpha)(1 - e^{-2\alpha t}))$, leading to an invariant distribution given by $N(m, \beta^2/2\alpha)$. Note that we would use the notation $\langle\cdot\rangle$, called the solvability condition, for the average with respect to the invariant distribution; that is, for an arbitrage function k,

$$\langle k\left(y_t\right)\rangle = \frac{1}{\sqrt{2\pi\nu^2}}\int_{-\infty}^{+\infty} k\left(y\right)e^{-(y-m)^2/2\nu^2}dy, \tag{4}$$

where $\nu^2 = \beta^2/2\alpha$. For the asymptotic analysis, the rate of mean reversion α in the process y_t is large and its inverse $\epsilon = 1/\alpha$ is the typical correlation of the process y_t with $\epsilon > 0$ being a small parameter. We assume that the order of ν^2 remains fixed in scale as ϵ becomes zero. Hence, we get

$$\alpha = \mathcal{O}\left(\epsilon^{-1}\right), \qquad \beta = \mathcal{O}\left(\epsilon^{-1/2}\right), \qquad \nu = \mathcal{O}\left(1\right). \tag{5}$$

From these assumptions, we replace the stochastic differential equation (SDE) (2) by

$$dy_t = \frac{1}{\epsilon}\left(m - y_t\right)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}dB_t^{(y)*}. \tag{6}$$

Note that the fast scale volatility factor has been considered as a singular perturbation case. Also, the volatility $\tilde{\sigma}_t$ has a slowly varying factor. Here, we choose as follows:

$$d\hat{z}_t = g\left(\hat{z}_t\right)dt + h\left(\hat{z}_t\right)d\hat{B}_t^{(z)*}, \tag{7}$$

where $\hat{B}_t^{(z)*}$ is a standard Brownian motion in the risk-neutral probability P^*. We assume that Lipschitz and growth conditions for the coefficients $g(z)$ and $h(z)$ are satisfied, respectively. For the asymptotic analysis, we use another small parameter $\delta > 0$ and then change time t to δt in $\hat{z}_t$. This means that $z_t \overset{d}{=} \hat{z}_{\delta t}$, so that this alteration replaces SDE (7) by

$$dz_t = \delta g\left(z_t\right)dt + \sqrt{\delta}h\left(z_t\right)dB_t^{(z)*}, \tag{8}$$

where $B_t^{(z)*}$ is another standard Brownian motion in the risk-neutral probability P^*. Note that the slow scale volatility factor has been considered as a regular perturbation situation. See, for example, Fouque et al. [11]. Consequently, the dynamics of the interest rate process r_t, intensity process λ_t, and stochastic volatility process $\tilde{\sigma}_t$ are given by the SDEs

$$\begin{aligned} dr_t &= (\theta_t - ar_t)\,dt + \sigma dB_t^{(r)*}, \\ d\lambda_t &= (\tilde{\theta}_t - \tilde{a}\lambda_t)\,dt + \tilde{\sigma}_t dB_t^{(\lambda)*}, \\ \tilde{\sigma}_t &= f(y_t, z_t), \\ dy_t &= \frac{1}{\epsilon}(m - y_t)\,dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dB_t^{(y)*}, \\ dz_t &= \delta g(z_t)\,dt + \sqrt{\delta} h(z_t)\,dB_t^{(z)*}, \end{aligned} \tag{9}$$

under the risk-neutral probability P^*, where $a, \tilde{a}$, and σ are constants, θ_t and $\tilde{\theta}_t$ are time-varying deterministic functions, and the standard Brownian motions $B_t^{(r)*}$, $B_t^{(\lambda)*}$, $B_t^{(y)*}$, and $B_t^{(z)*}$ are dependent on each other with the correlation structure given by

$$\begin{aligned} &d\left\langle B^{(r)*}, B^{(\lambda)*}\right\rangle_t = \rho_{r\lambda} dt, && d\left\langle B^{(r)*}, B^{(y)*}\right\rangle_t = \rho_{ry} dt, \\ &d\left\langle B^{(r)*}, B^{(z)*}\right\rangle_t = \rho_{rz} dt, && d\left\langle B^{(\lambda)*}, B^{(y)*}\right\rangle_t = \rho_{\lambda y} dt, \\ &d\left\langle B^{(\lambda)*}, B^{(z)*}\right\rangle_t = \rho_{\lambda z} dt, && d\left\langle B^{(y)*}, B^{(z)*}\right\rangle_t = \rho_{yz} dt. \end{aligned} \tag{10}$$

The zero-coupon defaultable bond price with the fractional recovery assumption at time t for an interest rate process level $r_t = r$, an intensity process level $\lambda_t = \lambda$, a fast volatility level $y_t = y$, and a slow volatility level $z_t = z$, denoted by $P(t, r, \lambda, y, z; T)$, is given by

$$\begin{aligned} &P(t, r, \lambda, y, z; T) \\ &= E^*\left[e^{-\int_t^T (r_s + (1-R)\lambda_s) ds} \mid r_t = r, \lambda_t = \lambda, y_t = y, z_t = z\right], \end{aligned} \tag{11}$$

and then using the four-dimensional Feynman-Kac formula, we obtain the Kolmogorov PDE

$$\begin{aligned} &\frac{\partial P}{\partial t} + (\theta_t - ar)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} + (\tilde{\theta}_t - \tilde{a}\lambda)\frac{\partial P}{\partial \lambda} \\ &\quad + \frac{1}{2} f^2(y,z)\frac{\partial^2 P}{\partial \lambda^2} + \frac{1}{\epsilon}(m - y)\frac{\partial P}{\partial y} + \frac{\nu^2}{\epsilon}\frac{\partial^2 P}{\partial y^2} + \delta g(z)\frac{\partial P}{\partial z} \\ &\quad + \frac{1}{2}\delta h^2(z)\frac{\partial^2 P}{\partial z^2} + \rho_{r\lambda}\sigma f(y,z)\frac{\partial^2 P}{\partial r \partial \lambda} + \rho_{ry}\sigma\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\frac{\partial^2 P}{\partial r \partial y} \\ &\quad + \rho_{rz}\sigma\sqrt{\delta} h(z)\frac{\partial^2 P}{\partial r \partial z} + \rho_{\lambda y} f(y,z)\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\frac{\partial^2 P}{\partial \lambda \partial y} \\ &\quad + \rho_{\lambda z} f(y,z)\sqrt{\delta} h(z)\frac{\partial^2 P}{\partial \lambda \partial z} + \rho_{yz}\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\sqrt{\delta} h(z)\frac{\partial^2 P}{\partial y \partial z} \\ &\quad - \{r + (1-R)\lambda\} P = 0 \end{aligned} \tag{12}$$

with the final condition $P(t, r, \lambda, y, z; T)|_{t=T} = 1$. Refer to Øksendal [16].

3. Asymptotic Analysis of Zero-Coupon Defaultable Bonds

In this section, we employ an asymptotic analysis for the solution of the PDE (12) and give an approximate solution for the zero-coupon defaultable bond price for the small independent parameters ϵ and δ.

The zero-coupon defaultable bond price function $P(t, r, \lambda, y, z; T)$ consists of the Feynman-Kac formula problems:

$$\begin{aligned} &\mathscr{L} P(t, r, \lambda, y, z; T) = 0, \quad t < T, \\ &P(t, r, \lambda, y, z; T)|_{t=T} = 1, \end{aligned} \tag{13}$$

where

$$\mathscr{L} := \frac{1}{\epsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_1 + \mathscr{L}_2 + \sqrt{\delta}\mathscr{M}_1 + \delta\mathscr{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathscr{M}_3, \tag{14}$$

and $\mathscr{L}_0, \mathscr{L}_1, \mathscr{L}_2, \mathscr{M}_1, \mathscr{M}_2$, and $\mathscr{M}_3$ are satisfied by

$$\begin{aligned} \mathscr{L}_0 &= (m - y)\frac{\partial}{\partial y} + \nu^2\frac{\partial^2}{\partial y^2}, \\ \mathscr{L}_1 &= \sqrt{2}\sigma\nu\rho_{ry}\frac{\partial^2}{\partial r \partial y} + \sqrt{2} f(y,z)\nu\rho_{\lambda y}\frac{\partial^2}{\partial \lambda \partial y}, \\ \mathscr{L}_2 &= \frac{\partial}{\partial t} + (\theta_t - ar)\frac{\partial}{\partial r} + (\tilde{\theta}_t - \tilde{a}\lambda)\frac{\partial}{\partial \lambda} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2} \\ &\quad + \frac{1}{2} f^2(y,z)\frac{\partial^2}{\partial \lambda^2} + \sigma f(y,z)\rho_{r\lambda}\frac{\partial^2}{\partial r \partial \lambda} \\ &\quad - \{r + (1-R)\lambda\}, \\ \mathscr{M}_1 &= \sigma h(z)\rho_{rz}\frac{\partial^2}{\partial r \partial z} + f(y,z) h(z)\rho_{\lambda z}\frac{\partial^2}{\partial \lambda \partial z}, \\ \mathscr{M}_2 &= g(z)\frac{\partial}{\partial z} + \frac{1}{2}h^2(z)\frac{\partial^2}{\partial z^2}, \\ \mathscr{M}_3 &= \sqrt{2}\nu h(z)\rho_{yz}\frac{\partial^2}{\partial y \partial z}. \end{aligned} \tag{15}$$

Here, $\alpha\mathscr{L}_0$ is the infinitesimal generator of the OU process y_t, $\mathscr{L}_1$ contains the mixed partial derivatives due to the correlation between r_t and y_t and between λ_t and y_t, $\mathscr{L}_2$ is the operator of the correlated Hull and White model with constant volatility at the volatility level $f(y,z)$, $\mathscr{M}_1$ includes the mixed partial derivatives due to the correlation between r_t and z_t and between λ_t and z_t, $\mathscr{M}_2$ is the infinitesimal generator with respect to z_t, and $\mathscr{M}_3$ contains the mixed partial derivative due to the correlation between y_t and z_t.

Now, we use the notation $P_{i,j}$ for the $\epsilon^{i/2}\delta^{j/2}$-order term, for $i = 0,1,2,\ldots$ and $j = 0,1,2,\ldots$. We first expand with respect to half-powers of δ and then for each of these terms we expand with respect to half-powers of ϵ. This choice is somewhat simpler than the reverse ordering. Hence, we consider an expansion of $P^{\epsilon,\delta}$:

$$P^{\epsilon,\delta} = P_0^{\epsilon} + \sqrt{\delta}P_1^{\epsilon} + \delta P_2^{\epsilon} + \cdots, \tag{16}$$

$$P_k^{\epsilon} = P_{0,k} + \sqrt{\epsilon}P_{1,k} + \epsilon P_{2,k} + \epsilon^{3/2}P_{3,k} + \cdots, \tag{17}$$

for $k = 0,1,2,\ldots$. The expansion (16) leads to the leading-order term P_0^{ϵ} and the next-order term P_1^{ϵ} given by the solutions of the PDEs

$$\left(\frac{1}{\epsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_1 + \mathscr{L}_2\right)P_0^{\epsilon} = 0, \quad t < T, \tag{18}$$

with the terminal condition $P_0^{\epsilon}(t,r,\lambda,y,z;T)|_{t=T} = 1$ and

$$\left(\frac{1}{\epsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_1 + \mathscr{L}_2\right)P_1^{\epsilon} = \left(\mathscr{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathscr{M}_3\right)P_0^{\epsilon}, \quad t < T, \tag{19}$$

with the terminal condition $P_1^{\epsilon}(t,r,\lambda,y,z;T)|_{t=T} = 0$, respectively.

3.1. Leading-Order Term $P_{0,0}$. We insert $k = 0$ into (17) as

$$P_0^{\epsilon} = P_{0,0} + \sqrt{\epsilon}P_{1,0} + \epsilon P_{2,0} + \epsilon^{3/2}P_{3,0} + \cdots. \tag{20}$$

Applying the expanded solution (20) to (18) leads to

$$\begin{aligned}&\frac{1}{\epsilon}\mathscr{L}_0 P_{0,0} + \frac{1}{\sqrt{\epsilon}}\left(\mathscr{L}_0 P_{1,0} + \mathscr{L}_1 P_{0,0}\right)\\ &\quad + \left(\mathscr{L}_0 P_{2,0} + \mathscr{L}_1 P_{1,0} + \mathscr{L}_2 P_{0,0}\right)\\ &\quad + \sqrt{\epsilon}\left(\mathscr{L}_0 P_{3,0} + \mathscr{L}_1 P_{2,0} + \mathscr{L}_2 P_{1,0}\right) + \cdots = 0.\end{aligned} \tag{21}$$

Theorem 1. *The leading order term $P_{0,0}$ of the expansion* (17) *with $k = 0$ is independent of the fast scale variable y and further it has the affine representation*

$$P_{0,0}(t,r,\lambda,z;T) = e^{F(t,T)-B(t,T)r-(1-R)C(t,T)\lambda}, \tag{22}$$

with $B(T,T) = C(T,T) = F(T,T) = 0$, where $B(t,T), C(t,T)$, and $F(t,T)$ are given by

$$\begin{aligned}&B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}, \qquad C(t,T) = \frac{1}{\tilde{a}}\left\{1 - e^{-\tilde{a}(T-t)}\right\},\\
&F(t,T)\\
&\quad = \ln\frac{P_{0,0}(0,T)}{P_{0,0}(0,t)} - C(t,T)\frac{\partial}{\partial s}\ln P_{0,0}(0,s)\Big|_{s=t}\\
&\qquad -(B(t,T)-C(t,T))\frac{\partial}{\partial s}\ln \mathscr{W}(0,s)\Big|_{s=t}\\
&\qquad + \frac{\sigma^2}{4a^3}\left\{\left(e^{-aT} - e^{-at}\right)^2 - a^2B^2(t,T)\right\}\\
&\qquad + \frac{\rho_{r\lambda}\sigma\overline{\sigma}(z)}{a\tilde{a}(a+\tilde{a})}(1-R)\\
&\qquad \times\left\{(a+\tilde{a})(T-t) - (\tilde{a}B(t,T)+1)(aC(t,T)+1)+1\right\}\\
&\qquad + \frac{\breve{\sigma}^2(z)}{4\tilde{a}^3}(1-R)\Big[\left(e^{-\tilde{a}T} - e^{-\tilde{a}t}\right)^2 - \tilde{a}^2C^2(t,T)\\
&\qquad\qquad - R\left\{2\tilde{a}(T-t) - (\tilde{a}C(t,T)+3)\right.\\
&\qquad\qquad\quad \left.\times(\tilde{a}C(t,T)-1)-3\right\}\Big],\end{aligned} \tag{23}$$

respectively. Here,

$$\begin{aligned}&\overline{\sigma}(z) := \langle f\rangle(z) = \frac{1}{\sqrt{2\pi\nu^2}}\int_{-\infty}^{+\infty} f(y,z)\,e^{-(y-m)^2/2\nu^2}dy,\\
&\breve{\sigma}^2(z) := \langle f^2\rangle(z) = \frac{1}{\sqrt{2\pi\nu^2}}\int_{-\infty}^{+\infty} f^2(y,z)\,e^{-(y-m)^2/2\nu^2}dy,\end{aligned} \tag{24}$$

and $\mathscr{W}(0,s)$ is the price of a zero-coupon default-free bond according to Hull and White [17].

Proof. Multiply (21) by ϵ and then let $\epsilon \to 0$. This gives the first two leading-order terms as

$$\begin{aligned}&\mathscr{L}_0 P_{0,0} = 0,\\ &\mathscr{L}_0 P_{1,0} + \mathscr{L}_1 P_{0,0} = 0.\end{aligned} \tag{25}$$

Recall that, because the infinitesimal operator $\mathscr{L}_0$ is the generator of the OU process y_t, the solution $P_{0,0}$ of (25) must be a constant with respect to the y variable; that is, $P_{0,0} = P_{0,0}(t,r,\lambda,z;T)$. Similarly, because $P_{0,0}$ does not rely on the y variable, we get $\mathscr{L}_1 P_{0,0} = 0$ and then $\mathscr{L}_0 P_{1,0} = 0$; that is, $P_{1,0} = P_{1,0}(t,r,\lambda,z;T)$. This means that the first two terms $P_{0,0}$ and $P_{1,0}$ do not depend on the current level y of the fast scale volatility driving the process y_t. In this way, we can continue to eliminate the terms of order $1, \sqrt{\epsilon}, \epsilon, \ldots$. For the order-1 term, we get $\mathscr{L}_0 P_{2,0} + \mathscr{L}_1 P_{1,0} + \mathscr{L}_2 P_{0,0} = 0$. This PDE becomes

$$\mathscr{L}_0 P_{2,0} + \mathscr{L}_2 P_{0,0} = 0, \tag{26}$$

because $\mathscr{L}_1 P_{1,0} = 0$. This PDE is a Poisson equation for $P_{2,0}$ with respect to the infinitesimal operator $\mathscr{L}_0$. It is well-known that a solution exists only if $\mathscr{L}_2 P_{0,0}$ is centered with respect to the invariant distribution of the stochastic volatility process y_t; that is,

$$\left\langle \mathscr{L}_2 P_{0,0} \right\rangle = 0, \quad t < T \tag{27}$$

with the terminal condition $P_{0,0}(t,r,\lambda,z;T)|_{t=T} = 1$. Because $P_{0,0}$ does not rely on the y variable, the solvability condition becomes

$$\left\langle \mathscr{L}_2 \right\rangle P_{0,0} = 0. \tag{28}$$

Here, $\langle \mathscr{L}_2 \rangle$ is a partial differential operator given by

$$\begin{aligned}\left\langle \mathscr{L}_2 \right\rangle = {} & \frac{\partial}{\partial t} + (\theta_t - ar)\frac{\partial}{\partial r} + \left(\tilde{\theta}_t - \tilde{a}\lambda\right)\frac{\partial}{\partial \lambda} \\ & + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial r^2} + \frac{1}{2}\breve{\sigma}^2(z)\frac{\partial^2}{\partial \lambda^2} + \sigma\overline{\sigma}(z)\rho_{r\lambda}\frac{\partial^2}{\partial r \partial \lambda} \\ & - \{r + (1-R)\lambda\}.\end{aligned} \tag{29}$$

Finally, substituting (22) into (28), we directly obtain the result of Theorem 1 using the results of Tchuindjo [7]. □

Note that in the stochastic volatility setting, we get the leading order term $P_{0,0}$, which is a generalization of the constant volatility result obtained by Tchuindjo [7]. If $f(y,z)$ is a constant function, then our results (22) reduce to those in Tchuindjo's study.

3.2. First Perturbation Term $P^{\epsilon}_{1,0}$. In this subsection, we precisely calculate $P^{\epsilon}_{1,0}$ by using the result of Theorem 1.

Theorem 2. *The correction term $P_{1,0}$ is independent of the variable y and $P^{\epsilon}_{1,0} := \sqrt{\epsilon}P_{1,0}(t,r,\lambda,z;T)$ is given by*

$$P^{\epsilon}_{1,0}(t,r,\lambda;T) = G(t,T)\,e^{F(t,T)-B(t,T)r-(1-R)C(t,T)\lambda}, \tag{30}$$

with $B(T,T) = C(T,T) = F(T,T) = G(T,T) = 0$, where $G(t,T)$ is given by

$$\begin{aligned} G(t,T) = {} & -\int_t^T \left[U^{\epsilon}_1\left\{(1-R)^2 B(s,T)C^2(s,T)\right\}\right. \\ & + U^{\epsilon}_2\left\{(1-R)B^2(s,T)C(s,T)\right\} \\ & \left. + U^{\epsilon}_3\left\{(1-R)^3 C^3(s,T)\right\}\right] ds, \\ U^{\epsilon}_1 := {} & \frac{\nu\sqrt{2}}{\sqrt{\alpha}}\rho_{ry}\sigma\left\langle\theta'\right\rangle(z) + \frac{\nu\sqrt{2}}{\sqrt{\alpha}}\rho_{\lambda y}\left\langle f\psi'\right\rangle(z), \\ U^{\epsilon}_2 := {} & \frac{\nu\sqrt{2}}{\sqrt{\alpha}}\rho_{ry}\sigma\left\langle\psi'\right\rangle(z), \\ U^{\epsilon}_3 := {} & \frac{\nu\sqrt{2}}{\sqrt{\alpha}}\rho_{\lambda y}\left\langle f\theta'\right\rangle(z). \end{aligned} \tag{31}$$

Here, $B(t,T)$, $C(t,T)$, and $F(t,T)$ are given by (23).

Proof. The order-$\sqrt{\epsilon}$ term in (21) leads to $\mathscr{L}_0 P_{3,0} + \mathscr{L}_1 P_{2,0} + \mathscr{L}_2 P_{1,0} = 0$, which is a Poisson equation for $P_{3,0}$ with respect to the infinitesimal operator $\mathscr{L}_0$ whose solvability condition is given by

$$\left\langle \mathscr{L}_1 P_{2,0} + \mathscr{L}_2 P_{1,0}\right\rangle = 0. \tag{32}$$

From (26) and (28), we obtain

$$P_{2,0} = -\mathscr{L}_0^{-1}\left(\mathscr{L}_2 - \left\langle\mathscr{L}_2\right\rangle\right)P_{0,0} + n(t,r,\lambda,z), \tag{33}$$

for some function $n(t,r,\lambda,z)$. Inserting (33) into (32), we derive a PDE for $P_{1,0}$ as follows:

$$\left\langle\mathscr{L}_2\right\rangle P_{1,0} = \left\langle \mathscr{L}_1\mathscr{L}_0^{-1}\left(\mathscr{L}_2 - \left\langle\mathscr{L}_2\right\rangle\right)\right\rangle P_{0,0}, \tag{34}$$

with the final condition $P_{1,0}(t,r,\lambda,z;T)|_{t=T} = 0$. Because our focus is only the first perturbation to $P_{0,0}$, we reset the PDE (34) with respect to $P^{\epsilon}_{1,0}(t,r,\lambda,z;T) := \sqrt{\epsilon}P_{1,0}(t,r,\lambda,z;T)$ as follows:

$$\begin{aligned}\left\langle\mathscr{L}_2\right\rangle P^{\epsilon}_{1,0} &= \mathscr{A}^{\epsilon}P_{0,0}, \\ \mathscr{A}^{\epsilon} &:= \frac{1}{\sqrt{\alpha}}\left\langle \mathscr{L}_1\mathscr{L}_0^{-1}\left(\mathscr{L}_2 - \left\langle\mathscr{L}_2\right\rangle\right)\right\rangle.\end{aligned} \tag{35}$$

From (15) and (29), we have

$$\begin{aligned}\mathscr{L}_2 - \left\langle\mathscr{L}_2\right\rangle = {} & \frac{1}{2}\left(f^2(y,z) - \left\langle f^2\right\rangle(z)\right)\frac{\partial^2}{\partial\lambda^2} \\ & + \rho_{r\lambda}\sigma\left(f(y,z) - \left\langle f\right\rangle(z)\right)\frac{\partial^2}{\partial r\partial\lambda}.\end{aligned} \tag{36}$$

Then, we introduce functions $\theta : R^2 \to R$ and $\psi : R^2 \to R$ defined by the solutions of

$$\begin{aligned}\mathscr{L}_0\theta(y,z) &= \frac{1}{2}\left(f^2(y,z) - \left\langle f^2\right\rangle(z)\right), \\ \mathscr{L}_0\psi(y,z) &= \rho_{r\lambda}\sigma\left(f(y,z) - \left\langle f\right\rangle(z)\right),\end{aligned} \tag{37}$$

respectively, and obtain the operator $\mathscr{L}^{\epsilon}$ denoted by

$$\mathscr{L}^{\epsilon} = U^{\epsilon}_1\frac{\partial^3}{\partial r\partial\lambda^2} + U^{\epsilon}_2\frac{\partial^3}{\partial r^2\partial\lambda} + U^{\epsilon}_3\frac{\partial^3}{\partial\lambda^3}. \tag{38}$$

Hence, we obtain (39) in the following form:

$$\begin{aligned} & \frac{\partial P^{\epsilon}_{1,0}}{\partial t} + (\theta_t - ar)\frac{\partial P^{\epsilon}_{1,0}}{\partial r} + \left(\tilde{\theta}_t - \tilde{a}\lambda\right)\frac{\partial P^{\epsilon}_{1,0}}{\partial\lambda} + \frac{1}{2}\sigma^2\frac{\partial^2 P^{\epsilon}_{1,0}}{\partial r^2} \\ & \quad + \frac{1}{2}\breve{\sigma}^2(z)\frac{\partial^2 P^{\epsilon}_{1,0}}{\partial\lambda^2} + \rho_{r\lambda}\sigma\overline{\sigma}(z)\frac{\partial^2 P^{\epsilon}_{1,0}}{\partial r\partial\lambda} - (r + (1-R)\lambda)P^{\epsilon}_{1,0} \\ & \quad = -\left[U^{\epsilon}_1\left\{(1-R)^2 B(t,T)C^2(t,T)\right\}\right. \\ & \qquad + U^{\epsilon}_2\left\{(1-R)B^2(t,T)C(t,T)\right\} \\ & \qquad \left. + U^{\epsilon}_3\left\{(1-R)^3 C^3(t,T)\right\}\right]P_{0,0}, \end{aligned} \tag{39}$$

with the terminal condition $P^{\epsilon}_{1,0}(t,r,\lambda,z;T)|_{t=T} = 0$. Finally, substituting (30) into (39), we obtain the result of Theorem 2 by direct computation. □

3.3. First Perturbation $P^{\delta}_{0,1}$. Using similar arguments to those in Section 3.2, we will derive the first perturbation term $P^{\delta}_{0,1}$. We insert $k=1$ in (17) as

$$P^{\epsilon}_{1} = P_{0,1} + \sqrt{\epsilon}P_{1,1} + \epsilon P_{2,1} + \epsilon^{3/2}P_{3,1} + \cdots. \tag{40}$$

Substituting the expansions (20) and (40) into (19), we get

$$\begin{aligned}
&\frac{1}{\epsilon}\mathscr{L}_0 P_{0,1} + \frac{1}{\sqrt{\epsilon}}\left(\mathscr{L}_0 P_{1,1} + \mathscr{L}_1 P_{0,1}\right) \\
&\quad + \left(\mathscr{L}_0 P_{2,1} + \mathscr{L}_1 P_{1,1} + \mathscr{L}_2 P_{0,1}\right) \\
&\quad + \sqrt{\epsilon}\left(\mathscr{L}_0 P_{3,1} + \mathscr{L}_1 P_{2,1} + \mathscr{L}_2 P_{1,1}\right) + \cdots \\
&\quad = \frac{1}{\sqrt{\epsilon}}\mathscr{M}_3 P_{0,0} + \left(\mathscr{M}_1 P_{0,0} + \mathscr{M}_3 P_{1,0}\right) \\
&\quad\quad + \sqrt{\epsilon}\left(\mathscr{M}_1 P_{1,0} + \mathscr{M}_3 P_{2,0}\right) + \cdots.
\end{aligned} \tag{41}$$

Theorem 3. *The correction term $P_{0,1}$ does not depend on the variable y and $P^{\delta}_{0,1} := \sqrt{\delta}P_{0,1}(t,r,\lambda,z;T)$ is given by*

$$P^{\delta}_{0,1}(t,r,\lambda;T) = H(t,T)\, e^{F(t,T)-B(t,T)r-(1-R)C(t,T)\lambda}, \tag{42}$$

with $B(T,T) = C(T,T) = F(T,T) = H(T,T) = 0$, where $H(t,T)$ is given by

$$\begin{aligned}
H(t,T) &= -\int_t^T \left\{V^{\delta}_1 B(s,T) + V^{\delta}_2 (1-R)\, C(s,T)\right\} ds, \\
V^{\delta}_1 &= \sqrt{\delta}\rho_{rz}\sigma h(z), \\
V^{\delta}_2 &= \sqrt{\delta}\rho_{\lambda z}\left\{h_z(z)\langle f\rangle(z) + h(z)\langle f\rangle_z(z)\right\}.
\end{aligned} \tag{43}$$

Here, $B(t,T)$, $C(t,T)$, and $F(t,T)$ are given by (23).

Proof. Multiply (41) by ϵ and let $\epsilon \to 0$. Then, we find the first two leading-order terms as follows:

$$\begin{aligned}
\mathscr{L}_0 P_{0,1} &= 0, \\
\mathscr{L}_0 P_{1,1} + \mathscr{L}_1 P_{0,1} &= \mathscr{M}_3 P_{0,0}.
\end{aligned} \tag{44}$$

Recall that, because $\mathscr{L}_0$ is the infinitesimal generator of the OU process y_t, the solution $P_{0,1}$ of (44) must be a constant with respect to the y variable; that is, $P_{0,1} = P_{0,1}(t,r,\lambda,z;T)$. The next order also generates $\mathscr{L}_0 P_{1,1} = 0$. Because $\mathscr{M}_3$ has a derivative with respect to the y variable and $P_{0,0}$ does not rely on the y variable, we get $\mathscr{M}_3 P_{0,0} = 0$ and $\mathscr{L}_1 P_{0,1} = 0$, respectively, as remarked above. Then, $P_{1,1}$ does not depend on the y variable; that is, $P_{1,1} = P_{1,1}(t,r,\lambda,z;T)$. This means that the first two terms $P_{0,1}$ and $P_{1,1}$ do not depend on the current level y of the fast scale volatility driving the process y_t. In this way, we can continue to eliminate terms of order $1, \sqrt{\epsilon}, \epsilon, \ldots$. For the order-1 term, we have $\mathscr{L}_0 P_{2,1} + \mathscr{L}_1 P_{1,1} + \mathscr{L}_2 P_{0,1} = \mathscr{M}_1 P_{0,0} + \mathscr{M}_3 P_{1,0}$. This PDE becomes $\mathscr{L}_0 P_{2,1} + \mathscr{L}_2 P_{0,1} = \mathscr{M}_1 P_{0,0}$, because $\mathscr{L}_1 P_{1,1} = 0$ and $\mathscr{M}_3 P_{1,0} = 0$. This PDE is a Poisson equation for $P_{2,1}$ with respect to the infinitesimal operator $\mathscr{L}_0$. It is well-known that a solution exists only if $\mathscr{L}_2 P_{0,1} = \mathscr{M}_1 P_{0,0}$ is centered with respect to the invariant distribution of the stochastic volatility process y_t. Because $P_{0,0}$ and $P_{0,1}$ do not depend on the y variable, we obtain

$$\langle \mathscr{L}_2 \rangle P_{0,1} = \mathscr{A}^{\delta} P_{0,0}, \quad t < T, \tag{45}$$

with the terminal condition $P_{0,1}(t,r,\lambda,z;T)|_{t=T} = 0$, where

$$\begin{aligned}
\mathscr{A}^{\delta} &= \sigma h_z(z)\rho_{rz}\frac{\partial}{\partial r} + \rho_{\lambda z}\left(h_z(z)\langle f\rangle(z)\right. \\
&\qquad \left. + h(z)\langle f\rangle_z(z)\right)\frac{\partial}{\partial \lambda},
\end{aligned} \tag{46}$$

$$\langle f\rangle_z(z) := \frac{\partial}{\partial z}\langle f\rangle(z), \qquad h_z(z) := \frac{\partial}{\partial z}h(z),$$

respectively. Because we concentrate on only the first perturbation to $P_{0,0}$, we reset the PDE (45) with respect to $P^{\delta}_{0,1}(t,r,\lambda,z;T) := \sqrt{\delta}P_{0,1}(t,r,\lambda,z;T)$ as follows:

$$\langle \mathscr{L}_2 \rangle P^{\delta}_{0,1} = \sqrt{\delta}\mathscr{A}^{\delta} P_{0,0}, \quad t < T, \tag{47}$$

with the final condition $P^{\delta}_{0,1}(t,r,\lambda,z;T)|_{t=T} = 0$. We then obtain the operator $\mathscr{L}^{\delta}$ expressed as

$$\mathscr{L}^{\delta} = V^{\delta}_1\frac{\partial}{\partial r} + V^{\delta}_2\frac{\partial}{\partial \lambda}. \tag{48}$$

Hence, we obtain (45) as follows:

$$\begin{aligned}
&\frac{\partial P^{\delta}_{1,0}}{\partial t} + \left(\theta_t - ar\right)\frac{\partial P^{\delta}_{1,0}}{\partial r} + \left(\tilde{\theta}_t - \tilde{a}\lambda\right)\frac{\partial P^{\delta}_{1,0}}{\partial \lambda} + \frac{1}{2}\sigma^2\frac{\partial^2 P^{\delta}_{1,0}}{\partial r^2} \\
&\quad + \frac{1}{2}\bar{\sigma}^2(z)\frac{\partial^2 P^{\delta}_{1,0}}{\partial \lambda^2} + \rho_{r\lambda}\sigma\bar{\sigma}(z)\frac{\partial^2 P^{\delta}_{1,0}}{\partial r\partial \lambda} - \left(r + (1-R)\lambda\right)P^{\delta}_{1,0} \\
&\quad = -\left\{V^{\delta}_1 B(t,T) + V^{\delta}_2 (1-R)\, C(t,T)\right\} P_{0,0},
\end{aligned} \tag{49}$$

with the terminal condition $P^{\delta}_{1,0}(t,r,\lambda,z;T)|_{t=T} = 0$. Finally, substituting (42) into (49), we obtain the result of Theorem 3 by direct computation. □

Note that the group parameters $U^{\epsilon}_1, U^{\epsilon}_2, U^{\epsilon}_3, V^{\delta}_1$, and V^{δ}_2, which contain model parameters, are needed for the pricing of the zero-coupon defaultable bond and simplify the estimation procedure. See Fouque et al. [15] for a general reference.

In summary, from (16), the asymptotic analysis of the zero-coupon defaultable bond price gives

$$\begin{aligned}
&P^{\epsilon,\delta}(t,r,\lambda,y,z;T) \\
&\quad \approx P_{0,0}(t,r,\lambda,z;T) + P^{\epsilon}_{1,0}(t,r,\lambda,z;T) + P^{\delta}_{0,1}(t,r,\lambda,z;T), \\
&\qquad t < T,
\end{aligned} \tag{50}$$

with the final condition $P^{\epsilon,\delta}(t,r,\lambda,y,z;T)|_{t=T} = 1$ and

$$\begin{aligned}
P_{0,0}(t,r,\lambda,z;T) &= e^{F(t,T)-B(t,T)r-(1-R)C(t,T)\lambda}, \\
P^{\epsilon}_{1,0}(t,r,\lambda,z;T) &= G(t,T)\, e^{F(t,T)-B(t,T)r-(1-R)C(t,T)\lambda}, \\
P^{\delta}_{0,1}(t,r,\lambda,z;T) &= H(t,T)\, e^{F(t,T)-B(t,T)r-(1-R)C(t,T)\lambda},
\end{aligned} \tag{51}$$

Table 1: Effect of the recovery rate.

Recovery rate (R)	0.1	0.2	0.3	0.5	0.7	0.8
Fast scale factor	0.35	0.09	−0.11	−0.29	−0.27	−0.20
Slow scale factor	−0.43	−0.26	−0.10	0.24	0.57	0.73
Multiple scale factors	−0.10	−0.20	−0.21	−0.03	0.34	0.58

where $P_{0,0}(t,r,\lambda,z;T)$, $P^{\epsilon}_{1,0}(t,r,\lambda,z;T)$, and $P^{\delta}_{0,1}(t,r,\lambda,z;T)$ are given by (22), (30), and (42), respectively. In terms of accuracy of the approximation, we refer to the error estimate obtained by Papageorgiou and Sircar [8].

3.4. Numerical Results. In this subsection, we conclude our paper with some sensitivity analyses of the model parameters. The parameter values used to obtain the results shown in Table 1 are $a = 0.2$, $\tilde{a} = 0.3$, $U^{\epsilon}_1 = 0.01$, $U^{\epsilon}_2 = -0.03$, $U^{\epsilon}_3 = 0.04$, $V^{\delta}_1 = 0.02$, $V^{\delta}_2 = -0.03$, $t = 0$, and $T = 1$. We calculate the magnitude of mispricing with respect to the zero-coupon defaultable bond as a percentage of the face value of the bond. That is, we show that the zero-coupon defaultable bond tends to be overpriced or underpriced in terms of the parameters involved. The mispricing, as a percentage of the zero-coupon defaultable bond price, caused by changes in the value of the recovery rate is shown in Table 1. The second row shows results when only the fast scale factor of the intensity ($V^{\delta}_1 = V^{\delta}_2 = 0$) is considered and the third row shows results when only the slow scale factor ($U^{\epsilon}_1 = U^{\epsilon}_2 = U^{\epsilon}_3 = 0$) is considered. The fourth row shows the results using multiple scale factors. We can see from Table 1 that the mispricing of the zero-coupon defaultable bond with a fast scale factor does not behave monotonically as the value of the recovery rate increases, but the mispricing of the zero-coupon defaultable bond with a slow scale factor monotonically increases with the value of the recovery rate. Additionally, the mispricing of the zero-coupon defaultable bond with multiple scale factors does not behave monotonically as the value of the recovery rate increases. From the above investigation, we note that the mispricing of the zero-coupon defaultable bond takes various forms depending on the recovery rate parameters involved.

Figures 1 and 2 show results for parameter values of $a = 0.2$, $\tilde{a} = 0.3$, $\sigma = 0.1$, $\overline{\sigma}(z) = 0.15$, $\breve{\sigma}^2(z) = 0.18$, $\rho_{r\lambda} = -0.2$, $U^{\epsilon}_1 = -0.07$, $U^{\epsilon}_2 = 0.01$, $U^{\epsilon}_3 = -0.06$, $V^{\delta}_1 = 0.01$, $V^{\delta}_2 = -0.04$, $r = 0.15$, $\lambda = 0.13$, $t = 0$, $T = 10$, and $R = 0.4$. These figures illustrate the defaultable bond price and the corresponding yield curve, respectively. Figures 1 and 2 have two types of curve. Case 1 corresponds to constant volatility in the intensity process, whereas Case 2 contains both fast and slow scale factors. Figure 1 shows that the defaultable bond prices with stochastic intensity become higher than those with constant volatility as the time to maturity increases. In addition, the hump-shaped yield curve, which matches those in structural models [10], appears in Case 2 in Figure 2. That is, the multiple time scales have both quantitative and qualitative effects. Hence, Figures 1 and 2 show the significant effect of multiple time scales in the stochastic intensity on both defaultable bond prices and yields.

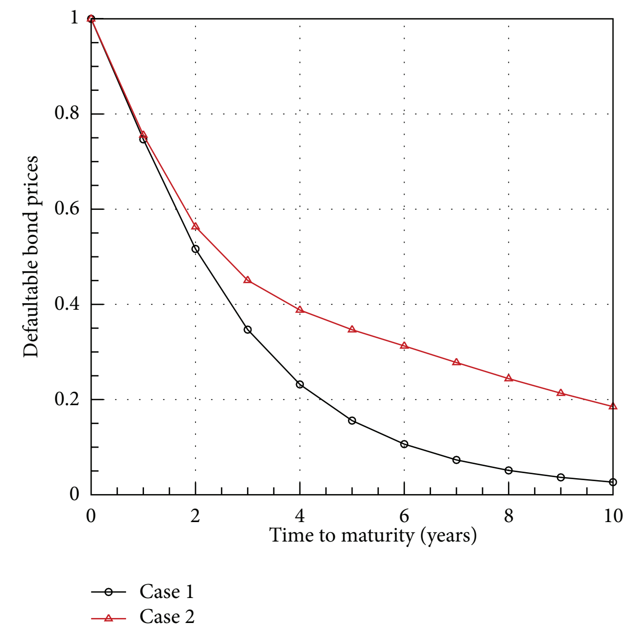

Figure 1: Defaultable bond prices (source: KIS Pricing).

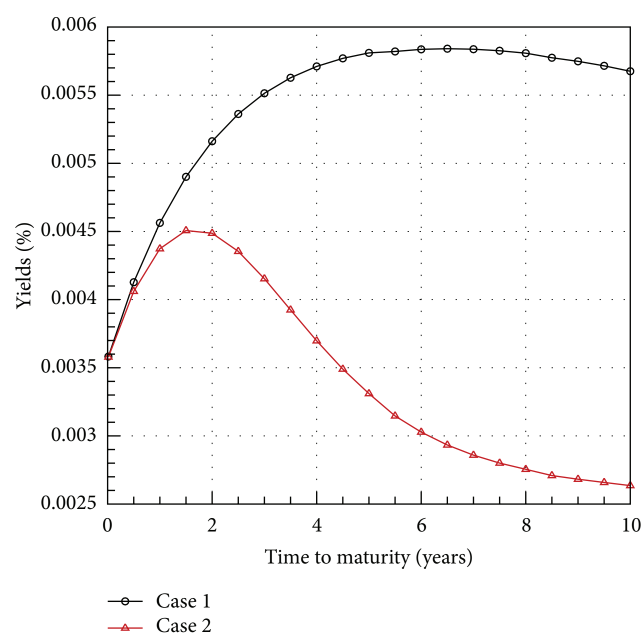

Figure 2: Yield curves (source: KIS Pricing).

4. Credit Default Swap and Bond Option

In this section, we derive a formula for the CDS rate and obtain another formula for options when the underlying asset is a zero-coupon defaultable bond using the results of Section 3.

4.1. Credit Default Swap

4.1.1. Preliminaries. A CDS is a bilateral contract in which one party (the protection buyer) pays a periodic, fixed premium to another (the protection seller) for protection related to credit events on a reference entity. If a credit event occurs, the protection seller is obliged to make a payment to the protection buyer to compensate him for any losses that he might otherwise incur. Then, the credit risk of the reference entity is transferred from the protection buyer to the protection seller. In particular, Ma and Kim [6] studied the problem of default correlation when the reference entity and the protection seller can default simultaneously. We assume the following.

(i) We consider a forward CDS rate, valuable after some initial time t_0 with $0 \le t_0 < t_1$, to use the results of Section 3.

(ii) Let T be the time-to-maturity of a forward CDS contract, $t_1 < \cdots < t_N = T$ the premium payment dates, and $\mathcal{T} = (t_1, \ldots, t_N)$ the payment tenor.

(iii) We assume that, in a credit event, the bond recovers a proportion R of its face value, and the protection seller provides the remaining proportion $1 - R$ to the protection buyer.

(iv) If the settlement takes place at a coupon date following the credit event occurring to the reference entity, then we do not take into account this accrued premium payment.

For a detailed explanation, refer to O'Kane and Turnbull [18].

Let us denote by $C(t, t_0; \mathcal{T})$ the price of the forward CDS rate. Note that the spread of a CDS rate is given by the spread of a forward CDS when $t_0 = t$. Let us first consider the case of the protection buyer. The premium leg is the series of payments of the forward CDS rate until maturity or until the first credit event τ. Let us denote by $C^{pb}(t, t_0; \mathcal{T})$ the price of the protection buyer paying $1_{(\tau > t)}$ at time t. This is given by

$$\begin{aligned} C^{pb}(t, t_0; \mathcal{T}) &= C(t, t_0; \mathcal{T}) \\ &\times \sum_{n=1}^{N} E^* \left[e^{-\int_t^{t_n} (r_s + \lambda_s) ds} \mid r_t = r, \lambda_t = \lambda, y_t = y, z_t = z \right]. \end{aligned} \tag{52}$$

Note that the payment made by the protection buyer is zero if $\tau \le t$. Here, we can easily solve (52) using the results of Section 3 when $R = 0$ (zero recovery). On the other hand, the price $C^{ps}(t, t_0; \mathcal{T})$ demanded by the protection seller at the credit event time τ is given by

$$\begin{aligned} C^{ps}(t, t_0; \mathcal{T}) &= (1 - R) E^* \\ &\times \left[\int_{t_0}^{t_N} e^{-\int_t^u (r_s + \lambda_s) ds} \lambda_u du \mid r_t = r, \lambda_t = \lambda, y_t = y, z_t = z \right], \end{aligned} \tag{53}$$

with $t < \tau$. Note that the payment demanded by the protection seller is zero if $\tau \le t$. Finally, from the no-arbitrage condition, based on the pricing of each counterparty's position by substituting (52) and (53), we obtain the forward CDS rate $C(t, t_0; \mathcal{T})$ as follows:

$$\begin{aligned} C(t, t_0; \mathcal{T}) &= (1 - R) \\ &\times \frac{E^* \left[\int_{t_0}^{t_N} e^{-\int_t^u (r_s + \lambda_s) ds} \lambda_u du \mid r_t = r, \lambda_t = \lambda, y_t = y, z_t = z \right]}{\sum_{n=1}^{N} E^* \left[e^{-\int_t^{t_n} (r_s + \lambda_s) ds} \mid r_t = r, \lambda_t = \lambda, y_t = y, z_t = z \right]}. \end{aligned} \tag{54}$$

Now, we will calculate (53) using the asymptotic analysis.

4.1.2. Asymptotic Analysis of the CDS Rate. For the protection seller payment, we put

$$\begin{aligned} \breve{C}(t, r, \lambda, y; T) &= E^* \left[\int_t^T e^{-\int_t^u (r_s + \lambda_s) ds} \lambda_u du \mid r_t = r, \lambda_t = \lambda, y_t = y, z_t = z \right], \end{aligned} \tag{55}$$

and then

$$C^{ps}(t, t_0; \mathcal{T}) = \breve{C}(t, r, \lambda, y; T) - \breve{C}(t, r, \lambda, y; t_0). \tag{56}$$

Using the four-dimensional Feynman-Kac formula, we have the Kolmogorov PDE

$$\begin{gathered} \mathcal{L}^{ps} \breve{C}(t, r, \lambda, y, z; T) + \lambda = 0, \quad t < T, \\ \breve{C}(t, r, \lambda, y, z; T)\big|_{t=T} = 0, \end{gathered} \tag{57}$$

where

$$\begin{aligned} \mathcal{L}^{ps} &:= \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2^{ps} + \sqrt{\delta} \mathcal{M}_1 + \delta \mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}} \mathcal{M}_3, \\ \mathcal{L}_2^{ps} &= \frac{\partial}{\partial t} + (\theta_t - a r) \frac{\partial}{\partial r} + (\widetilde{\theta}_t - \widetilde{a} \lambda) \frac{\partial}{\partial \lambda} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial r^2} \\ &\quad + \frac{1}{2} f^2(y) \frac{\partial^2}{\partial \lambda^2} + \sigma f(y) \rho_{r\lambda} \frac{\partial^2}{\partial r \partial \lambda} - (r + \lambda), \end{aligned} \tag{58}$$

where $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ are defined in Section 3. Here, the operator $\mathcal{L}_2^{ps}$ is equal to the operator $\mathcal{L}_2$ when $R = 0$.

Now, we use the notation $\breve{C}_{i,j}$ for the $\epsilon^{i/2} \delta^{j/2}$-order term for $i = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$. We first expand with respect to half-powers of δ and then for each of these terms we expand with respect to half-powers of ϵ. The expansion of $\breve{C}^{\epsilon,\delta}$ is

$$\begin{aligned} \breve{C}^{\epsilon,\delta} &= \breve{C}_0^{\epsilon} + \sqrt{\delta} \breve{C}_1^{\epsilon} + \delta \breve{C}_2^{\epsilon} + \cdots, \\ \breve{C}_k^{\epsilon} &= \breve{C}_{0,k} + \sqrt{\epsilon} \breve{C}_{1,k} + \epsilon \breve{C}_{2,k} + \epsilon^{3/2} \breve{C}_{3,k} + \cdots, \end{aligned} \tag{59}$$

for $k = 0, 1, 2, \ldots$. The expansion (59) leads to the leading-order term $\check{C}_0^{\epsilon}$ and the next-order term $\check{C}_1^{\epsilon}$ is defined by

$$\left(\frac{1}{\epsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_1 + \mathscr{L}_2^{ps}\right)\check{C}_0^{\epsilon} = 0, \quad t < T, \tag{60}$$

with the terminal condition $\check{C}_0^{\epsilon}(t, r, \lambda, y, z; T)|_{t=T} = 0$ and

$$\left(\frac{1}{\epsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_1 + \mathscr{L}_2^{ps}\right)\check{C}_1^{\epsilon} = \left(\mathscr{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathscr{M}_3\right)\check{C}_0^{\epsilon}, \quad t < T, \tag{61}$$

with the terminal condition $\check{C}_1^{\epsilon}(t, r, \lambda, y, z; T)|_{t=T} = 0$, respectively. We insert $k = 0$ in (59) as

$$\check{C}_0^{\epsilon} = \check{C}_{0,0} + \sqrt{\epsilon}\check{C}_{1,0} + \epsilon\check{C}_{2,0} + \epsilon^{3/2}\check{C}_{3,0} + \cdots. \tag{62}$$

Applying the expanded solution (62) to (60) leads to

$$\begin{aligned}&\frac{1}{\epsilon}\mathscr{L}_0\check{C}_{0,0} + \frac{1}{\sqrt{\epsilon}}\left(\mathscr{L}_0\check{C}_{1,0} + \mathscr{L}_1\check{C}_{0,0}\right)\\&\quad + \left(\mathscr{L}_0\check{C}_{2,0} + \mathscr{L}_1\check{C}_{1,0} + \mathscr{L}_2^{ps}\check{C}_{0,0}\right)\\&\quad + \sqrt{\epsilon}\left(\mathscr{L}_0\check{C}_{3,0} + \mathscr{L}_1\check{C}_{2,0} + \mathscr{L}_2^{ps}\check{C}_{1,0}\right) + \cdots = 0.\end{aligned} \tag{63}$$

Using similar processes as in Section 3, we will calculate the leading-order term $\check{C}_{0,0}$ and the first perturbation term $\check{C}_{1,0}$. That is, we let

$$\left\langle \mathscr{L}_2^{ps}\right\rangle \check{C}_{0,0} + \lambda = 0, \tag{64}$$

with the terminal condition $\check{C}_{0,0}(t, r, \lambda, z; T)|_{t=T} = 0$ and

$$\begin{aligned}&\left\langle \mathscr{L}_2^{ps}\right\rangle \check{C}_{1,0}^{\epsilon} = \mathscr{A}^{ps}\check{C}_{0,0}, \qquad \check{C}_{1,0}^{\epsilon} := \sqrt{\epsilon}\check{C}_{1,0},\\&\mathscr{A}^{ps} := \frac{1}{\sqrt{\alpha}}\left\langle \mathscr{L}_1\mathscr{L}_0^{-1}\left(\mathscr{L}_2^{ps} - \left\langle \mathscr{L}_2^{ps}\right\rangle\right)\right\rangle,\end{aligned} \tag{65}$$

with the terminal condition $\check{C}_{1,0}^{\epsilon}(t, r, \lambda, z; T)|_{t=T} = 0$. Let $\check{C}_{0,0}(t, r, \lambda, z; T)$ be equal to the leading-order term $P_{0,0}(t, r, \lambda, z; T)$ in (22) with zero recovery; that is,

$$\check{C}_{0,0}(t, r, \lambda, z; T) = e^{F(t,T) - B(t,T)r - C(t,T)\lambda}, \tag{66}$$

where $B(t, T)$, $C(t, T)$, and $F(t, T)$ are given by (23). Then, the solution of the inhomogeneous PDE (64) is given by

$$\check{C}_{0,0}(t, r, \lambda, z; T) = \lambda\int_t^T P_{0,0}(s, r, \lambda, z; T)\, ds, \tag{67}$$

with the terminal condition $\check{C}_{0,0}(t, r, \lambda, z; T)|_{t=T} = 0$ and the solution of the PDE (65) is given by

$$\begin{aligned}&\check{C}_{1,0}^{\epsilon}(t, r, \lambda; T)\\&\quad = \lambda\int_t^T\int_s^T\left[U_1^{\epsilon}\left\{B(s,h)C^2(s,h)\right\} + U_2^{\epsilon}\left\{B^2(s,h)C(s,h)\right\}\right.\\&\qquad\left. + U_3^{\epsilon}\left\{C^3(s,h)\right\}\right]\check{C}_{0,0}(s, r, \lambda, z; h)\, dh\, ds,\end{aligned} \tag{68}$$

where $U_1^{\epsilon}, U_2^{\epsilon}$, and U_3^{ϵ} are given by (30). Also, we insert $k = 1$ in (59) as

$$\check{C}_1^{\epsilon} = \check{C}_{0,1} + \sqrt{\epsilon}\check{C}_{1,1} + \epsilon\check{C}_{2,1} + \epsilon^{3/2}\check{C}_{3,1} + \cdots. \tag{69}$$

Substituting the expansions (62) and (69) into (61), we get

$$\begin{aligned}&\frac{1}{\epsilon}\mathscr{L}_0\check{C}_{0,1} + \frac{1}{\sqrt{\epsilon}}\left(\mathscr{L}_0\check{C}_{1,1} + \mathscr{L}_1\check{C}_{0,1}\right)\\&\quad + \left(\mathscr{L}_0\check{C}_{2,1} + \mathscr{L}_1\check{C}_{1,1} + \mathscr{L}_2^{ps}\check{C}_{0,1}\right)\\&\quad + \sqrt{\epsilon}\left(\mathscr{L}_0 P_{3,1} + \mathscr{L}_1\check{C}_{2,1} + \mathscr{L}_2^{ps}\check{C}_{1,1}\right) + \cdots\\&\quad = \frac{1}{\sqrt{\epsilon}}\mathscr{M}_3\check{C}_{0,0} + \left(\mathscr{M}_1\check{C}_{0,0} + \mathscr{M}_3\check{C}_{1,0}\right)\\&\qquad + \sqrt{\epsilon}\left(\mathscr{M}_1\check{C}_{1,0} + \mathscr{M}_3\check{C}_{2,0}\right) + \cdots.\end{aligned} \tag{70}$$

Using similar processes as in Section 3, we will calculate the first perturbation term $\check{C}_{0,1}^{\delta}$. That is, we let

$$\left\langle \mathscr{L}_2^{ps}\right\rangle \check{C}_{0,1}^{\delta} = \sqrt{\delta}\mathscr{M}_1\check{C}_{0,0}, \qquad \check{C}_{0,1}^{\delta} := \sqrt{\delta}\check{C}_{0,1}, \tag{71}$$

with the terminal condition $\check{C}_{0,1}^{\delta}(t, r, \lambda, z; T)|_{t=T} = 0$. Then, the solution of the PDE (71) is given by

$$\begin{aligned}&\check{C}_{0,1}^{\delta}(t, r, \lambda; T)\\&\quad = \lambda\int_t^T\int_s^T\left(V_1^{\delta}B(s,h) + V_2^{\delta}C(s,h)\right)\\&\qquad \times \check{C}_{0,0}(s, r, \lambda, z; h)\, dh\, ds,\end{aligned} \tag{72}$$

with the terminal condition $\check{C}_{1,0}^{\delta}(t, r, \lambda, z; T)|_{t=T} = 0$, where V_1^{δ} and V_2^{δ} are given by (43). Therefore, combining (67), (68), and (72), we derive an asymptotic expression for the protection seller, which is given by

$$\begin{aligned}&\check{C}^{ps}(t, r, \lambda, y; T)\\&\quad \approx \check{C}_{0,0}(t, r, \lambda; T) + \check{C}_{1,0}^{\epsilon}(t, r, \lambda; T) + \check{C}_{0,1}^{\delta}(t, r, \lambda; T).\end{aligned} \tag{73}$$

4.2. Bond Option Pricing. We use T and T_0, $t \le T_0 < T$, to denote the maturity of the zero-coupon defaultable bond and the maturity of the option written on that zero-coupon defaultable bond, respectively. We assume that the option becomes invalid when a credit event occurs before T_0. Then, the option price with the fractional recovery assumption, denoted by $X(t, r, \lambda, y, z; T_0, T)$, is given by

$$\begin{aligned}&X(t, r, \lambda, y, z; T_0, T)\\&\quad = E^*\left[e^{-\int_t^{T_0} r_s ds} l\left(P\left(T_0, r_{T_0}, \lambda_{T_0}, y_{T_0}, z_{T_0}; T\right)\right) \mid r_t = r,\right.\\&\qquad\left. \lambda_t = \lambda, y_t = y, z_t = z\right],\end{aligned} \tag{74}$$

under risk-neutral probability P^*, where the bond price $P(T_0, r_{T_0}, \lambda_{T_0}, y_{T_0}, z_{T_0}; T)$ is

$$P\left(T_0, r_{T_0}, \lambda_{T_0}, y_{T_0}, z_{T_0}; T\right) = E^*\left[e^{-\int_{T_0}^{T}(r_s+(1-R)\lambda_s)ds} \mid r_{T_0}, \lambda_{T_0}, y_{T_0}, z_{T_0}\right], \tag{75}$$

and $l(P(T_0, r_{T_0}, \lambda_{T_0}, y_{T_0}, z_{T_0}; T))$ is the payoff function of the option at time T_0. Here, for simplicity, we assume that the payoff function l is at best linearly growing at infinity and is smooth. In fact, the nonsmoothness assumption on l can be treated by a nontrivial regularization argument, as presented in Fouque et al. [19].

From the four-dimensional Feynman-Kac formula, we obtain $X(t, r, \lambda, y, z; T_0, T)$ as a solution of the Kolmogorov PDE

$$\begin{aligned}
&\frac{\partial X}{\partial t} + (\theta_t - ar)\frac{\partial X}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 X}{\partial r^2} + \left(\tilde{\theta}_t - \tilde{a}\lambda\right)\frac{\partial X}{\partial \lambda} \\
&\quad + \frac{1}{2}f^2(y,z)\frac{\partial^2 X}{\partial \lambda^2} + \frac{1}{\epsilon}(m-y)\frac{\partial X}{\partial y} + \frac{\nu^2}{\epsilon}\frac{\partial^2 X}{\partial y^2} + \delta g(z)\frac{\partial X}{\partial z} \\
&\quad + \frac{1}{2}\delta h^2(z)\frac{\partial^2 X}{\partial z^2} + \rho_{r\lambda}\sigma f(y,z)\frac{\partial^2 X}{\partial r\partial \lambda} + \rho_{ry}\sigma\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\frac{\partial^2 X}{\partial r\partial y} \\
&\quad + \rho_{rz}\sigma\sqrt{\delta}h(z)\frac{\partial^2 X}{\partial r\partial z} + \rho_{\lambda y}f(y,z)\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\frac{\partial^2 P}{\partial \lambda\partial y} \\
&\quad + \rho_{\lambda z}f(y,z)\sqrt{\delta}h(z)\frac{\partial^2 X}{\partial \lambda\partial z} + \rho_{yz}\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\sqrt{\delta}h(z)\frac{\partial^2 X}{\partial y\partial z} \\
&\quad - \{r + (1-R)\lambda\}X = 0,
\end{aligned} \tag{76}$$

with the final condition $X(t, r, \lambda, y, z; T_0, T)|_{t=T_0} = l(P(T_0, r_{T_0}, \lambda_{T_0}, y_{T_0}, z_{T_0}; T))$. Hence, keeping the notation used in the pricing of zero-coupon defaultable bonds with fractional recovery in Section 3, but with a different terminal condition, the option price $X(t, r, \lambda, y, z; T_0, T)$ contains the Feynman-Kac formula problems:

$$\begin{gathered}
\mathscr{L}X(t, r, \lambda, y, z; T_0, T) = 0, \quad t < T_0, \\
X(t, r, \lambda, y, z; T_0, T)|_{t=T_0} = l\left(P\left(T_0, r_{T_0}, \lambda_{T_0}, y_{T_0}, z_{T_0}; T\right)\right),
\end{gathered} \tag{77}$$

with

$$\begin{aligned}
X &= X_0^\epsilon + \sqrt{\delta}X_1^\epsilon + \delta X_2^\epsilon + \cdots, \\
X_k^\epsilon &= X_{0,k} + \sqrt{\epsilon}X_{1,k} + \epsilon X_{2,k} + \epsilon^{3/2}X_{3,k} + \cdots,
\end{aligned} \tag{78}$$

for $k = 0, 1, 2 \ldots$. Using the assumed smoothness of the payoff function l, the terminal condition (77) can be expanded as follows:

$$X(t, r, \lambda, y, z; T_0, T)|_{t=T_0} \tag{79}$$

$$= l\left(P_{0,0}\left(T_0, r_{T_0}, \lambda_{T_0}, y_{T_0}, z_{T_0}; T\right)\right)$$

$$+ P_{1,0}^\epsilon\left(T_0, r_{T_0}, \lambda_{T_0}, y_{T_0}, z_{T_0}; T\right) \times l'\left(P_{0,0}\left(T_0, r_{T_0}, \lambda_{T_0}, y_{T_0}, z_{T_0}; T\right)\right) \tag{80}$$

$$+ P_{0,1}^\delta\left(T_0, r_{T_0}, \lambda_{T_0}, y_{T_0}, z_{T_0}; T\right) \times l'\left(P_{0,0}\left(T_0, r_{T_0}, \lambda_{T_0}, y_{T_0}, z_{T_0}; T\right)\right), \tag{81}$$

where $P_{0,0}$, $P_{1,0}^\epsilon$, and $P_{0,1}^\delta$ are given by (22), (30), and (42), respectively. Using a similar argument as in Section 3, the terms of order $1/\epsilon$ and $1/\sqrt{\epsilon}$ have y-independence in $X_{0,0}$, $X_{1,0}$, and $X_{0,1}$. The order-1 terms give a Poisson equation in $X_{2,0}$, with which the solvability condition $\langle \mathscr{L}_2 \rangle X_{0,0} = 0$ is satisfied. From the solution calculated in Section 3.1, we have

$$\begin{aligned}
&X_{0,0}(t, r, \lambda, z; T_0, T) \\
&\quad = E^*\left[e^{-\int_t^{T_0} r_s ds}h\left(P_{0,0}\left(T_0, r_{T_0}, \lambda_{T_0}, z_{T_0}; T\right)\right) \mid r_t = r, \right. \\
&\qquad \left. \lambda_t = \lambda, z_t = z\right],
\end{aligned} \tag{82}$$

with the terminal condition (79), where $P_{0,0}(T_0, r_{T_0}, \lambda_{T_0}, z_{T_0}; T)$ is given by (22) at time $t = T_0$. The order-$\sqrt{\epsilon}$ terms give a Poisson equation in $X_{3,0}$, with which the solvability condition $\langle \mathscr{L}_1 X_{2,0} + \mathscr{L}_2 X_{1,0}\rangle = 0$. If we put $X_{1,0}^\epsilon = \sqrt{\epsilon}X_{1,0}$, then this solvability condition leads to the PDE

$$\langle \mathscr{L}_2 \rangle X_{1,0}^\epsilon(t, r, \lambda, z; T_0, T) = \sqrt{\epsilon}\mathscr{A}^\epsilon X_{0,0}(t, r, \lambda, z; T_0, T), \quad t < T_0, \tag{83}$$

with the terminal condition (80), where the infinitesimal operator $\mathscr{A}^\epsilon$ is given by (35). Hence, by applying the Feynman-Kac formula to (82) and (83), we get the following probabilistic representation of the first perturbation term:

$$\begin{aligned}
&X_{1,0}^\epsilon(t, r, \lambda, z; T_0, T) \\
&= E^*\left[e^{-\int_t^{T_0} r_s ds}P_{1,0}^\epsilon\left(T_0, r_{T_0}, \lambda_{T_0}, z_{T_0}; T\right)\right. \\
&\qquad \times l'\left(P_{0,0}\left(T_0, r_{T_0}, \lambda_{T_0}, z_{T_0}; T\right)\right) \\
&\qquad - \int_t^{T_0} e^{-\int_t^u r_s ds}\mathscr{A}^\epsilon X_{0,0} \\
&\qquad\quad \left. \times (u, r_u, \lambda_u, z_u; T_0, T)\,du \mid r_t = r, \lambda_t = \lambda, Z_t = z\right].
\end{aligned} \tag{84}$$

The order-$\sqrt{\delta}$ terms give a Poisson equation $X_{2,1}$, with which the solvability condition $\langle \mathscr{L}_2 X_{0,1} \rangle = \langle \mathscr{A}^{\delta} X_{0,0} \rangle$. If we put $X_{0,1}^{\delta} := \sqrt{\delta} X_{0,1}$, then this solvability condition leads to the PDE

$$\langle \mathscr{L}_2 \rangle X_{0,1}^{\delta} (t, r, \lambda, z; T_0, T) = \sqrt{\delta} \mathscr{A}^{\delta} X_{0,0} (t, r, \lambda, z; T_0, T), \quad t < T_0, \tag{85}$$

where the operator $\mathscr{A}^{\delta}$ is obtained by (46). Hence, by applying the Feynman-Kac formula to (82) and (85), we have the following probabilistic representation of the first perturbation term:

$$\begin{aligned} X_{0,1}^{\delta} & (t, r, \lambda, z; T_0, T) \\ = E^* \Big[& e^{-\int_t^{T_0} r_s ds} P_{0,1}^{\delta} \left(T_0, r_{T_0}, \lambda_{T_0}, z_{T_0}; T \right) \\ & \times l' \left(P_0 \left(T_0, r_{T_0}, \lambda_{T_0}, z_{T_0}; T \right) \right) \\ & - \int_t^{T_0} e^{-\int_t^u r_s ds} \mathscr{A}^{\delta} X_{0,0} \\ & \quad \times (u, r_u, \lambda_u, z_u; T_0, T)\, du \mid r_t = r, \lambda_t = \lambda, z_t = z \Big]. \end{aligned} \tag{86}$$

In summary, we have an asymptotic expression for the zero-coupon defaultable bond option price with fractional recovery, which is given by

$$\begin{aligned} X & (t, r, \lambda, y, z; T_0, T) \\ & \approx X_{0,0} (t, r, \lambda, z; T_0, T) + X_{1,0}^{\epsilon} (t, r, \lambda, z; T_0, T) \\ & \quad + X_{0,1}^{\delta} (t, r, \lambda, z; T_0, T), \end{aligned} \tag{87}$$

where $X_{0,0}(t, r, \lambda, z; T_0, T)$, $X_{1,0}^{\epsilon}(t, r, \lambda, z; T_0, T)$, and $X_{0,1}^{\delta}(t, r, \lambda, z; T_0, T)$ are given by (82), (84), and (86), respectively.

5. Final Remarks

In this paper, we have studied the effect of applying stochastic volatility to the default intensity in zero-coupon defaultable bonds. To model this, we considered the correlated Hull and White model, developed by Tchuindjo [7], with multiple time scales in the stochastic volatility of the intensity process. Using asymptotic analysis, we obtained approximate solutions to price zero-coupon defaultable bonds, credit default swap rates, and bond options when ϵ and δ are independently small parameters. To understand multiple time scales, we provided numerical examples to price the zero-coupon defaultable bonds as well as the yield curve. This showed how these multiple time scales can have both quantitative and qualitative effects. In future work, we will provide an efficient tool for calibrating the zero-coupon defaultable bond intensity models from market yield spreads.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work was supported by Korea Institute of Science and Technology Information (KISTI) under Project no. K-14-L06-C14-S01.

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